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OPTIMAL ERROR ESTIMATES FOR FINITE ELEMENTS ON MESHES CONTAINING BANDS OF CAPS

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Abstract: In this short note we provide an optimal analysis of finite element convergence on meshes containing a so-called band of caps. These structures consist of a zig-zag arrangement of 'degenerating' triangles which violate the maximum angle condition. A necessary condition on the geometry of such a structure for various H^1-convergence rates was previously given by Kučera. Here we prove that the condition is also sufficient, providing an optimal analysis of this special case of meshes. In the special case of optimal $O(h)$-convergence of finite elements, the analysis states that such optimal convergence is possible if and only if the height of the band of caps is at least Ch^2 for some constant C. Numerical experiments confirm this result.

Keywords: Finite element method, error estimates, maximum angle condition

MSC: 65N30, 65N15, 65N50

1. Introduction

The finite element method is the golden standard of current methods for partial differential equations. Much work has been devoted over the past 60 years to develop various error estimates for this method applied a wide range of problems. It may therefore seem surprising that the simplest basic question remains unanswered to this day: What is a necessary and sufficient condition on triangular meshes for piecewise linear finite elements to converge? Even in the simplest of all settings – Poisson's problem and estimates in the corresponding $H^1(\Omega)$ energy norm, this is still an open problem.

The basic textbook result is that if the meshes satisfy the *minimum angle condition*, then finite elements will exhibit optimal $O(h)$ convergence in the energy norm. This condition requires that all angles of all elements in the mesh(es) are uniformly bounded away from zero. A slightly more advanced result is that $O(h)$ convergence occurs under the more general *maximum angle condition*, which requires that the

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maximal angles of all triangles are uniformly bounded away from π. This sufficient condition was generally assumed to also be necessary – the confusion was caused by the misleading title "The maximum angle condition is essential" from the original paper [1]. The title refers to a counterexample provided in the paper, where finite elements do not converge on a special mesh consisting only of 'degenerating' elements. As it turns out, the maximum angle condition is not necessary for $O(h)$ convergence of the finite element method, cf. [3]. Since then, another counterexample was analyzed in the paper [6], where a single structure, a so-called band of caps, contained in the mesh destroys finite element convergence. The analysis leads to conditions on the proportions and geometry of the band of caps that is necessary for $O(h)$ convergence, and more generally $O(h^\alpha)$ convergence for some $\alpha \in [0, 1]$.

The purpose of this short note is to show that the condition on the band of caps derived in [6] is optimal, i.e. both necessary and sufficient for $O(h^\alpha)$ convergence. Although the question of a general necessary and sufficient condition for the convergence of the finite element method still remains open, at least there is a second special case that can be analyzed optimally. The main result of the analysis is that $O(h)$ convergence of finite elements occurs if and only if the height of the band of caps is at least Ch^2 for some constant C. This is important, as a band of caps is a natural triangulation of a (straight) interface. In 2D, an interface is a 1D object, and it is natural to approximate it using very flat triangles in a mesh. The theorem states that the triangles approximating the interface can be flatter and flatter as we refine the mesh, as long as their height is at least Ch^2. We present numerical experiments that confirm this result, and also indicate that a height of at least Ch^2 is also necessary and sufficient for $O(h^2)$-convergence in the L^2-norm, a result that we are unable to prove rigorously.

2. Finite element method

As a model problem, we will be focused on Poisson's problem in $\mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with Lipschitz boundary $\partial\Omega$. We solve the problem

$$-\Delta u = f \text{ on } \Omega, \quad u|_{\partial\Omega} = 0 \tag{1}$$

with the weak form: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where $H_0^1(\Omega)$ is the standard Sobolev space of functions with square integrable derivatives and a zero trace on $\partial\Omega$, while $(f, v) = \int_\Omega f v \,\mathrm{d}x$ is the L^2 scalar product.

In the finite element method, we consider a conforming triangulation $\mathcal{T}_h$ of Ω, i.e. a partition into triangles (elements) with mutually disjoint interiors such that the intersection of two neighboring elements is either a single vertex or a whole edge.

