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IMPROVED FLUX RECONSTRUCTIONS IN ONE DIMENSION

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Abstract: We present an improvement to the direct flux reconstruction technique for equilibrated flux a posteriori error estimates for one-dimensional problems. The verification of the suggested reconstruction is provided by numerical experiments.

Keywords: a posteriori error estimates, flux reconstructions, numerical experiments

MSC: 65N15, 65N30

1. Introduction

A posteriori error estimates play an important role in the numerical solution of PDEs. They enable to provide the information about the discretization error for the current choice of discretization parameters and also enable localization of the sources of errors that can be exploited in possible adaptive strategies. For the survey of main a posteriori techniques for PDE discretizations see e.g.[1], [3], [7], [12], [14] and references cited therein.

Important class of approaches for deriving guaranteed a posteriori upper bounds is based on the Hyper-circle theorem, see [11]. This theorem assumes the reconstruction of the fluxes to be in $H(\text{div})$. Such a property can be gained by global procedures that are very accurate but also very expensive, see e.g. [12]. Among the local procedures, the local mixed finite element technique is very popular, since it enables to reconstruct the fluxes based on local, relatively cheap problems. The theoretical results devoted to these mixed finite element reconstructions can be found in [5] and [9]. The paper [15] presents even more simple, more direct and cheaper reconstructions based on the natural degrees of freedom for the Raviart-Thomas spaces inspired by [8], where a similar idea is applied to the discontinuous Galerkin discretizations.

Although a posteriori error estimates based on the direct evaluation presented in [15] are reliable and robust, their accuracy gets slightly worse in some situations, especially for even degree polynomial approximations. The reason behind this

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behavior may possibly come from a rather naive choice how to define the flux reconstructions on the boundary of elements. Therefore, we present a suggestion for an improvement of the definition of the flux reconstruction on the boundary of elements for one-dimensional problems. Our suggestion is supported with numerical experiments.

2. Continuous problem and its discretization

2.1. Continuous problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain with Lipschitz continuous boundary $\partial\Omega$. Most of the presented results hold true in any dimension. Nevertheless, the final result will be presented for one-dimensional problems only, i.e. $d = 1$. We use standard notation for Lebesque and Sobolev spaces, respectively. Let us consider the following boundary value problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla \cdot (\nabla u - bu) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1}$$

where $f \in L^2(\Omega)$ and $b \in W^{1,\infty}(\Omega)^d$ such that $\nabla \cdot b = 0$. Let us denote weak derivative of u by u' for $d = 1$.

Let $(.,.)$ and $\|.\|$ be the $L^2(\Omega)$ scalar product and norm, respectively.

Definition 1. *We say that a function $u \in H_0^1(\Omega)$ is a weak solution of (1), if*

$$(\nabla u - bu, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{2}$$

According to the Lax-Milgram lemma, there exists a unique solution of problem (2).

2.2. Discrete problem

We consider a space partition $\mathcal{T}_h$ consisting of a finite number of closed, d-dimensional simplices K with mutually disjoint interiors and covering $\overline{\Omega}$, i.e., $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$. We denote the edges (or faces) by e. In the rest of the paper we speak about boundary objects of co-dimension 1 as about edges, but we mean vertices, edges or faces depending on the dimension d. For each edge e, let $n = n_e$ denote a unit normal vector to e with arbitrary but fixed direction for the inner edges and with outer direction on $\partial\Omega$. The unit outward normal to K will be denoted by n_K. We assume conforming properties of the mesh, i.e., neighbouring elements share an entire edge. We set $h_K = \text{diam}(K)$ and $h = \max_K h_K$. We assume shape regularity of elements, i.e., $h_K/\rho_K \le C$ for all $K \in \mathcal{T}_h$, where ρ_K is the radius of the largest d-dimensional ball inscribed into K and constant C does not depend on $\mathcal{T}_h$ for $h \in (0, h_0)$. Moreover, we assume the local quasi-uniformity of the mesh, i.e. we assume $h_K \le C h_{K'}$ for neighbouring elements K and K' and constant C does not depend on $\mathcal{T}_h$ for $h \in (0, h_0)$ again.

In order to simplify the notation, we set $(.,.)_M$ and $\|.\|_M$ to be the local $L^2(M)$-scalar products and norms, respectively, where $M \subset \overline{\Omega}$ is some union of elements K or edges e. We denote a sum over all elements as $\sum_K$.

We define classical finite element space

$$V_h = \{v \in H_0^1(\Omega) \colon v|_K \in P^p(K)\}, \tag{3}$$

where the space $P^p(K)$ denotes the space of polynomials on K up to the degree $p \geq 1$.

