SEVERAL CLASSES OF UNIFORM SPACES CONNECTED WITH BANACH VALUED MAPPINGS

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In 1975, Z.Frolík, J.Pelant and the author wrote a paper concerning rings of uniformly continuous functions and extensions of uniformly continuous (real valued) functions ([7]), which appeared in the last volume of Seminar Uniform Spaces. The aim of this note is to examine to what extent similar results are valid for uniformly continuous functions into infinite dimensional Banach spaces.

All uniform spaces are supposed to be separated, all locally convex (topological vector) spaces are supposed to be over the field of reals endowed with its natural translation invariant uniformity. By $U(X,Y)$ we denote the set of all uniformly continuous mappings from X into Y; if Y is the real line (with its natural metrizable uniformity), we write simply $U(X)$. Further we use the following symbols: R for the real line, I for a compact interval, $H(A)$ for a hedgehog over a set A. Recall that $H(A)$ is the set of all $\langle a,x\rangle$, $a\in A$, $0\leq x\leq 1$, where we identify $0=\langle a,0\rangle=\langle b,0\rangle$ for all $a,b\in A$, with the metric

$$d(\langle a,x\rangle,\langle b,y\rangle)=\begin{cases}|x-y| & \text{for } a=b\\ x+y & \text{otherwise}\end{cases}.$$

We recall that $H(A)$ is an injective uniform space ([12]), which means that all uniformly continuous mappings ranging in $H(A)$ extend from arbitrary subspaces to a uniformly continuous mapping. If X is a uniform space, by $X\text{-}t_f$ we shall understand the class of all uniform spaces Y such that each $f\in U(Y,X)$ remains uniformly continuous into the corresponding (topologically) fine uniformity $t_f X$. It follows from [15] that $X\text{-}t_f$ always forms a coreflective subcategory of uniform spaces.

At first we recall the main result of paper [7]:

Theorem 0: The following properties of a uniform space X are equivalent:

(1) X is $H(\omega)$-t_f

(2) X is $H(m)$-t_f for arbitrary infinite cardinal m

(3) X is hereditarily R-t_f

(4) X is hereditarily R^n-t_f for any natural number n

(5) For each subspace Y of X, $U(Y)$ is a ring (under pointwise operations)

(6) If $\{f_n; n\in\omega\}$ is a countable family of bounded uniformly continuous functions on X such that the family $\{\text{coz } f_n; n\in\omega\}$ is uniformly discrete in X, then the function $\sum\{f_n; n\in\omega\}$ is uniformly continuous. (In other words the family $\{f_n; n\in\omega\}$ is uniformly equicontinuous.)

(7) If the families $\{B_n; n\in\omega\}$, $\{A_{n_i}; i=1,2,\dots,k_n\}$ are both uniformly discrete in X, $B_n = \bigcup\{A_{n_i}; i=1,2,\dots,k_n\}$, then the family $\{A_{n_i}; n\in\omega, i=1,2,\dots,k_n\}$ is uniformly discrete in X

(8) $U(X)$ is a ring and each uniformly continuous function on a subspace of X has a uniform extension over X.

Moreover the described class is the largest coreflective subclass contained in the class Ext consisting of all X such that for each subspace Y of X and $f\in U(Y)$, there is $\bar{f}\in U(X)$ extending f.

The present paper is divided into three parts. The first one will generalize the property (7) to countable uniformly discrete unions of arbitrary uniformly discrete families and show that it is an important property for some Banach valued mappings to be distally continuous. The second part will examine those spaces X, for which $U(X,E)$ is a module over $U(X)$ for any Banach space E. It will appear that spaces having hereditarily this property have a very nice description in terms of countable sums of Banach valued mappings and also in terms of a sort of local fineness. The last part contains some results on extensions of uniformly-continuous Banach valued mappings. Throughout the paper some problems are stated.

ω - hyperdistal spaces.

Let X be a uniform space. We shall call a family $\{A_\iota; \iota \in I\}$ of subsets of X hyperdiscrete, if $I = \bigcup\{I_a; a \in J\}$ and the families

$$\{\bigcup\{A_\iota;\ \iota \in I_a\}; a \in J\},\ \{A\ ;\ \iota \in I_a\}$$

are uniformly discrete in X for all $a \in J$. If J is supposed to have cardinality ω, we call the family $\{A_\iota\}$ ω - hyperdiscrete. A uniform space X is called hyperdistal (resp. ω - hyperdistal), if each hyperdiscrete (ω - hyperdiscrete) family in X is uniformly discrete. (Hyperdistal spaces were defined originally by Z.Frolík in [5].) For illustration we present the following result:

Proposition 1: Let M be a metrizable uniform space. Then each hyperdiscrete family in M is ω - hyperdiscrete.