Here h denotes the length of the longest edge in the triangulation. This partition defines the continuous piecewise linear finite element space

$$V_h = \{v_h \in C(\overline{\Omega}); v_h|_K \in P^1(K) \text{ for all } K \in \mathcal{T}_h\}, \tag{3}$$

where $P^1(K)$ is the space of linear functions on the triangular element $K \in \mathcal{T}_h$.

The finite element method is then defined as follows: Find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \,\mathrm{d}x = (f, v_h), \quad \forall v_h \in V_h. \tag{4}$$

It is desirable to obtain estimates for the error $u - u_h$. To this end, Céa's lemma, cf. [2], gives us an estimate in the $H^1(\Omega)$-seminorm:

$$|u - u_h|_{H^1(\Omega)} = \inf_{v_h \in V_h} |u - v_h|_{H^1(\Omega)}, \tag{5}$$

where $|u|_{H^1(\Omega)} = \sqrt{\int_\Omega |\nabla u|^2 \mathrm{d}x}$. We note that for other problems, one can expect an inequality in (5) and a problem-dependent constant in the upper bound.

Standard finite element estimates are typically derived by taking the piecewise linear Lagrange interpolation $\Pi_h u$ as v_h in (5). This is defined element-wise: on each element $K \in \mathcal{T}_h$ the function $\Pi_h u|_K = \Pi_K u \in P^1(K)$ coincides with u at the vertices of K. Such a locally defined function naturally gives a globally continuous piecewise linear function in V_h.

For triangles, there is an *optimal* estimate for the interpolation error $u - \Pi_K u$ in seminorms in the general Sobolev space $W^{1,p}(\Omega)$. We will need this estimate only in the special case of $p = \infty$. Consider an arbitrary triangle $K \subset \mathbb{R}^2$. Denote the length of its longest edge as h_K and its height perpendicular to this edge as $\overline{h}_K$. Finally, define R_K as the circumradius of K, i.e. the radius of the circumscribed circle to K. We have the following optimal estimate, cf. [4], [5].

Lemma 1 (Circumradius estimate)**.** *Let $K \subset \mathbb{R}^2$ be an arbitrary triangle. Let $u \in W^{2,p}(K)$, $1 \le p \le \infty$, and let $\Pi_K u$ be the linear Lagrange interpolation of u on K. Then there exists a constant C_c independent of u and K such that*

$$|u - \Pi_K u|_{W^{1,p}(K)} \le C_c R_K |u|_{W^{2,p}(K)} \le C_c \frac{h_K^2}{\overline{h}_K} |u|_{W^{2,p}(K)}. \tag{6}$$

One is especially interested in optimal convergence results of the order $O(h)$ in the $H^1(\Omega)$-seminorm, via (5). A sufficient (but not necessary!) condition for this to happen is when $R_K \le \widetilde{C}h$ for all $K \in \mathcal{T}_h$ with some constant $\widetilde{C}$ independent of h. Geometrically, this is equivalent to satisfying the *maximum angle condition.* This condition requires that all maximal angles α_K of all triangles $K \in \mathcal{T}_h$ are smaller than some $\alpha_0 < \pi$. Then we have the following element-wise estimate, which can then be applied in (5).

Lemma 2 (Maximum-angle condition)**.** *Let $K \subset \mathbb{R}^2$ be a triangle satisfying the maximum angle condition: $\alpha_K \leq \alpha_0 < \pi$ for some fixed α_0. Let $u \in H^2(K)$ and let $\Pi_K u$ be the linear Lagrange interpolation of u on K. Then there exists a constant C_I depending only on α_0 such that*

$$|u - \Pi_K u|_{H^1(K)} \leq C_I h |u|_{H^2(K)}. \tag{7}$$

By taking the piecewise linear element-wise Lagrange interpolation in Céa's lemma (5) one immediately obtains the following error estimate from Lemma (2).

Theorem 3 (Basic error estimate)**.** *Let $u \in H^2(\Omega)$ be the solution of (2) and $u_h \in V_h$ the finite element solution of (4). If $\alpha_K \leq \alpha_0 < \pi$ for all $K \in \mathcal{T}_h$, we have*

$$|u - u_h|_{H^1(\Omega)} \leq C_I h |u|_{H^2(\Omega)}, \tag{8}$$

where C_I is the constant from Lemma 2.