Although the functions from V_h are globally continuous, we will need to work with piece-wise continuous functions as well. We define one-sided values, jumps and mean values on the inner edges

$$\begin{aligned} v(x-) &= \lim_{s\to 0+} v(x - ns), \qquad & v(x+) &= \lim_{s\to 0+} v(x+ns), \\ [v](x) &= v(x-) - v(x+), \qquad & \langle v\rangle(x) &= \frac{1}{2}(v(x-)+v(x+)). \end{aligned} \tag{4}$$

For the boundary edges we define

$$v(x-) = \langle v\rangle(x) = \lim_{s\to 0+} v(x-ns), \qquad [v](x) = 0. \tag{5}$$

Finally, we define the finite element solution of problem (2).

Definition 2. *We say that a function $u_h \in V_h$ is a discrete solution of* (2), *if*

$$(\nabla u_h - bu_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{6}$$

The existence and uniqueness of the discrete solution follows again from the Lax-Milgram lemma.

2.3. Discontinuous Galerkin method

The justification of the presented result is based on the discontinuous Galerkin method. Therefore, we briefly define the interior penalty discontinuous Galerkin discretization of problem (2) using the same notation as in Section 2.2. In order to simplify forthcoming considerations, we assume here purely diffusion problems, i.e. $b = 0$, only. Then the interior penalty discontinuous Galerkin method reads: find $u_h \in X_h$ such that

$$\begin{aligned} \sum_K (\nabla u_h, \nabla v_h)_K - \sum_e (\langle \nabla u_h\rangle \cdot n, [v_h])_e + \theta(\langle \nabla v_h\rangle \cdot n, [u_h])_e \\ + \sum_e (\alpha[u_h],[v_h])_e = (f, v_h) \quad \forall v_h \in X_h, \end{aligned} \tag{7}$$

where $\alpha > 0$ is penalization parameter that should be chosen large enough to ensure positivity of the resulting problem and the space X_h is defined as

$$X_h = \{v \in L^2(\Omega) \colon v|_K \in P^p(K)\}. \tag{8}$$

Parameter θ distinguishes different variants, where the most common variants are symmetric (SIPG, $\theta = 1$), nonsymmetric (NIPG, $\theta = -1$) and incomplete (IIPG, $\theta = 0$). For more details about the discontinuous Galerkin method and its properties see e.g. [6].

The same discretization can be denoted with the aid of the numerical fluxes similarly as in the finite volume method. Then the general discontinuous Galerkin discretization can be expressed as

$$\sum_K (\nabla u_h, \nabla v_h)_K - (\hat{\sigma} \cdot n_K, v_h)_{\partial K} + (\hat{u} - u_h, \nabla v_h \cdot n_K)_{\partial K} = (f, v_h) \quad \forall v_h \in X_h, \tag{9}$$

where the numerical fluxes $\hat{\sigma}$ and $\hat{u}$ approximate ∇u_h and u_h on the edges, respectively. For example, the choice for the numerical fluxes corresponding to IIPG is

$$\hat{u} = u_h, \qquad \hat{\sigma} = \langle \nabla u_h \rangle - \alpha [u_h] n. \tag{10}$$

The connection between the primal discontinuous Galerkin formulations and the formulations with the numerical fluxes is described in [2].

3. A posteriori error bound

3.1. Flux reconstruction

Since the discretization by the finite element method is conforming, the exact solution u as well as the discrete solution u_h belong to common space $H_0^1(\Omega)$. This quality no longer holds for the flux of the solution $\sigma(u) = \nabla u - bu$, since $\sigma(u) \in H(\mathrm{div}, \Omega)$ and $\sigma(u_h) \notin H(\mathrm{div}, \Omega)$ in general. Our aim is to find a suitable reconstruction $\sigma_h = \sigma_h(u_h) \in H(\mathrm{div}, \Omega)$ such that $\sigma_h \approx \sigma(u_h)$.