Proof: Let $\{X_{ab}; a \in A_b\ ,\ b \in B\}$ be a hyperdiscrete family in M and let M have a metric d. For each $b \in B$ we let

$$\varepsilon_b = \inf\{d(X_{ab}\ ,\ X_{a'b}); a, a' \in A_b\}.$$

Obviously $\varepsilon_b > 0$. Then we let

$$B_1 = \{b \in B;\ \tfrac{1}{2} < \varepsilon_b\}\ ,\ B_n = \{b \in B;\ \tfrac{1}{n+1} < \varepsilon_b \le \tfrac{1}{n}\}\ \text{for}\ n > 1\ ,$$

$$A_n = \bigcup\{A_b; b \in B_n\}\ \text{for natural}\ n\ .$$

The rest of the proof is evident.

Observe that the concept of an ω - hyperdistal space depends only on the structure of uniformly discrete families (distality) of the space. One can see that this concept is a strengthening of the property (7) in Theorem 0, so each ω - hyperdistal space enjoys the properties of Theorem 0.

Let α be an infinite cardinal number. We shall denote by $D(\alpha)$ the uniform space on the set $\alpha \times \omega$ with basis $\{\mathcal{U}_n\}_{n\in\omega}$, where

$$\mathcal{U}_n = \{\{\langle\iota,k\rangle\};\ \iota\in\alpha,\ k<n\}\cup\{\alpha\times\{k\};k>n\}.$$

All spaces $D(\alpha)$ are complete metrizable zero dimensional topologically discrete uniform spaces. Recall that a mapping between uniform spaces is called distally continuous if preimages of uniformly discrete families are uniformly discrete.

Theorem 1: The following properties of a uniform space X are equivalent:

(1) X is ω - hyperdistal

(2) X is hereditarily $D(\alpha)$-t_f for any cardinal number α

(3) For any cardinal number α the following holds: whenever $f_n : X \to \ell_\infty(\alpha)$ is a countable family of uniformly continuous bounded mappings such that $\{\text{supp } f_n; n\in\omega\}$ forms a uniformly discrete family, then the mapping $\sum\{f_n;n\in\omega\}$ is distally continuous

(4) The condition (3) assuming only distal continuity of the mappings f_n

(5) For any cardinal number α and for each subspace Y of X, if $f\in U(Y,\ell_\infty(\alpha))$, then f^2 is distally continuous

(6) For any Banach space E, $f_n\in U(X,E)$ bounded, the mapping $\sum\{f_n;n\in\omega\}$ is distally continuous provided that the family $\{\text{supp } f_n;n\in\omega\}$ is uniformly discrete

(7) Condition (6) assuming only distal continuity of the mappings f_n

(8) For any Banach algebra E and for any subspace Y of X, if $f \in U(X,E)$, then f^2 is distally continuous.

Proof: Using the well known fact that each Banach space is embeddable into some space $\ell_\infty(\alpha)$ as a topological vector subspace and each Banach algebra is embeddable into some $\ell_\infty(\alpha)$ as a Banach subalgebra (see [1]), we have immediately that (3)$\Leftrightarrow$(6), (4)$\Leftrightarrow$(7), (5)$\Leftrightarrow$(8). (1)$\Rightarrow$(4): Suppose $\{A_\iota ; \iota \in I\}$ is a uniformly discrete family in $\ell_\infty(\alpha)$. We may assume that the point 0 is not contained in any of the sets A_ι. For each n the family $\{f_n^{-1}[A_\iota] ; \iota \in I\}$ is uniformly discrete in X, its union is a part of $\operatorname{supp} f_n$, hence the family $\{f_n^{-1}[A_\iota] ; n \in \omega, \iota \in I\}$ is ω - hyperdiscrete, and hence uniformly discrete. If we put $f = \sum\{f_n ; n \in \omega\}$, we have $f^{-1}[A_\iota] = \bigcup\{f_n^{-1}[A_\iota] ; n \in \omega\}$, hence the family $\{f^{-1}[A_\iota] ; \iota \in I\}$ is uniformly discrete. This implies the distal continuity of f.

