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Classifying stationary sets: a survey

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by

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In this second lecture I will describe results obtained last year by Eric van Douwen and myself, which will appear in [vDL]. I will continue to use the definitions, notations and conventions of the first lecture in which the collection cub(κ) and the notion of a stationary set in κ were introduced. Please keep in mind that κ always denotes a regular uncountable cardinal (i.e. $\kappa = cf(\kappa) > \omega_0$) and that κ is identified with the set $[0, \kappa)$ of all ordinals less than κ.

In the first lecture we saw that the class of all stationary subsets of regular cardinals classifies all non-paracompact generalized ordered spaces. Once that is realized, it becomes a question of some interest whether, except for cardinality, there are really distinct ways in which a generalized ordered space can be non-paracompact. Stated more precisely, and in a special case, suppose a non-paracompact generalized ordered space X has cardinality ω_1; then we know X must contain a stationary subset of ω_1. Must it contain a copy of ω_1 ? (The answer is no; see [L].) Or is there a fixed bistationary subset S of ω_1 such that either X contains a copy of ω_1 or else a copy of S ? (Again the answer is no; see Theorem K, below.) Finally, can two bistationary sets be topologically distinct, and if so, how many non-homeomorphic bistationary subsets exist in ω_1 ? (See Theorem M, below.)

Today's lecture will have three parts. I will begin by describing how to recognize a stationary subset of κ. Then I will describe the kind of functions suitable for the investigation of stationary sets. Finally I will give a reasonably simple criterion for recognizing when two stationary sets are equivalent and that will enable me to determine the number of equivalence classes of stationary subsets of κ. I begin with an elementary observation.

A. Lemma: If $\mathbb{C} \subset \mathrm{cub}(\kappa)$ and card $(\mathbb{C}) < \kappa$ then $\bigcap \mathbb{C} \in \mathrm{cub}(\kappa)$

The following well-known result, called the Diagonal Intersection Lemma, will be needed at two crucial junctures in the lecture.

B. Lemma: For each $\alpha < \kappa$ let C_α be a member of $\mathrm{cub}(\kappa)$. Then the set $D = \{\beta < \kappa \mid$ if $\alpha < \beta$ then $\beta \in C_\alpha\}$ also belongs to $\mathrm{cub}(\kappa)$.

Proof: It is easily seen that D is closed in κ. To establish that D is cofinal in κ, fix $\gamma < \kappa$. Since the set $\bigcap\{C_\alpha \mid \alpha < \gamma\}$ being the intersection of fewer than κ members of $\mathrm{cub}(\kappa)$, must belong to $\mathrm{cub}(\kappa)$, it is possible to choose γ_1, the first element of $\bigcap\{C_\alpha \mid \alpha < \gamma\}$ which is larger than γ. Inductively choose ordinals γ_n satisfying

(a) $\gamma < \gamma_1 < \dots < \gamma_n < \gamma_{n+1}$,

(b) γ_{n+1} belongs to $\bigcap\{C_\alpha \mid \alpha < \gamma_n\}$.

Let $\delta = \sup\{\gamma_n \mid n \geq 1\}$. Then $\delta \in D$ as required. $\square$.

In order to give the first characterization theorem, I must first introduce four special notations. Let S be any cofinal subset of κ. (Equivalently, let $S \subset \kappa$ have $\mathrm{card}(S) = \kappa$.) Then:

$\mathbb{P}(S) = \{T \mid T \subset S\}$;

$\mathrm{cub}(S) = \{T \subset S \mid \mathrm{card}(T) = \kappa$ and T is relatively closed in $S\}$;

$\mathbb{S}(S) = \{T \subseteq S \mid T$ is stationary in $\kappa\}$;

$\mathbb{M}(S) = \{T \subseteq S \mid$ either T or $S - T$ contains a member of $\mathrm{cub}(S)\}$.

Obviously cub(S) $\neq \emptyset$ while Ⓢ(S) may be empty. The members of Ⓜ(S) are called measurable subsets of S. (In case S is stationary in ω_1, then it is known that Ⓜ(S) is precisely the collection of Borel subsets of S.) The first characterization theorem is

C. Theorem: Let S be a cofinal subset of κ. Then the following are equivalent:

(1) S is not stationary;

(2) if T is a cofinal subset of S then some relatively closed, discrete subset D of S is cofinal in S and has $D \subset T$;

(3) cub(S) contains two disjoint members;

(4) Ⓜ(S) = Ⓟ(S);

(5) there is a regressive function $f : S \longrightarrow \kappa$ such that for each $y \in \kappa$, the set $f^{-1}\{y\}$ is non--stationary.