Let $RT_p(K)$ be the local Raviart-Thomas space of order p for element $K \in \mathcal{T}_h$, i.e. $RT_p(K) = P_p(K)^d + x P_p(K)$. For the details about Raviar-Thomas spaces and about FEM-like spaces for approximation $H(\mathrm{div}, \Omega)$ in general see e.g. [4]. We define the reconstruction σ_h element-wise. We seek $\sigma_h|_K \in RT_p(K)$ such that

$$\begin{aligned} \sigma_h|_e \cdot n &= \phi_e \quad \forall e \subset K, \\ (\sigma_h, z_h)_K &= (\nabla u_h - b u_h, z_h)_K \quad \forall z_h \in P_{p-1}(K)^d, \end{aligned} \tag{11}$$

where $\phi_e \in P_p(e)$ is a suitable function. The conditions in (11) represent the natural degrees of freedom for $RT_p(K)$, see [4, Proposition 2.3.4]. Applying the basis corresponding to these degrees of freedom enables to assemble σ_h directly without the necessity to solve any local linear problems which results in extremely cheap evaluation of the reconstruction σ_h. This property is demonstrated in [15, Lemma 5.1] for $d = 1$.

We point out that the resulting function σ_h has globally continuous normal components and therefore the sum of local contributions of σ_h is globally in $H(\mathrm{div}, \Omega)$.

Important property of σ_h is the orthogonality of $f + \nabla \cdot \sigma_h$ to functions from V_h that follows from the discrete problem formulation (6) and from (11)

$$\begin{aligned}(f + \nabla \cdot \sigma_h, v_h) =&(f, v_h) - (\sigma_h, \nabla v_h)\\ =&(f, v_h) - (\nabla u_h - b u_h, \nabla v_h) = 0 \quad \forall v_h \in V_h.\end{aligned} \tag{12}$$

3.2. Upper bound

We define the error measure as the dual norm of residual

$$\mathrm{Err}(w) = \sup_{0 \neq v \in H_0^1(\Omega)} \frac{(f, v) - (\nabla w - bw, \nabla v)}{\|\nabla v\|}. \tag{13}$$

For the most simple case $b = 0$, the error measure is equivalent to H^1-seminorm, i.e. $\mathrm{Err}(w) = \|\nabla u - \nabla w\|$.

An upper bound to the error measure $\mathrm{Err}(u_h)$ can be derived similarly as in [15]. Here, we present the final result.

Theorem 1. *Let $u_h \in V_h$ be the discrete solution obtained by (6) and σ_h be the reconstruction obtained from u_h by (11). Then*

$$Err(u_h)^2 \leq \eta^2 = \sum_K (\eta_{R,K} + \eta_{F,K})^2, \tag{14}$$

where the local error estimators are

$$\begin{aligned}\eta_{R,K} &= C_P h_K \|f + \nabla \cdot \sigma_h\|_K,\\ \eta_{F,K} &= \|\sigma_h - \sigma(u_h)\|_K = \|\sigma_h - \nabla u_h + b u_h\|_K.\end{aligned} \tag{15}$$

The constant C_P is the Poincare constant and can be bounded by $C_P \leq 1/\pi$, cf. [10]. It shall be pointed out that all the terms in (14) are cheaply computable.

3.3. Choice of ϕ_e

A posteriori error estimate (14) holds regardless of the choice of ϕ_e in (11). On the other hand, the quality of the estimate (14), i.e. how much the estimator η overestimates the error $\mathrm{Err}(u_h)$, depends on the choice of ϕ_e.

The paper [15] discusses the most naive possibility $\phi_e = \langle \nabla u_h \rangle \cdot n$ and the numerical experiments provided in the paper [15] show that this choice is far from optimal in some cases, most importantly for even degree polynomial approximations.

The goal of this paper is to show a suggestion for some more accurate choice of ϕ_e. Since we will only consider one-dimensional problems, we may simplify the domain Ω as the interval $(0, 1)$ and we can denote the partition nodes $0 = e_0 < e_1 < \ldots < e_N = 1$ and the corresponding elements $K_k = [e_{k-1}, e_k]$. Then the suggested choice for ϕ_e is following

$$\begin{aligned}\phi_{e_N} &= -(f, x) - (b u_h, 1) = -\int_0^1 x f(x) + b(x) u_h(x) \mathrm{d}x,\\ \phi_{e_k} &= \phi_{e_{k+1}} + (f, 1)_{K_{k+1}}, \quad k = N-1, \ldots, 0.\end{aligned} \tag{16}$$

The idea of element-wise flux reconstruction similar to (11) was already applied with success for the interior penalty discontinuous Galerkin a posteriori error estimates, see e.g. [8]. It is possible to find out by careful comparison that the choice for boundary degrees of freedom ϕ_e in [8] corresponds to the numerical fluxes $\hat{\sigma}$, cf. Section 2.3.

Our idea for the choice (16) follows from imitating the discontinuous Galerkin technique, where the finite element method is expressed as a variant of the discontinuous Galerkin method. More precisely, we modify the IIPG numerical flux $\hat{\sigma}$ from (10) in such a way that the resulting IIPG solution with this modified flux is identical to the finite element solution.