(4)$\Rightarrow$(3) is self evident. (3)$\Rightarrow$(5): At first we observe the following easy fact:

Lemma: Let $f : X \to Y$ be a mapping between uniform spaces. Suppose for some finite uniform cover $\mathcal{P}$ of the space X the mappings $f|P$ are distally continuous for all $P \in \mathcal{P}$. Then f is distally continuous.

Proof of the lemma: Recall that f is distally continuous if and only if the f-pre-image of each finite-dimensional uniform cover is uniform. Now the lemma follows immediately.

Let $B(r)$ denote the closed ball in $\ell_\infty(\alpha)$ centred in 0 with the radius r (for a positive real r), $B(0) = \emptyset$. Take arbitrary $f \in U(X, \ell_\infty(\alpha))$. For any natural n we denote

$$X_n = f^{-1}\left[B(n + \tfrac{1}{2}) \setminus B(n-1)\right] .$$

The families $\{X_n ; n \text{ odd}\}$, $\{X_n ; n \text{ even}\}$ are both uniformly discrete in X. For all n we define $f_n \in U(X, \ell_\infty(\alpha))$ bounded in the following manner: At first we put

$$f_n|X_n = f|X_n, f_n|f^{-1}[\ell_\infty(\alpha)\setminus B(n+\tfrac{2}{3})] = 0, f_n|f^{-1}[B(n-\tfrac{7}{6})] = 0.$$

The family

$$\{X_n, f^{-1}[\ell_\infty(\alpha)\setminus B(n+\tfrac{2}{3})], f^{-1}[B(n-\tfrac{7}{6})]\}$$

is uniformly discrete in X, f_n partly defined on its union is uniformly continuous into $B(n+\frac{1}{2})$. So we can find its uniformly continuous extension $f_n \in U(X, B(n+\frac{1}{2}))$, because a closed ball in the space $\ell_\infty(\alpha)$ is an injective uniform space (see Isbell [9]). Each f_n is uniformly continuous and bounded, hence each f_n^2 is again bounded and uniformly continuous and the following families of mappings:

$$\{f_n^2; n \text{ odd}\}, \{f_n^2; n \text{ even}\}$$

fulfill the assumptions of condition (3), hence the mappings

$$F = \sum\{f_n^2; n \text{ odd}\}, \quad G = \sum\{f_n^2; n \text{ even}\}$$

are distally continuous.

Now if we denote $A = \bigcup\{X_n; n \text{ odd}\}$, $B = \bigcup\{X_n; n \text{ even}\}$, the cover $\{A, B\}$ is a finite uniform cover of X, the restrictions $f^2|A = F|A$, $f^2|B = G|B$ are distally continuous, hence using our lemma the mapping f^2 is also distally continuous. The rest follows from the obvious hereditariness of the property (3).

(5)$\Rightarrow$(2): We define the embedding j of the space $D(\alpha)$ into $\ell_\infty(\alpha)$ as follows:

$$j(\langle \iota, n\rangle) = \{x_a\}, \text{ where } x_a = \begin{cases} n & \text{for } \iota \neq a \\ n + \frac{1}{2n} & \text{for } \iota = a \end{cases}$$

of course, j is a uniform embedding of $D(\alpha)$ into $\ell_\infty(\alpha)$. If Y is a subspace of X, $f \in U(Y, D(\alpha))$, then the mapping $(jf)^2 = j^2 f$ is distally continuous, $j^2(D(\alpha))$ is uniformly discrete, hence $f^{-1}[D(\alpha)]$ is uniformly discrete, and hence $f \in U(Y, t_f D(\alpha))$.

(2)$\Rightarrow$(1) is obvious, and finishes the proof.

Remarks: a) The class of all ω - hyperdistal spaces is coreflective in uniform spaces. This is easy to see, for example verifying that condition (3) is closed under uniform sums and quotients.

b) The classes from Theorem 0 and Theorem 1 differ. For example if we take the space $D(\omega)$, the coreflection of it in $H(\omega)$-t_f has for basis covers, the trace of which on each column $\{n\} \times \omega$ is a finite partition, so it is not uniformly discrete, while the ω - hyperdiscrete coreflection of $D(\omega)$ is uniformly discrete.

c) If we consider the ω - hyperdistal coreflection only in the structure of distal spaces, we can compare it formally with the e-locally fine coreflection ([6]) in uniform spaces, just as the hyperdistal coreflection in distal spaces is comparable formally with the locally fine coreflection in uniform spaces (see [9]). One can construct these coreflections step by step using transfinite induction, as it is shown for a hyperdistal coreflection of a given distal space in [5] .

