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Stationary sets and paracompactness in ordered spaces: a survey

by

D.J. Lutzer

This is the text of the first of three survey talks presented during the spring of 1977 when I was a guest of the Czechoslovak Academy of Sciences, under an exchange program sponsored by the American N.A.S. and the ČSAV. I wish to express my gratitude to those two organizations for their support.

In this first lecture I will introduce the notion of a stationary set and give several examples of one way that stationary sets can be used in topology. In addition, I hope to explain the motivation for the results described in the second lecture.

Throughout this talk, κ will denote a regular uncountable cardinal, i.e., an initial ordinal satisfying $\kappa = cf(\kappa) > \omega_0$. I will usually identify κ with the set $[0,\kappa)$ of all ordinals smaller than κ, and that set of ordinals will always carry the usual order topology. A subset S of κ will be topologized by the relative (or subspace) topology; that topology is usually different from the order topology which would be generated by the ordering which S inherits from κ.

By $cub(\kappa)$ I mean the set of all closed unbounded subsets of κ. Because κ is regular, a subset S of κ is unbounded (= cofinal) exactly when $card(S) = \kappa$. A subset S of κ is stationary in κ provided $S \cap C \neq \emptyset$ for each $C \in cub(\kappa)$. Because $\kappa = cf(\kappa) > \omega_0$, it is clear that any two members of $cub(\kappa)$ have nonvoid intersection so that any (superset of a) member of $cub(\kappa)$ is stationary. Indeed, any intersection of fewer than κ members of $cub(\kappa)$ belongs to $cub(\kappa)$, a fact which will be used later.

One immediate consequence is the observation that if $\bigcup\{S_\alpha : \alpha\in A\}$ is a stationary subset of κ, where $\mathrm{card}(A)<\kappa$, then some set S_α is stationary.

Those elementary observations can be used to give an easy proof that in ω_1 there are stationary sets which are much more complicated than supersets of members of $\mathrm{cub}(\omega_1)$. This proof is due to Mary Ellen Rudin [R].

A. <u>Theorem:</u> There is a subset S of ω_1 such that both S and $\omega_1 - S$ are stationary.

<u>Proof:</u> Suppose not. Then for each $S\subset\omega_1$, either S or $\omega_1 - S$ contains a member of $\mathrm{cub}(\omega_1)$.

Fix any 1-1 function f from ω_1 into $\mathbb{R}$, the usual space of real numbers. For each n let $\mathbb{G}(n)$ be a countable covering of $\mathbb{R}$ by sets of diameter $< 1/n$.

Let $n\geq 1$. I assert that for some $G\in\mathbb{G}(n)$, $f^{-1}[G]$ contains a member of $\mathrm{cub}(\omega_1)$. For otherwise, for each $G\in\mathbb{G}(n)$ I could choose a set $C(G)\in\mathrm{cub}(\omega_1)$ having $C(G)\subset\omega_1 - f^{-1}[G]$. But then, $\mathbb{G}(n)$ being countable,

$$\emptyset\neq\bigcap\{C(G)\mid G\in\mathbb{G}(n)\}\subset\omega_1-\bigcup\{f^{-1}[G]\mid G\in\mathbb{G}(n)\}=$$
$$=\omega_1-f^{-1}[\mathbb{R}]\quad=\emptyset\,.$$

Therefore I may choose $G_n\in\mathbb{G}(n)$ and $C_n\in\mathrm{cub}(\omega_1)$ such that $C_n\subset f^{-1}[G_n]$. But then the set $\bigcap\{C_n\mid n\geq 1\}$ has at least two points (it belongs to $\mathrm{cub}(\omega_1)$) and yet

$$\bigcap\{C_n\mid n\geq 1\}\subset\bigcap\{f^{-1}[G_n]\mid n\geq 1\}=f^{-1}[\bigcap\{G_n\mid n\geq 1\}]$$

which is impossible because f is 1-1 and $\bigcap\{G_n\mid n\geq 1\}$ is at most a single point. $\square$.

Subsets S of κ such that both S and $\kappa - S$ are stationary in κ are called <u>bistationary sets.</u> The proof of Theorem A obviously generalizes to certain higher cardinals, but there is another source for bistationary sets in any cardinal - the Ulam--Solovay theorem [U][S] .

B. Theorem: Let S be a stationary subset of a regular cardinal κ. Then there is a collection Ⓣ of subsets of S satisfying:

(1) Ⓣ is a pairwise disjoint collection;

(2) each $T \in$ Ⓣ is stationary in κ ;

(3) card(Ⓣ) = κ.

That theorem will be used once in this first lecture (to split a stationary set into two disjoint stationary subsets) and in its full force during the second lecture.

The most important tool in working with stationary sets is the Pressing-Down Lemma (PDL). This lemma grew out of a long chain of results due to Alexandroff, Urysohn, Neumer and Fodor (see [J]) and is often called PDL, probably to avoid disagreements over its parentage. There is an easy proof, which I will give in my second lecture.