Still, there is a work to be done concerning precise numerical analysis, e.g. IIPG error norm justification or IIPG a posteriori error analysis including efficiency analysis.

4. Numerical experiments

The aim of this section is to show how accurate, reliable and robust are a posteriori error estimates based on (11) and (16). The numerical experiments in paper [15], where the naive choice of ϕ_e as $\phi_e = \langle \nabla u_h \rangle \cdot n$ is discussed, show that the estimates are slightly worse in some situations, especially for even polynomial degrees. We want to show that the choice of ϕ_e according to (16) improves this behavior and the resulting estimates are accurate regardless of the situation.

Although the individual error estimator can be computed directly, the evaluation of the error measure can be difficult even in simplified situations, where the exact solution is known, since the defining formula (13) represents the supremum over infinite-dimensional space. Therefore, we approximate the error measure $\mathrm{Err}(w)$ by

$$\mathrm{Err}^+(w) = \sup_{0 \neq v \in V_h^+} \frac{(f,v) - (\nabla w - bw, \nabla v)}{\|\nabla v\|}, \tag{17}$$

where V_h^+ is chosen adaptively and $V_h \subset V_h^+ \subset H_0^1(\Omega)$. The error measure simplifies to $\mathrm{Err}(w) = \|\nabla u - \nabla w\|$ for purely diffusion problems ($b = 0$) and no approximation of the error measure is needed in these situations.

4.1. Purely diffusion problem

We study the error estimate (14) with respect to the mesh refinement and with respect to the changing polynomial degree. We assume the purely diffusion problem ($b = 0$) on the domain $\Omega = (0,1)$ and we set the right-hand side $f = \pi^2 \sin(\pi x)$.

Since the paper [15] shows that there are two different regimes for odd and even polynomial degrees, we provide the tests with equidistant meshes for refining mesh-size h starting at $h = 1$ and fixed polynomial degrees $p = 2$ and $p = 3$.

We set fixed $h = 0.25$ for the changing polynomial degree tests.

Tables 1–3 show that the estimate (14) provides extremely accurate upper bounds. The estimator η_R converges faster to 0 than the error and the second estimator η_F

$1/h$	$\|u' - u_h'\|$	η	Eff	η_R	η_F
1	$2.6718-1$	$3.1054-1$	1.16	$5.4235-2$	$2.5631-1$
2	$1.9719-1$	$2.0686-1$	1.05	$1.3166-2$	$1.9369-1$
4	$5.0620-2$	$5.1238-2$	1.01	$8.4125-4$	$5.0396-2$
8	$1.2739-2$	$1.2778-2$	1.00	$5.2868-5$	$1.2724-2$
16	$3.1900-3$	$3.1924-3$	1.00	$3.3088-6$	$3.1891-3$
32	$7.9783-4$	$7.9787-4$	1.00	$2.0687-7$	$7.9777-4$
64	$1.9948-4$	$1.9949-4$	1.00	$1.2930-8$	$1.9947-4$

Table 1: Global h-performance, diffusion, $p = 2$

$1/h$	$\|u' - u_h'\|$	η	Eff	η_R	η_F
1	$2.6718-1$	$3.1054-1$	1.16	$5.4235-2$	$2.5631-1$
2	$2.6332-2$	$2.7382-2$	1.04	$1.3086-3$	$2.6073-2$
4	$3.3650-3$	$3.3984-3$	1.01	$4.1667-5$	$3.3567-3$
8	$4.2295-4$	$4.2400-4$	1.00	$1.3082-6$	$4.2269-4$
16	$5.2941-5$	$5.2974-8$	1.00	$4.0928-8$	$5.2933-5$
32	$6.6199-6$	$6.6211-6$	1.00	$1.4696-9$	$6.6197-6$
64	$8.2751-7$	$8.2778-7$	1.00	$3.3135-10$	$8.2756-7$

Table 2: Global h-performance, diffusion, $p = 3$

as expected. On the other hand, the results show that the estimator η_F is not able to provide upper bound without the correction from the estimator η_R. Moreover, Tables 1–3 show that there is no longer any significant difference between odd and even polynomial degrees, compare with [15].