The classes Mod and HerMod:

If X is a uniform space and E is a Banach space, recall that for $f \in U(X,E)$, $g \in U(X)$ the product $f \cdot g$ is uniformly continuous whenever both f,g are bounded, but fails to be uniformly continuous in general. We shall denote by Mod the class of all uniform spaces X such that for any Banach space $E, U(X,E)$ is a module over $U(X)$. HerMod will denote the class of all spaces having hereditarily the property Mod.

Theorem 2: The following properties of a uniform space X are equivalent:

(1) X has the property Mod

(2) $U(X, \ell_\infty(\alpha))$ is a module over $U(X)$ for any cardi-

nal α

(3) For any Banach space E, $f \in U(X,E)$, the mapping $x \to f(x)\|f(x)\|$ is uniformly continuous

(4) The condition (3) restricted to spaces $\ell_\infty(\alpha)$ only

(5) For any cardinal number α, $f \in U(X, \ell^+_\infty(\alpha))$, $g \in U(X,R^+)$, the mapping $f \cdot g$ is uniformly continuous. (ℓ^+_∞, R^+ stand for respective positive cones.)

Proof: (2)$\Rightarrow$(1) for the same reasons as in Theorem 1, the implications (1)$\Rightarrow$(3), (3)$\Rightarrow$(4) are evident.

(4)$\Rightarrow$(5): We prove at first that condition (4) implies that the product of any two positive real valued uniformly continuous functions u,v on X is uniformly continuous. The condition (4) gives for $\alpha = 1$ immediately the uniform continuity of functions u^2, v^2 and $(u+v)^2$. The assertion follows from the identity

$$u \cdot v = \frac{1}{2}\left((u+v)^2 - u^2 - v^2\right) .$$

Now taking arbitrary mappings $f \in U(X, \ell^+_\infty(\alpha))$, $g \in U(X,R^+)$, we define $G \in U(X, \ell^+_\infty(\alpha))$ as the diagonal mapping:

$$G(x) = \{g(x), g(x), \dots \} .$$

Observing that $\|f(x) + G(x)\| = \|f(x)\| + g(x)$, we have

$$f(x)\cdot g(x) = (f(x) + G(x))\cdot\|f(x) + G(x)\| \quad f(x)\cdot\|f(x)\| - - G(x)\cdot\|f(x)\| - G(x)\cdot g(x) .$$

The first two summands on the right side are uniformly continuous immediately from (4), the others are uniformly continuous because of uniform continuity of real valued functions g^2 and $x \to g(x)\cdot\|f(x)\|$ hence also the mapping $f \cdot g$ is uniformly continuous.

(5)$\Rightarrow$(2): Take any $f \in U(X, \ell_\infty(\alpha))$ and rewrite it as $f_+ - f_-$, where f_+, f_- are from $U(X, \ell^+_\infty(\alpha))$. Similarly we represent any $g \in U(X)$. The assertion follows now from the equality:

$$f \cdot g = f_+ \cdot g_+ + f_- \cdot g_- - f_- \cdot g_+ - f_+ \cdot g_- .$$

Remarks: a) One can easily verify that the class Mod is closed unde uniform sums and quotients, so it is a coreflective subclass of uniform spaces.

b) If Y is any locally convex space, Y can be embedded into some product of Banach spaces as a topological linear subspace. Therefore if $X \in$ Mod, then $U(X,Y)$ is a module over $U(X)$ for all locally convex spaces Y.

The class Mod is not closed under subspaces, even each uniform space can be embedded into some space in Mod. We shall turn to the class HerMod of all spaces being hereditarily in Mod now and we show that it has a very nice description.