Proof: The equivalence of (1), (2) and (3) is straightforward, and (2) is easily seen to imply (4). We show that (4) implies (1). To that end, suppose (4) holds and yet S is stationary. According to the Ulam-Solovay theorem (Lecture 1, Theorem C) there are disjoint sets U, $V \subset S$ both of which are stationary in κ. According to (4), both U and V belong to Ⓜ(S). However, it is easily seen that if W is a stationary subset of κ which belongs to Ⓜ(S), then W cannot be disjoint from any member of cub(S), so that W must contain a member of cub(S). Applying that observation to the sets U and V, one obtains (3) which is impossible because (3) and (1) are equivalent.

The proof that (1) implies (5) is also straightforward. To complete the proof of Theorem C, I will prove that if S is stationary then the PDL holds, as promised in the first lecture. So let S be stationary and suppose $f : S \longrightarrow \kappa$ is a regressive function with non-stationary fibers. For each $y \in \kappa$ let $C_y \in \mathrm{cub}(\kappa)$ be disjoint from $f^{-1}\{y\}$. According to Lemma B, the set $D = \{x \in \kappa \mid \text{if } y < x \text{ then } x \in C_y\}$ belongs to $\mathrm{cub}(\kappa)$. Then $D \cap S \neq \emptyset$. Let $x \in D \cap S$. Then $f(x) = y < x$ so that $x \in C_y \cap f^{-1}\{y\} = \emptyset$. $\square$.

D. Corollary: If S is a cofinal subset of κ, then $M(S)$ is a σ-algebra.

Proof: The hard case occurs when S is not stationary in κ and in that case $M(S) = P(S)$ by Theorem C. $\square$.

It is possible to give characterizations of stationary sets which do not depend on ordering, e.g.,

E. Theorem: Let S be a cofinal subset of κ. Then S is stationary in κ if and only if whenever $f : S \longrightarrow M$ is a continuous mapping of S into a metric space M, then $\mathrm{card}(f[S]) < \kappa$.

Proof: Sufficiency is obvious. To prove necessity, suppose a continuous mapping $f : S \longrightarrow M$ is given, and yet $\mathrm{card}(f[S]) = \kappa$. Replacing M by $f[S]$, we may assume that f is surjective.

Let $y \in M$. If $f^{-1}\{y\}$ is stationary in κ, then (f being continuous) $f^{-1}\{y\} \in \mathrm{cub}(S)$. Write $M - \{y\} = \bigcup\{F_n \mid n \geq 1\}$ where each F_n is closed in M. Then each set $f^{-1}[F_n]$ is closed in S and is disjoint from the set $f^{-1}\{y\}$, showing that $f^{-1}[F_n]$ cannot be cofinal in S. Because $\mathrm{cf}(\kappa) > \omega_0$, there is some $\lambda < \kappa$ having $\bigcup\{f^{-1}\{F_n\} \mid n \geq 1\} \subset [0, \lambda)$. But then $\bigcup\{F_n \mid n \geq 1\}$ has cardinality at most $\mathrm{card}(\lambda) < \kappa$, whence $\mathrm{card}(M) < \kappa$. Therefore no set $f^{-1}\{y\}$ is stationary in κ.

Let $\mathcal{B} = \bigcup\{\mathcal{B}(n) \mid n \geq 1\}$ be a σ-discrete base for M. For each $y \in M$ the set $N(y) = \{n \geq 1 \mid y \in \bigcup \mathcal{B}(n)\}$ is non-void. For each $n \in N(y)$, let $B(n,y)$ be the unique member of $\mathcal{B}(n)$ containing y. Because $f^{-1}\{y\} = \bigcap \{f^{-1}[cl_M(B(n,y))] \mid n \in N(y)\}$, one of the sets $f^{-1}[cl_M(B(n,y))]$ must be a bounded subset of κ. Let $K(y)$ be the first integer $n \in N(y)$ with that property.