4.2. Convection-diffusion problem

We study convection-diffusion equation

$$-\epsilon u'' + bu' = f, \tag{18}$$

where $\Omega = (0, 1)$, $b = 1$, $f = 1$ and $\epsilon > 0$ is a constant. For more information about convection-diffusion problems see [13]. We present the performance of the estimate (14) with respect to h for fixed $\epsilon = 0.01$, $p = 1$ and successively refined equidistant meshes starting with $h = 0.1$ and with respect to ϵ for the fixed equidistant mesh with $h = 0.025$ and decreasing parameter ϵ.

Tables 4 and 5 show that the accuracy of the estimate is preserved either for convection or diffusion dominated situation and the estimate is accurate and robust with respect to h as well as ϵ.

Moreover, it is possible to study the local distribution of errors and corresponding estimates. Figure 1 presents the exact solution u and the discrete solution u_h for the convection dominated situation on the equidistant mesh with $h = 0.1$ and $\epsilon = 0.01$. The corresponding distribution of estimates is presented in Figure 2. We can find

p	$\|u' - u_h'\|$	η	Eff	η_R	η_F
1	$4.9851-1$	$5.0603-1$	1.02	$1.2655-2$	$4.9338-1$
2	$5.0620-2$	$5.1238-2$	1.01	$8.4125-4$	$5.0396-2$
3	$3.3650-3$	$3.3984-3$	1.01	$4.1667-5$	$3.3567-3$
4	$1.6667-4$	$1.6806-4$	1.01	$1.6459-6$	$1.6641-4$
5	$6.5836-6$	$6.6304-6$	1.01	$5.3935-8$	$6.5765-6$
6	$2.1766-7$	$2.1911-7$	1.01	$4.2163-9$	$2.1617-7$

Table 3: Global p-performance, diffusion, $h = 0.25$

$1/h$	$\mathrm{Err}^+(u_h)$	η	Eff
10	$2.0665-1$	$2.0770-1$	1.01
20	$1.0155-1$	$1.0206-1$	1.01
40	$5.0775-2$	$5.1031-2$	1.01
80	$2.5388-2$	$2.5516-2$	1.01
160	$1.2694-2$	$1.2758-2$	1.01

Table 4: Global h-performance, convection-diffusion, $\epsilon = 0.01$

ϵ	$\mathrm{Err}^+(u_h)$	η	Eff
$1.0-0$	$7.4691-3$	$7.5067-3$	1.01
$1.0-1$	$1.6057-2$	$1.6138-2$	1.01
$1.0-2$	$5.0775-2$	$5.1031-2$	1.00
$1.0-3$	$1.6159-1$	$1.6164-1$	1.00
$1.0-4$	$9.1726-1$	$9.1727-1$	1.00

Table 5: Global ϵ-performance, convection-diffusion, $h = 0.025$

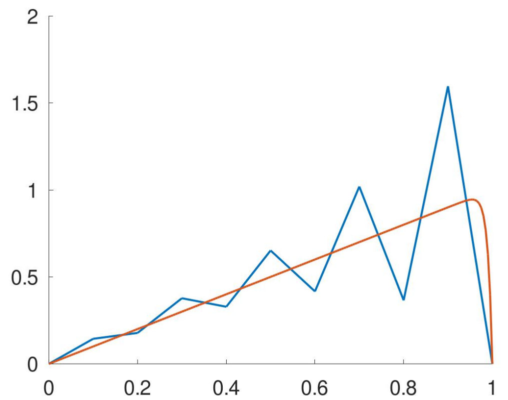

Figure 1: Exact and discrete solution

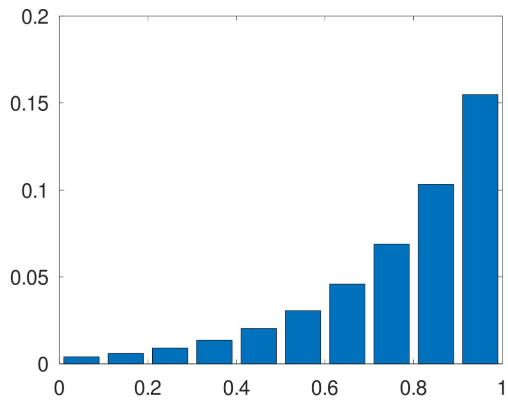

Figure 2: Element-wise error estimates

out comparing Figures 1 and 2 that the distribution of the error matches very well with the distribution of the local error estimates.

5. Conclusion

We suggested an improvement of the flux reconstruction for a posteriori error estimates from [15] for one-dimensional problems and provided numerical experiments verifying the accuracy, robustness and reliability of the suggested reconstruction. The main drawback lies in the fact that it is not obvious how to extend presented result to multi-dimensional problems. Moreover, precise analysis is still missing as well. These topics will be part of the future research.

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