The maximum angle condition has a long and complicated history, being discovered independently by several groups, e.g. [1]. In [3] it was proven that this condition is not necessary for $O(h)$ convergence. In fact $\mathcal{T}_h$ can contain many 'bad' triangles violating the maximum angle condition while still exhibiting optimal $O(h)$ convergence. In other words, the finite element method can converge optimally even when the Lagrange interpolation error goes to infinity. This is especially important when we have a sequence of meshes obtained e.g. by refinement and let $h \to 0$. In this situation one usually considers a set of triangulations $\mathcal{T}_h$, $h \in (0, h_0)$ for some $h_0 > 0$.

Apart from the paper [3], paper [6] has dealt with sufficient as well as necessary conditions for $O(h)$ convergence, or more generally, for $O(h^\alpha)$ estimates with $0 \leq \alpha \leq 1$. Specifically, the so-called *band of caps* has been identified as the basic (but not only) villain preventing optimal convergence of the finite element method. The band of caps consists of triangles in a zigzag pattern, cf. Figure 1, where all of the elements violate the maximum angle condition with the given α_0. Specifically, we shall consider such a band of length L and height $\bar{h}$ consisting of identical isosceles triangles with diameters h, cf. Figure 1. We assume that every $\mathcal{T}_h$ we consider contains one such band, while all other elements satisfy the maximum angle condition with a fixed maximal angle α_0. It is important to note that the length L of the band can also depend on h (e.g. $L \sim \sqrt{h}$, etc.), although the most important case in our situation is that $L \sim 1$ is independent of h.

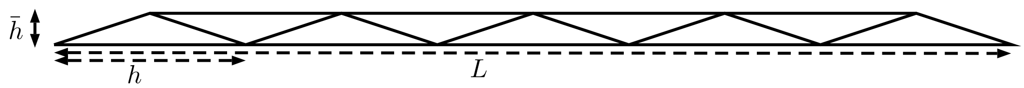

Figure 1: Band of caps of length L and height $\bar{h}$.

The band of caps is important as a model for an approximated interface within the mesh $\mathcal{T}_h$. This is because it is an essentially 1D object (as an interface in 2D would be) with some nonzero thickness $\overline{h}$. It is then desirable to have the thickness of the approximate interface as small as possible without affecting the convergence rate of the finite element method. Due to the regular structure, the finite element error can be analyzed on meshes containing these bands of caps. Specifically, what are conditions on the geometry parameters L and $\overline{h}$ in order to preserve $O(h)$ convergence, or more generally $O(h^\alpha)$ convergence for some $\alpha \in [0,1]$. In [6], the following result is proved as a special case of the main theorem of the paper dealing with a band of general elements (cf. estimate (64) in the cited paper).

Theorem 4. *Let $u \in W^{2,\infty}(\Omega)$ and let $\alpha \in [0,1]$. Let $\mathcal{T}_h, h \in (0, h_0]$ each contain a band of caps $\mathcal{B}$ of length L and height $\overline{h}$. Let $L \geq C_L h^{2\alpha/5}$, where C_L is a sufficiently large constant. Then a necessary condition for the estimate*

$$|u - u_h|_{H^1(\Omega)} \leq \widehat{C} h^\alpha \tag{9}$$

to hold with some $\widehat{C}$ independent of h, is

$$\overline{h} \geq \widetilde{C} h^{4-2\alpha} L \tag{10}$$

for some $\widetilde{C} > 0$.

In the special case of a band of caps of length $L \sim 1$, the condition says that for $O(h)$ convergence of the finite element method, we must necessarily have $\overline{h} \geq \widetilde{C} h^2$ for some $\widetilde{C} > 0$. And for (even arbitrarily slow) convergence of the finite element method, i.e. the limiting case of $\alpha = 0$, we must necessarily have $\overline{h} \geq \widetilde{C} h^4$ for some $\widetilde{C} > 0$. In the next section, we will show that these conditions are both necessary and sufficient.