C. Pressing Down Lemma: Suppose S is stationary in κ and f : $S \to \kappa$ is a function satisfying $f(x) < x$ for each $x \in S - \{0\}$. Then for some $y \in \kappa$ the set $f^{-1}\{y\}$ is a stationary subset of κ.

A function f having $f(x) < x$ is often called a regressive function. In applications it is often enough to know that $f^{-1}\{y\}$ is cofinal in κ but there are times when the full force of the PDL is needed (e.g., see Theorem I, below).

The applications which I will describe today are all based on a theorem proved in 1974 by Ryszard Engelking and myself characterizing paracompactness in generalized ordered spaces. Recall that a generalized ordered (GO) space is any topological space X which can be topologically embedded in a linearly ordered topological space (LOTS) Y , i.e., Y is a linearly ordered set equipped with the usual order topology. The characterization theorem given in [EL] is:

D. Theorem: A generalized ordered space X is not paracompact if and only if some closed subspace of X is homeomorphic to a stationary set in some regular uncountable cardinal κ .

Half of the proof of Theorem D (constructing the closed subset of X , given that X is not paracompact) follows from the Q-gap theory of Gillman and Henriksen [GH] and requires too many definitions to explain here. The other half of the proof is a nice introduction to the use of PDL and might be worth a minute of our time. We all know that if a space X is paracompact then so is each of its closed subspaces. Therefore it is enough to prove:

E. Lemma: If S is stationary in κ, then S is not paracompact.
Proof: It is easily seen that if S is stationary in κ, then so is S^d, the set of non-isolated points of the space S. (This elementary fact will be used repeatedly in this first lecture.)

Now consider $\mathcal{U} = \{S \cap [0,x) \mid x \in S\}$, an open cover of the space S . Suppose that $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. For each $x \in S^d$ choose $V(x) \in \mathcal{V}$ with $x \in V(x)$. Since $x \in S^d$ there is a first ordinal $f(x) \in S$ such that $f(x) < x$ and $[f(x), x] \cap S \subset V(x)$. The function f is regressive so that for some

$y \in S$ the set $f^{-1}\{y\}$ is stationary in κ (cofinal will do). But then the point y of S belongs to κ distinct members of Ⓥ because no member of Ⓥ is cofinal in κ, and that i impossible since Ⓥ is a locally finite collection. □.

Now I'll present some applications. The first two, along with some more technical results, appear in the paper [EL]. I know the fondness that topologists here in Prague have for uniform spaces, so it is appropriate that the first application relate somehow to such spaces. Let us say that a topological space is Dieudonné complete if it has a Cauchy-complete uniformity compatible with its topology. (I think of uniformities as certain collections of subsets of $X \times X$, each of which is a neighborhood of the diagonal. A filter-base Ⓕ in X is Cauchy with respect to a uniformity Ⓤ if for each $U \in$ Ⓤ some $F \in$ Ⓕ has $F \times F \subset U$, and the uniformity Ⓤ is complete if $\bigcap$Ⓕ $\neq \emptyset$ whenever Ⓕ is a filter base of closed sets which is Ⓤ-Cauchy.) The next result was obtained by Ishii [I] for LOTS.

F. Theorem: Any Dieudonné-complete generalized ordered space X is paracompact.

Proof: Suppose not. Then some stationary set S in a regular uncountable cardinal κ is homeomorphic to a closed subspace of X. But then S is also Dieudonné-complete, say with respect to the uniformity Ⓤ. Fix $U \in$ Ⓤ. As in Lemma E, the set S^d of non-isolated points of S is stationary in κ, and for each $x \in S^d$ there is a first ordinal $f_U(x) \in S$ such that $f_U(x) < x$ and $([f_U(x), x] \cap S)^2 \subset$ $\subset U$. The function f_U is regressive so that, by PDL, there is a $y_U \in S$ such that $f_U^{-1}\{y_U\}$ is stationary in κ (cofinal in κ would be enough). But then $([y_U, \kappa) \cap S)^2 \subset U$. Therefore the collection

Ⓕ $=\{[x,\kappa)\cap S : x\in S\}$ is a Ⓤ-Cauchy filterbase of closed sets, and yet $\bigcap$Ⓕ $= \emptyset$, which is impossible. □.

The second application also uses PDL and also the fact that the cardinal κ in Theorem D has $cf(\kappa) > \omega_0$. Recall that a space X is perfect if each closed subset of X is a G_δ -set. The next result was proved in [L_1], but by a much harder proof.

G. Theorem: Any perfect GO space is paracompact.

Proof: Suppose not. Then, by Theorem D, some stationary set S in a regular uncountable cardinal κ is a perfect space. With S^d as above, S^d must be a G_δ -set, say $S^d = \bigcap\{G(n) | n\geq 1\}$ where each G(n) is open in S. For each $n\geq 1$ and each $x\in S^d$ let $f_n(x)$ be the first element of S having $f_n(x) < x$ and $[f_n(x),x]\cap S\subset G(n)$. According to PDL, there is a point $y_n\in S$ such that $f_n^{-1}\{y_n\}$ is stationary in κ . Then $[y_n,\kappa)\cap S\subset G(n)$. Because $cf(\kappa) > \omega_0$, some $z\in S$ has $z > y(n)$ for every $n\geq 1$. But then $S\cap[z,\kappa)\subset$ $\subset \bigcap\{G(n)|n\geq 1\} = S^d$ and that is obviously impossible. □.