For each $k \geq 1$, define $M(k) = \{y \in M \mid K(y) = k\}$. Then $M = \bigcup\{M(k) \mid k \geq 1\}$ so that, because $cf(\kappa) > \omega_0$, some $k_0 \geq 1$ has the property that $card(M(k_0)) = \kappa$. The collection $\mathcal{T} = \{f^{-1}[cl_m(B(k_0,y))] \mid y \in M(k_0)\}$ is a closed, discrete collection in the space S so that $S_0 = \bigcup \mathcal{T}$ is a closed subset of S. Furthermore, $card(S_0) \geq card(M(k_0)) = \kappa$ so that $S_0 \in cub(\kappa)$. But it is an easy consequence of the Pressing Down Lemma that no stationary subset of κ admits a discrete covering by bounded sets, and this contradiction establishes Theorem E. $\square$.

One immediate consequence of Theorem E is the fact that the property "S is a stationary subset of κ" is a topological, as opposed to an order theoretic, property of S.

F. Corollary: Let S be stationary in κ and suppose $f : S \longrightarrow \kappa$ is a continuous mapping having $card(f[S]) = \kappa$. Then $f[S]$ is a stationary subset of κ.

Proof: If $f[S]$ is not stationary in κ, then there is a metric space M having $card(M) = \kappa$ and a continuous surjective mapping $g : f[S] \longrightarrow M$. But then $g \circ f : S \longrightarrow M$ is also continuous and surjective, which is impossible by Theorem E. $\square$.

Corollary F can also be deduced from Theorem D and that approach, while not as elegant as the one I gave a moment ago, suggests that continuous mappings are not the best kind of mappings to use in the study of stationary sets. For example, it would obviously be enough to know that, given $f : S \to \kappa$, if D_1 and D_2 are disjoint members of $\mathrm{cub}(f[S])$, then $f^{-1}[D_1]$ and $f^{-1}[D_2]$ each contain a member of $\mathrm{cub}(S)$. Observations of that type lead to the following definitions.

<u>Definition:</u> Let S be a cofinal subset of κ. A function $f : S \to \kappa$ is measurable if $f^{-1}[D] \in M(S)$ whenever $D \in M(\kappa)$ and f is strongly measurable if f is measurable and for each $y \in \kappa$, the set $f^{-1}\{y\}$ is not stationary.

Let me pause to describe a simple example showing the reason for considering strongly measurable functions instead of measurable functions.

<u>Example:</u> Given any nonvoid $T \subset \kappa$, there is a measurable mapping $f : \kappa \longrightarrow \kappa$ having $f[\kappa] = T$. For let S be the set of limit ordinals in κ and let t_0 be the first point of T. Let g be any function from $\kappa - S$ onto $T - \{t_0\}$, and define $f : \kappa \longrightarrow \kappa$ by

$$f(x) = \begin{cases} t_0 & \text{if } x \in S \\ g(x) & \text{if } x \in \kappa - S . \end{cases}$$

Then f is measurable. Furthermore, note that if $\mathrm{card}(T) = \kappa$ then the function g can be one-to-one so that $f^{-1}\{t_0\}$ is the only fiber of f having more than one point. $\square$.

The next lemma shows that the range of f is not important in the definition of a measurable function.

G. <u>Lemma</u>: Suppose S and T are cofinal subsets of κ and suppose $f : S \longrightarrow T$. Then f is measurable if and only if $f^{-1}[D] \in M(S)$ whenever $D \in M(T)$.

Furthermore, measurable and continuous functions are easily related.

H. <u>Lemma</u>: Let S be a cofinal subset of κ. Any continuous $f : S \longrightarrow \kappa$ is measurable and any continuous $f : S \longrightarrow \kappa$ for which $\mathrm{card}(f[S]) = \kappa$ must be strongly measurable.

The utility of strongly measurable functions may be seen from the next theorem which is valid for functions having stationary domains.

I. <u>Theorem</u>: Let $S \subset \kappa$ be stationary and let $f : S \longrightarrow \kappa$ have the property that the set $T = f[S]$ is cofinal in κ. Then the following are equivalent:

(1) f is strongly measurable;

(2) if $A \in S(S)$ then $f[A]$ is stationary in κ;

(3) there is a set $F \in \mathrm{cub}(S)$ such that $f(x) = x$ for every $x \in F$.