Theorem 3: The following properties of a uniform space X are equivalent:

(1) X is HerMod

(2) For any cardinal α and for any subspace Y of X, $U(Y, \ell_\infty(\alpha))$ is a module over $U(Y)$

(3) $X \in$ Mod and for each subspace Y of X, $f \in U(Y)$, there is $\bar{f} \in U(X)$ extending f

(4) X is simultaneously Mod and $H(\omega)$-t_f

(5) If $f_n \in U(X, \ell_\infty(\alpha))$ (α is an arbitrary cardinal number) is a countable family of bounded uniformly continuous mappings such that the family $\{\operatorname{supp} f_n; n \in \omega\}$ is uniformly discrete in X, then the mapping $f = \sum\{f_n; n \in \omega\}$ is uniformly continuous

(6) Condition (5) for $f_n \in U(X,E)$ for each Banach space E

(7) For any cardinal number α and for any subspace Y of X, $U(Y, \ell_\infty(\alpha))$ is a ring

(8) Condition (7) for any Banach algebra

(9) Each cover of the form $\{U_n \cap V_a^n\}_{n \in \omega, a \in A}$ is a uniform cover of X, provided that $\{U_n\}_{n \in \omega}$ is a finite-dimensional countable uniform cover and for each

$n \in \omega$, $\{V_a^n\}_{a \in A}$ is a uniform cover of X .

Proof: Again we can easily observe the equivalences (1)$\Leftrightarrow$(2), (5)$\Leftrightarrow$(6), (7)$\Leftrightarrow$(8). The rest of the proof will follow the schema:

$$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (9) \Rightarrow (7) \Rightarrow (2) .$$

(2)$\Rightarrow$(3): If Y is a subspace of X , then obviously $U(Y)$ is a ring, and the assertion follows from Theorem 0.

(3)$\Rightarrow$(4): This is immediate from Theorem 0.

(4)$\Rightarrow$(5): $\mathrm{Mod} \cap H(\omega)-t_f$ is an intersection of two coreflective classes, hence it itself is a coreflective subclass of uniform spaces. Let us denote for a while F the corresponding coreflector.

Take $f_n \in U(X, \ell_\infty(\alpha))$ bounded with the uniformly discrete family of their supports. We may and shall assume that none of them is an identical zero mapping. We put:

$$g_n(x) = \frac{f_n(x)}{2^n \cdot \|f_n\|} \quad , \text{ where } \|f_n\| = \sup\{\|f_n(x)\|; x \in X\} .$$

The mappings g_n converge uniformly to 0, hence

$$g = \sum \{g_n; n \in \omega\}$$

is uniformly continuous and bounded, so it has all its values in some closed ball B. According to the injectivity of $H(\omega)$ and uniform discreteness of the family $\{\mathrm{supp}\ f_n\}_n$ we can find $b \in U(X, H(\omega))$ such that for each $x \in \mathrm{supp}\ f_n$ there is $b(x) = \langle n,1 \rangle$. We denote $\vartheta = g \times b$ the cartesian product of the mappings g, b. There is $\vartheta \in U(X, B \times H(\omega))$, hence there is also $\vartheta \in U(X, F(B \times H(\omega)))$. If we denote by π_1 , π_2 the corresponding projections onto $B, H(\omega)$, we have:

$$\pi_1 \in U(F(B \times H(\omega)), \ell_\infty(\alpha)), \pi_2 \in U(F(B \times H(\omega)), t_g H(\omega)).$$

Now, because $t_f H(\omega)$ allows extensions of real valued functions, we can find the function $h \in U(t_f H(\omega))$ such that $h(\langle n,1 \rangle) = \|f_n\| \cdot 2^n$. Using the fact that $F(B \times H(\omega)) \in \mathrm{Moc}$, the mapping

$$G = ((h \circ \pi_2) \cdot \pi_1) \circ \vartheta$$

from X into $\ell_\infty(\alpha)$ is uniformly continuous. ($\circ$ stands for composition of mappings, $\cdot$ for an algebraic product.) It is easy to verify that

$$G(x) = \begin{cases} f_n(x) & \text{for } x \in \operatorname{supp} f_n \\ 0 & \text{otherwise} \end{cases} .$$

Therefore $G = \sum\{f_n; n \in \omega\}$ is uniformly continuous. (Notice that instead of supposing that $X \in \mathrm{Mod}$, it was sufficient to suppose that the product of each bounded uniformly continuous mapping into $\ell_\infty(\alpha)$ with any function from $U(X)$ is uniformly continuous.)