3. Optimal error estimate for a band of caps

Here we will show that the condition (10) on $\overline{h}$ from Theorem 4 is not only necessary for $O(h^\alpha)$-convergence, but also sufficient. It turns out that unlike the lengthy technical proof of Theorem 4, this is a simple application of the circumradius estimate. In the following, C will be a generic constant independent of u and h.

Theorem 5. *Let $u \in W^{2,\infty}(\Omega)$ and let $\alpha \in [0,1]$. Let $\mathcal{T}_h$ contain a band of caps $\mathcal{B}$ of length L and height $\overline{h}$, while all other elements in $\mathcal{T}_h$ satisfy the maximum angle condition with some α_0. Let there exist $\widetilde{C} > 0$ such that*

$$\overline{h} \geq \widetilde{C} h^{4-2\alpha} L. \tag{11}$$

Then there exists a constant C independent of u and h, such that

$$|u - u_h|_{H^1(\Omega)} \leq C h^\alpha |u|_{W^{2,\infty}(\Omega)}. \tag{12}$$

Proof. From Céa's lemma we have

$$|u-u_h|^2_{H^1(\Omega)} \le |u-\Pi_h u|^2_{H^1(\Omega)} = |u-\Pi_h u|^2_{H^1(\Omega\setminus\mathcal{B})} + |u-\Pi_h u|^2_{H^1(\mathcal{B})}, \tag{13}$$

due to additivity of integrals. The first term in (13) uses standard estimates (all elements of $\Omega \setminus \mathcal{B}$ satisfy the maximum angle condition):

$$|u-\Pi_h u|^2_{H^1(\Omega\setminus\mathcal{B})} \le Ch^2|u|^2_{H^2(\Omega)} \le Ch^2|\Omega||u|^2_{W^{2,\infty}(\Omega)}. \tag{14}$$

The second term in (13) is estimated using the circumradius estimate (6):

$$\begin{aligned} |u-\Pi_h u|^2_{H^1(\mathcal{B})} &= \int_{\mathcal{B}} |\nabla u - \nabla \Pi_h u|^2 \,\mathrm{d}x \le |u-\Pi_h u|^2_{W^{1,\infty}(\mathcal{B})}|\mathcal{B}| \\ &\le C\left(\frac{h^2}{\overline{h}}\right)^2 |u|^2_{W^{2,\infty}(\mathcal{B})}|\mathcal{B}| \le C\frac{h^4}{\overline{h}}L|u|^2_{W^{2,\infty}(\Omega)}, \end{aligned} \tag{15}$$

since $|\mathcal{B}| \le \overline{h}L$. Using assumption (11) on $\overline{h}$ in the right-hand side of (15), we get

$$|u-\Pi_h u|^2_{H^1(\mathcal{B})} \le C\frac{h^4}{\widetilde{C}h^{4-2\alpha}L}L|u|^2_{W^{2,\infty}(\Omega)} = Ch^{2\alpha}|u|^2_{W^{2,\infty}(\Omega)}. \tag{16}$$

Combining estimates (13), (14), and (16), and taking the square root gives us the desired estimate. □

If we are specifically interested in the most interesting case of $L \sim 1$ independent of h, and $\alpha = 1$ (i.e. $O(h)$-convergence), we get the following theorem. It states that the height $\overline{h}$ of the band of caps can in fact go to zero as fast as h^2 without influencing the $O(h)$ convergence rate of the finite element method. For the simulation of interfaces this is good news, since it allows for a finer resolution of the interface (which technically has zero height).