<u>Proof</u>: I will show $(1) \longrightarrow (2) \longrightarrow (3) \longrightarrow (1)$. First suppose f is strongly measurable and let $A \subset S$ be stationary in κ. I need a lemma.

J. <u>Lemma</u>: Suppose S is a stationary subset of κ and suppose T is a disjoint collection of non-stationary subsets of κ having $S \subset \bigcup T$. Then there is a set $C \in \mathrm{cub}(S)$ such that $\mathrm{card}(C \cap T) \leq 1$ for each $T \in T$.

<u>Proof</u>: For each $x \in S$ let $T(x)$ be the unique member of T containing x and let $C(x)$ be a member of $\mathrm{cub}(\kappa)$ which is disjoint from $T(x)$. For $x \in \kappa - S$ let $C(x) = \kappa$. Then, according to Lemma B

of this lecture, the set $D = \{x \in \kappa \mid \text{if } y < x \text{ then } x \in C(y)\}$ belongs to $\text{cub}(\kappa)$. The set $C = S \cap D$ is the required set. $\square$.

Now I return to the proof of $(1) \to (2)$ in Theorem I. Applying Lemma J to the collection $T = \{f^{-1}\{y\} \mid y \in T\}$, I obtain a set $C \in \text{cub}(S)$ having at most one point in common with each set $f^{-1}\{y\}$. Then the function $g = f|_C$ is measurable and one-to-one. Furthermore the set $A \cap C$ is stationary in κ so that, by Theorem D, some $B \subset A \cap C$ does not belong to $M(A \cap C)$. Because $B = g^{-1}[g[B]]$, and because g is measurable, $g[B] \notin M(g[A \cap C])$ so that, again by Theorem D, $g[A \cap C]$ is stationary in κ. Hence so is the even larger set $f[A]$.

Next, I show that $(2) \to (3)$. Let $S_0 = \{x \in S \mid f(x) < x\}$. Each fiber of the function $f|_{S_0}$ is non-stationary (indeed, for each $y \in T$ the fiber $f^{-1}\{y\}$ is non-stationary or else, by (2), some singleton in T would be stationary). According to the equivalence of (1) and (5) in Theorem D, the set S_0 is non-stationary. Let $C_0 \in \text{cub}(\kappa)$ have $C_0 \cap S_0 = \emptyset$. Let $S_1 = \{x \in S \mid f(x) > x\}$ and let $T_1 = f[S_1]$. Define $g : T_1 \to S_1$ by the rule that $g(y)$ is the first element of $S_1 \cap f^{-1}\{y\}$. Then g is regressive and one-to-one so that T_1 cannot be stationary (by Theorem D, again). According to (2), S_1 cannot be stationary either. Let $C_1 \in \text{cub}(\kappa)$ have $C_1 \cap S_1 = \emptyset$. But then $C_0 \cap C_1 \in \text{cub}(\kappa)$ so that the set $F = (C_0 \cap C_1) \cap S$ belongs to $\text{cub}(S)$, and $f(x) = x$ for each $x \in F$.

Finally I show that $(3) \to (1)$. Obviously no fiber $f^{-1}\{y\}$ can be stationary since any stationary subset of S must meet $F \in \text{cub}(S)$. Thus it remains only to show that f is measurable and to do that it will be sufficient to show that if $D \in \text{cub}(T)$ then $f^{-1}[D]$ contains a member of $\text{cub}(S)$. If $D \in \text{cub}(T)$ then $E = \text{cl}_\kappa(D)$ belongs to $\text{cub}(\kappa)$ so that $E \cap F \in \text{cub}(S)$. But if $x \in E \cap F$ then $x = f(x) \in T$ so that $x \in E \cap T = D$, showing that $E \cap F \subset f^{-1}[D]$ as required. $\square$.

The equivalence of (1) and (3) in Theorem I allows me to prove the next theorem, which is the first step in counting the number of equivalence classes of stationary sets, up to measurable isomorphism (defined later).

K. Theorem: Let S and T be stationary subsets of κ. Then the following are equivalent:

(1) $S - T$ is not stationary;

(2) there is a strongly measurable mapping from S onto T.