(5)⟹(9): Take $\{U_n\}_{n \in \omega}$ an m-discrete uniform cover of X, that is, $\omega = \bigcup\{A_k; k=1,2,\dots,m\}$, $\{U_n; n \in A_k\}$ is a uniformly discrete family for each $k=1,2,\dots,m$. Let us take further for each $n \in \omega$ a uniform cover $\{V^n_a\}_a$ of the space X. We shall denote $\mathcal{W} = \{U_n \cap V^n_a\}_{n,a}$, $\mathcal{W}$ is a cover of X. Now choose some $1 \le k \le m$. For each $n \in A_k$ we find some f_n bounded uniformly continuous from X into some ℓ_∞ such that the family $\{\operatorname{supp} f_n; n \in A_k\}$ is uniformly discrete in X and the cover $f_n^{-1}(\mathcal{G}(1)) | U_n$ refines $\{U_n \cap V^n_a\}_a$ for all $n \in A_k$. ($\mathcal{G}(1)$ denotes the usual metric cover of the space ℓ_∞ with unit balls.) This is possible, because $\{U_n; n \in A_k\}$ is uniformly discrete in X and the closed balls in ℓ_∞ are injective uniform spaces. The condition (5) gives the uniform continuity of the mappings $F_k = \sum\{f_n; n \in A_k\}$; $k=1,2,\dots,m$. Now we observe that the cover

$$\{\bigcup\{U_n; n \in A_k\}; k=1,2,\dots,m\}$$

is a finite uniform cover of X, for each k the cover

$$F_k^{-1}(\mathcal{G}(1)) | \bigcup\{U_n; n \in A_k\} \text{ refines } \mathcal{W} | \bigcup\{U_n; n \in A_k\},$$

hence the uniform cover

$$\wedge\{F_k^{-1}(\mathcal{G}(1)); k=1,2,\dots,m\} \text{ refines } \mathcal{W}.$$

(9)⟹(7): Let $B(r)$ for a positive real number r denote again the closed ball in ℓ_∞ centred in 0 with radius r, $B(0) = \emptyset$. Take arbitrary $f \in U(X, \ell_\infty)$ and denote for each natural n:

$$X_n = f^{-1}\left[B(n + \tfrac{1}{2}) \setminus B(n-1)\right]$$

$\{X_n\}_n$ is a 2-discrete countable uniform cover of X. For each $\varepsilon > 0$

and for each n the cover $(f^2)^{-1}(\varphi(\mathcal{E}))$ is uniform on X_n, so there is a uniform cover $\{V_a^n\}_a$ of X refining it on X_n. So the cover $(f^2)^{-1}(\varphi(\mathcal{E}))$ is refined by a cover $\{X_n \cap V_a^n\}_{n,a}$, the latter being uniform on X according to (9). Therefore the mapping f^2 is uniformly continuous.

The rest follows from the identity

$$f \cdot g = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

and from the evident hereditariness of the property (9).

(7)⟹(2): According to Theorem 2 it remains to prove that for $f \in U(X, \ell_\infty)$ the mapping $x \to f(x) \cdot \|f(x)\|$ is uniformly continuous. But this is easy, as one can take the diagonal mapping

$$(x \to \{\|f(x)\|, \|f(x)\|, \dots\}) \in U(X, \ell_\infty)$$

instead of $(x \to \|f(x)\|) \in U(X)$ and (7) gives the desired uniform continuity.

Remarks: a) The condition (4) implies immediately that the class HerMod is coreflective in uniform spaces, the property (9) allows the construction by use of transfinite induction for each X its coreflection in HerMod. The method is described for instance in [9] for constructing the locally fine coreflection.

b) If r is a set-preserving reflector in uniform spaces (i.e. the corresponding reflective class is closed under products, all subspaces and contains a compact interval), the space X is called r-locally fine (following Z. Frolík [6]), if each cover $\{U_a \cap V_b^a\}_{a,b}$ is uniform on X, whenever $\{U_a\}_a$ is a uniform cover of rX and for all a the covers $\{V_b^a\}_b$ are uniform on X. So if we denote as usual D^1 the separable distal reflector (i.e. D^1X chooses finite-dimensional countable uniform covers, (see for instance [4]), the condition (9) says nothing else than that the class HerMod is exactly the class of all D^1-locally fine uniform spaces. Also quite interesting is the class of e-locally fine spaces, where e denotes the separable reflector in uniform spaces, called by M.Rice locally sub-metric-fine spaces and studied in [13], [14]. Obviously the class of all e-locally fine spaces is contained in HerMod (from condition (9)), but it seems to be an open problem, whether these two classes coincide or not.

c) Looking through Theorems 1 and 3, we can immediately see that each space in HerMod is ω - hyperdistal. The converse is not true, as the following example shows: Let X be a uniform space on the set R^ω, the base of which is formed by all open finite-dimensional covers of the product space R^ω. X is a separable space, uniformly discrete families of which are the same as in $t_f X = t_f R^\omega$, hence X is ω - hyperdistal. On the other hand $X \neq t_f X$ and from the condition (9) of Theorem 3 it directly follows that X is not HerMod.

d) For the same reasons as in the Remark b) to Theorem 2, we can see that if X is HerMod and E any locally convex space, then U(Y,E) is a module over U(Y) for any subspace Y of X. Similar remarks can be added for other conditions in Theorem 3.