Theorem 6. *Let $u \in W^{2,\infty}(\Omega)$. Let $\mathcal{T}_h$ contain a band of caps $\mathcal{B}$ of length $L \sim 1$ and height $\overline{h}$, while all other elements in $\mathcal{T}_h$ satisfy the maximum angle condition with some α_0. Let there exist $\widetilde{C} > 0$ such that*

$$\overline{h} \ge \widetilde{C}h^2. \tag{17}$$

Then there exists a constant C independent of h and u, such that

$$|u-u_h|_{H^1(\Omega)} \le Ch|u|_{W^{2,\infty}(\Omega)}. \tag{18}$$

4. Numerical experiments

In this section we use numerical experiments to confirm that condition (17) is necessary and sufficient for $O(h)$-convergence. Namely we consider problem (2) on $\Omega = [-1,1]^2$ with the manufactured solution $u(x,y) = \cos(\pi x)\sin(\pi y)$. We consider

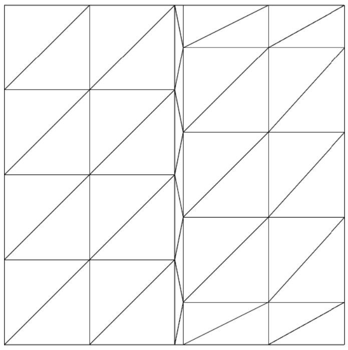

Figure 2: Test mesh containing a vertical band of caps.

meshes with a single vertical band of caps in the center of the domain which spans the whole height of Ω, cf. Figure 2. The mesh outside of the band of caps is very regular, consisting of right-angled triangles. We construct a series of 11 such meshes with decreasing h. We also construct reference meshes which do not contain a band of caps but only identical right-angled triangles throughout the entire mesh.

To test the sharpness of condition (17), we consider meshes containing bands of caps with height $\overline{h} = h^k$ for $k = 2, 2.5$, and 3, where $k = 2$ corresponds to the case of $\overline{h} = h^2$ from Theorem 6. According to the theorem, higher exponents than $k = 2$ should result in slower than $O(h)$-convergence of the finite element method.

The H^1-errors are plotted in Figure 3. We observe that the convergence curves on the 'nice' reference meshes and on the meshes with a band of caps with height $\overline{h} = h^2$ are essentially indistinguishable. On the other hand, the curve corresponding to $k = 3$, i.e. $\overline{h} = h^3$, clearly exhibits a slower convergence rate as $h \to 0$. For $k = 2.5$ the curve also exhibits a decrease in convergence rate, although not as dramatic as $k = 3$. Testing exponents even closer to $k = 2$, e.g. $k = 2.1$ would require extremely fine meshes to observe the slow-down of convergence. Nevertheless, we view Figure 3 as a confirmation of Theorem 6.

Finally, we have also tested convergence in the $L^2(\Omega)$-norm. The results are in Figure 4. In the $L^2(\Omega)$-norm, we expect the error to be $O(h^2)$ under ideal circumstances (e.g. provided the maximum angle condition). This convergence rate can be seen in the convergence curve for the reference meshes. Although we are unable to prove this, we get this optimal convergence rate also in the presence of bands of caps of height $\overline{h} = h^2$ (i.e. the case satisfying Theorem 6). Again the two curves are essentially identical. And as for the H^1-seminorm, the convergence rate in the L^2-norm decreases for the higher exponents $k = 3$ and $k = 2.5$.

We note that we are currently unable to prove the optimal convergence rates in L^2 in the presence of bands of caps, since the proof of Theorems 6 and 5 is based essentially on $L^\infty(\Omega)$ estimates of the gradients and second derivatives. Proving these optimal convergence rates would require the application of the Aubin-Nitsche duality argument in L^∞-based norms, which leads to technical issues that we were so-far unable to circumvent.