Proof: Suppose $f : S \to T$ is a surjective, strongly measurable mapping. Let $F \in \mathrm{cub}(S)$ have $f(x) = x$ for each $x \in F$. Then $(S - T) \cap F = \emptyset$ so that $S - T$ is non-stationary.

Conversely suppose $S - T$ is non-stationary. Let $C \in \mathrm{cub}(\kappa)$ have $C \cap (S - T) = \emptyset$. Then $C \cap S \subset C \cap T$, and $C \cap S \in \mathrm{cub}(S)$. Let D be the set of non-isolated points of the space $C \cap S$. Then $D \in \mathrm{cub}(S)$. Let g be any one-to-one mapping of $S - D$ onto $T - D$ and define $f : S \to T$ by

$$f(x) = \begin{cases} x & \text{if } x \in D \\ g(x) & \text{if } x \in S - D . \end{cases}$$

Then f is strongly measurable since $f(x) = x$ for each $x \in D \in \mathrm{cub}(S)$ and f maps S bijectively to T. $\square$.

Suppose S and T are cofinal subsets of κ. We will say that a function $f : S \to T$ is a measurable isomorphism if f is a bijection having the property that $f[C] \in \mathcal{M}(T)$ if and only if $C \in \mathcal{M}(S)$. Measurable isomorphisms and measurable bijections are not the same things, as the next example shows.

Example: There are two stationary subsets S and T such that S and T are not measurably isomorphic and yet there is a one-to-one measurable mapping from S onto T. For let S be any bistationary subset of κ (i.e., both S and $\kappa - S$ are stationary in κ) and let $T = \kappa$. The mapping $f : S \rightarrow T$ defined in the proof of Theorem K is a measurable bijection and yet, because $T - S = \kappa - S$ is stationary, $f^{-1}: T \rightarrow S$ cannot be measurable. $\square$.

However, there is an easy way to recognize when two stationary sets are measurably isomorphic, given in the next theorem. The proof of that theorem uses the ideas in the proof of Theorem K and so is omitted.

L. Theorem: Let S and T be stationary subsets of κ. Then there is a measurable isomorphism from S onto T if and only if the set $S \triangle T = (S - T) \cup (T - S)$ is non-stationary. $\square$

Of course, it now follows from the Ulam-Solovay theorem (Theorem B of the first lecture) that κ contains a family D of stationary sets such that $\text{card}(D) = \kappa$ and such that no two members of D are measurably isomorphic. However, by an elementary trick, one can get a much better result, namely:

M. Theorem: There is a collection E of stationary subsets of κ such that:

(1) $\text{card}(E) = 2^{\kappa}$;

(2) if $S \neq T$ are members of E then there is no strongly measurable mapping from S into T;

(3) if $S \neq T$ belong to E then S and T are not measurably isomorphic;

(4) if $S \neq T$ belong to $\mathfrak{E}$ then S and T are not homeomorphic - indeed there is no continuous $f : S \rightarrow T$ having $\mathrm{card}(f[S]) = \kappa$.

Proof: Obviously (3) and (4) follow from (2). Consider the disjoint collection $\mathfrak{D}$ of stationary subsets of κ guaranteed by the Ulam-Solovay theorem. Write $\mathfrak{D} = \mathfrak{S} \cup \mathfrak{T}$ where $\mathfrak{S}$ and $\mathfrak{T}$ are disjoint collections, each with cardinality κ . Index (without repetitions) $\mathfrak{S}$ and $\mathfrak{T}$ as $\mathfrak{S} = \{S_\alpha \mid \alpha \in \kappa\}$ and $\mathfrak{T} = \{T_\alpha \mid \alpha \in \kappa\}$. For each $A \subset \kappa$ define $U(A) = (\cup\{S_\alpha \mid \alpha \in A\}) \cup \cup(\cup\{T_\alpha \mid \alpha \in \kappa - A\})$. Observe that if $A \neq B$ are subsets of κ , then $U(A) - U(B)$ contains some member of $\mathfrak{D}$ so that no strongly measurable mapping from $U(A)$ onto any subset of $U(B)$ exists. Then let $\mathfrak{E} = \{U(A) \mid A \in P(\kappa)\}$. $\square$.

References

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