<u>Extensions of Banach valued mappings:</u>

If $\mathbb{K}$ is a class of Banach spaces, we shall use the following notation: Ext($\mathbb{K}$) will denote the class of all uniform spaces X such that whenever Y is a subspace of X, E a space from $\mathbb{K}$, $f \in U(Y,E)$, then there is a uniformly continuous extension $\bar{f} \in U(X,E)$ of f. $Ext_d(\mathbb{K})$ is a class of all such X, where for any uniformly discrete subspace D of X, $E \in \mathbb{K}$, $f : D \to E$ there is $\bar{f} \in U(X,E)$ extending f. Finally $Ext^*(\mathbb{K})$ will denote the class of all uniform spaces X, where all bounded uniformly continuous mappings from arbitrary subspaces of X ranging in some $E \in \mathbb{K}$ have a bounded uniform extension to the whole X.

We shall denote here $\mathcal{B}$ the class of all Banach spaces, $\mathcal{F}$ the class of all finite-dimensional Banach spaces. It was an open problem for some time, whether $Ext^*(\mathcal{B})$ is the class of all uniform spaces, or, equivalently, if a (closed) unit ball in each Banach space is an injective uniform space. This problem was answered negatively by J. Lindenstrauss in [11], where there is constructed a Banach space not having a uniformly injective unit ball. However many Banach spaces do have uniformly injective unit balls. (For examples see [10], [11].) We shall denote here $\mathcal{L}$ the class of all Banach spaces whose closed balls are injective uniform spaces. (Equivalently the class of all E such that $Ext^*(\{E\})$ are all uniform spaces.)

Our aim is to study what natural coreflective conditions allow extensions of uniformly continuous Banach valued mappings in the above cases. At first we find a natural condition for a space to be in $Ext(\mathcal{B})$.

Recall that a space X is called metric-fine (resp.(complete metric)-fine), if each uniformly continuous mapping from X into any metric (resp. complete metric) space M remains uniformly continuous into $t_f M$. (See for example [3], [5], [8].) We recall as least that both classes are coreflective in uniform spaces and that (complete metric)-fine spaces are exactly all subspaces of metric-fine spaces.

<u>Theorem 4:</u> If X is (complete metric)-fine, then $X \in Ext(\mathcal{B})$.

<u>Proof:</u> The proof will follow from the classical topological theorem (Dugundji [2]):

If M is a metric space, $A \subset M$ a closed topological subspace,

$f : A \to E$ a continuous mapping into a locally convex topological vector space, then f can be extended to a continuous $\bar{f}$: $M \to E$. (In fact one can find an extension $\bar{f}$ such that $\bar{f}(M)$ is contained in the convex hull of $f(A)$ in E.)

Take X (complete metric)-fine, so we can find Y metric-fine containing X as a subspace. Take a subspace A of X , $f \in U(A,E)$, where $E \in \mathcal{B}$. E is metrizable, hence there is a uniformly continuous pseudometric d on Y , such that f is uniformly continuous from (A,d) into E . According to the completeness of E , there is f': $\overline{(A,d)}^{(Y,d)} \to E$, the uniformly continuous extension to the closure. Now using the Theorem of Dugundji, we obtain $\bar{f} : (Y,d) \to E$ a continuous extension of f'.

The space Y is metric-fine, hence the identity mapping $i : Y \to (Y,d)$ remains uniformly continuous into $t_f(Y,d)$ and simultaneously $\bar{f} \in U(t_f(Y,d),E)$. Therefore $\bar{f} \in U(Y,E)$ and extends f, $\bar{f}|X$ is the desired uniformly continuous extension of f to the space X .

Remarks: a) Observe that the proof needs only completeness and metrizability of the space E.

b) Theorem 4 gives a coreflective subclass of $Ext(\mathcal{B})$. It is not known to me, whether it is the largest one, or whether the largest coreflection contained in $Ext(\mathcal{B})$ exists.