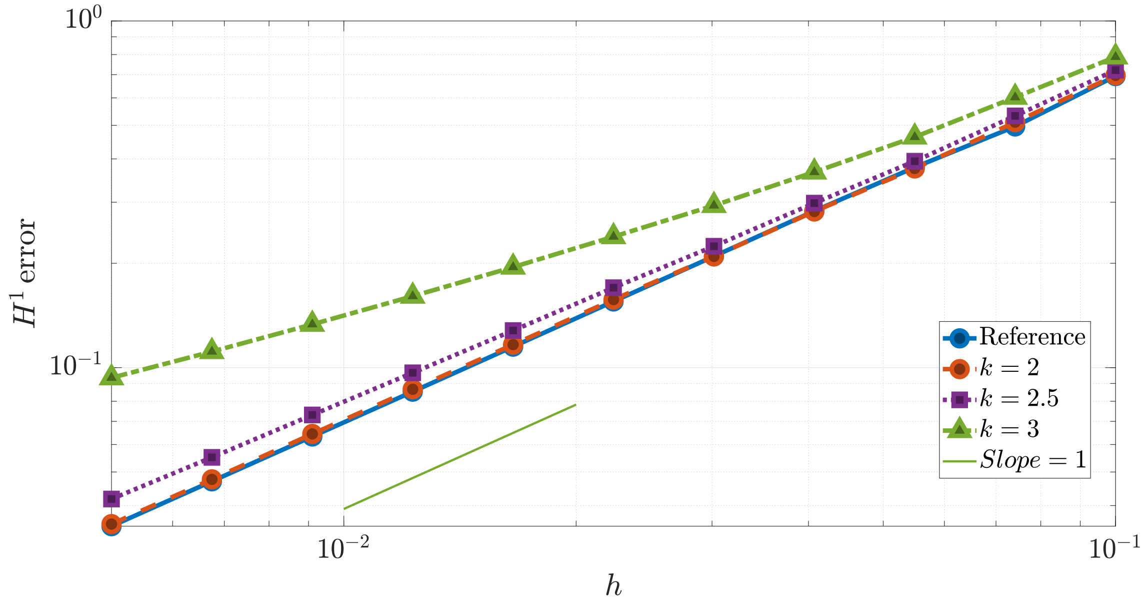

Figure 3: Convergence of the finite element method in the H^1-seminorm. Convergence on a regular reference mesh and convergence on meshes containing bands of caps with height $\overline{h} = h^k$ with $k = 2, 2.5, 3$.

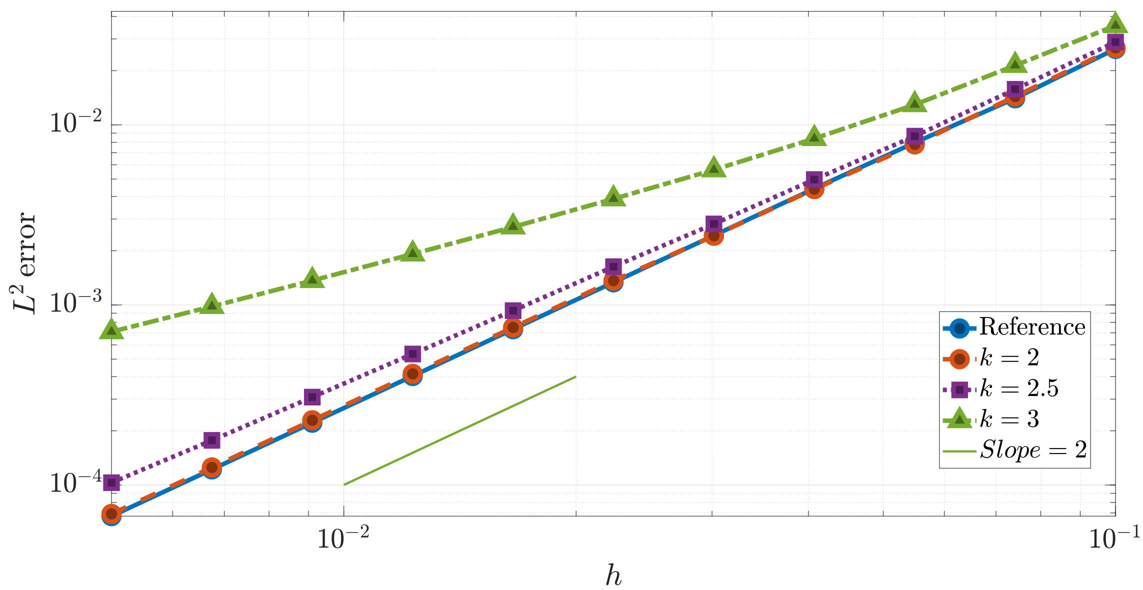

Figure 4: Convergence of the finite element method in the L^2-norm. Convergence on a regular reference mesh and convergence on meshes containing bands of caps with height $\overline{h} = h^k$ with $k = 2, 2.5, 3$.

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