The discussion of the case Ext_d is simpler. It is proved in [7] that $H(\omega)-t_f$ is the largest coreflective subclass contained in $Ext_d(\mathcal{F})$. We shall prove that even:

Proposition 2: The class $H(\omega)-t_f$ is contained in $Ext_d(\mathcal{B})$.

Proof: Take $X \in H(\omega)-t_f$, D any uniformly discrete subspace of X . According to condition (2) in Theorem 0 there is $X \in H(D)-t_f$. Take arbitrary $f \in U(D,E)$, where $E \in \mathcal{B}$. Because of the injectivity of the space $H(D)$ we can find $g \in U(X,H(D))$ such that for any $x \in D$ there is $g(x) = \langle x,1 \rangle$. We have $g \in U(X,t_f H(D))$, $t_f H(D)$ is in $Ext(\mathcal{B})$ (Theorem 4), hence there is $h \in (t_f H(D),E)$ such that $h(\langle x,1 \rangle) = f(x)$ for each $x \in D$. The mapping hg is the desired uniformly continuous extension of f.

Remarks: a) Proposition 2 together with the remark before it gives the result, that $H(\omega)-t_f$ is the largest coreflective subclass in both $Ext_d(\mathcal{B})$, $Ext_d(\mathcal{F})$.

b) Theorem 4 together with the appendix in the Dugundji theorem mentioned in the proof gives the result that complete metric-fine spaces are contained in both $Ext(\mathcal{B})$ and $Ext^*(\mathcal{B})$; however we do not know anything about the largest coreflective subclasses there.

c) Theorem 0 contains the result that $H(\omega)-t_f$ is the largest coreflective subclass contained in $Ext(\mathcal{F})$.

The only nontrivial case is the case $Ext(\mathcal{L})$ now. The best result we are able to prove about this important case is the following:

Theorem 5: The class HerMod is contained in $Ext(\mathcal{L})$.

Proof: We use the property (5) of Theorem 3. Take a space E. We denote again $B(r)$, for a positive real number r, the closed ball in E centred in 0 with radius r, $B(0) = \emptyset$. Take arbitrary $X \in$ HerMod, Y its uniform subspace, $f \in U(Y,E)$.

For each natural n we put

$$Y_n = f^{-1}[B(n) \setminus B(n-1)] .$$

The family $\{Y_n; n \text{ even}\}$ is uniformly discrete in Y. For each n even we find $f'_n \in U(X,E)$ bounded such that $f'_n | Y_n = f | Y_n$ and the family $\{\text{supp } f'_n; n \text{ even}\}$ is uniformly discrete in X. This is le to find, because each $B(n)$ is an injective uniform space. This implies that the mapping $f' = \sum \{f'_n; n \text{ even}\}$ is uniformly continuous hence the mapping $g = f - f'|Y$ is uniformly continuous from Y into E. The mapping g can be written as $\sum\{g_n; n \text{ odd}\}$, where $\operatorname{supp} g_n \subset Y_n$ for all n odd.

Now we choose the family $\{h_n; n \text{ odd}\}$, $h_n \in U(X,E)$ bounded such that the family $\{\operatorname{supp} h_n; n \text{ odd}\}$ is uniformly discrete and for each odd n there is $h_n|Y_n = g_n|Y_n$. The mapping $h = \sum\{h_n; n \text{ odd}\}$ is again uniformly continuous, so $f' + h \in U(X,E)$ and moreover for $y \in Y_n$ there is:

$$f'(y) + h(y) = \begin{cases} f(y) + 0 = f(y) & \text{for } n \text{ even} \\ f'(y) + f(y) - f'(y) = f(y) & \text{for } n \text{ odd}. \end{cases}$$

This finishes the proof.

Remarks: a) Again we are not able to find some larger coreflective subclass of $\operatorname{Ext}(\mathscr{L})$, even if we restrict to mappings into spaces of the type ℓ_∞ only. On the other hand we do not know if it is not the largest one.

b) Theorems 3 and 5 show that if X is in the class HerMod, the structure of uniformly equicontinuous point bounded families on X has some very nice properties: It is closed under some special countable sums (condition (5) of Theorem 3), we can extend them from arbitrary subspaces, and others. These properties have good applications in the theory of free uniform measures, but we shall not go into details here.

c) Theorem 5 says more for spaces in $\mathscr{L}$ than Theorem 4, because for instance in [14] it is shown that even the class of all e-locally fine spaces is much larger then the class of all (complete metric)-fine spaces.

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