You will note the elementary pattern of the last two proofs: if a GO space X can have property Ⓟ and yet fail to be paracompact, then for some κ , there is a stationary $S\subset\kappa$ having property Ⓟ, and that is impossible, usually by PDL. What makes this pattern viable is the fact that most properties which imply paracompactness are closed-hereditary. But that is not always the case, and just to show that we have learned something since 1974, let me tell you about a lemma in a recent paper by Bennett and myself. The paper [BL_1] studies the notion of a σ-minimal base in a GO space.

Recall that a collection Ⓒ of subsets of X is minimal or irreducible if $\bigcup$Ⓓ $\neq \bigcup$Ⓒ whenever Ⓓ $\subsetneq$ Ⓒ. Equivalently,

Ⓒ is minimal if each $C \in$ Ⓒ contains a point $x(C)$ belonging to no other member of Ⓒ. It is crucial to realize that Ⓒ is <u>not</u> required to cover X. The notion of a <u>σ-minimal base</u>, (i.e., a base which is a countable union of minimal collections) was introduced by C.E.Aull [Au] who asked about the relation of σ-minimal bases to quasidevelopability [B]. The paper [BL_1] grew out of the surprising observation (surprising to me, at least) that the lexicographically ordered square has a σ-minimal base [BB]. It is known, and easily proved, that the lexicographic square cannot have a σ-minimal base whose members are intervals, however, and this pathology must be kept in mind. It is also known that the property of having a σ-minimal base is <u>not</u> closed-hereditary. (The Alexandroff "double arrow" $A = [0,1] \times \{0,1\}$ is a closed subspace of the lexicographic square and does not have a σ-minimal base for its subspace topology.) Nonetheless, one can prove:

H. <u>Theorem:</u> Any GO space X with a σ-minimal base is (hereditarily) paracompact.

<u>Proof:</u> <u>Hereditary</u> paracompactness in such a space follows from paracompactness of the space once it is proved that X is first-countable (an easy lemma) [L_2, Prop. C]. Let me sketch the proof for paracompactness of X. If X is not paracompact then there is a κ and a stationary set S in κ which is homeomorphic to a closed subspace of X. Furthermore (and this was not mentioned in my statement of Theorem D but is noted in [EL]) the homeomerphism can be taken to be strictly monotonic. Therefore we may assume $S \subset X$ and that the order inherited by S from X coincides with the usual ordering of S. Also, since S is closed in X, S has no supremum in X; we write κ for the ideal element of X (i.e., for the gap of X) which lies at the "top" of S. For $x \in S$, let x' be the first element

of S which is larger than x.

Now let $\textcircled{B} = \bigcup\{\textcircled{B}(n) \mid n \geq 1\}$ be a σ-minimal base for X. For each $n \geq 1$ let $S'(n) = \{x \in S \mid \text{some fixed } B_n(x) \in \textcircled{B}(n) \text{ has } x \in B_n(x) \subset (\leftarrow, x')\}$. (Here I am writing $(\leftarrow, x')$ to mean $\{y \in X \mid y < x'\}$.) Let $M = \{n \geq 1 \mid S'(n) \text{ is stationary}\}$. Because $S = \bigcup\{S'(n) \mid n \geq 1\}$ (which follows from the fact that $\textcircled{B}$ is a base), $M \neq \emptyset$.

For each $n \in M$ and each $x \in S^d$ (notation as in Lemma E) let $f_n(x)$ be the first point of S having $f_n(x) < x$ and $[f_n(x), x] \subset B_n(x)$. According to the PDL there is a $y_n \in S$ such that $f_n^{-1}\{y_n\}$ is stationary. Because $cf(\kappa) > \omega_0$ there is a $z \in S$ having $z > y_n$ for each $n \in M$.

Define $T = [z', \kappa) \cap S$ and let

$T_n = \{x \in T \mid \text{for some fixed } C(x) \in \textcircled{B}(n),\ x \in C(x) \subset (z, x')\}$.

Again because $\textcircled{B}$ is a base, $T = \bigcup\{T_n \mid n \geq 1\}$ so that, T being stationary, some T_m is stationary. Because $T_m \subset S_m$, $m \in M$ and hence the function f_m and the point y_m are defined. Fix any $x \in T_m$ and consider any point y of $C(x)$ which belongs to no other member of $\textcircled{B}(m)$. Since T_m is stationary there are points $z_1, z_2 \in T_m$ with $y' < z_1 < z_1' < z_2$. But then $B_m(z_1)$ and $B_m(z_2)$ are distinct members of $\textcircled{B}(m)$ and both contain y, which is impossible. $\square$.

As a final application, let me describe a lemma from another paper by Bennett and myself [BL_2] in which we study GO spaces which are hereditarily p-spaces in the sense of Arhangel'skiĭ [Ar] (i.e., GO spaces whose every subspace is a p-space). In our paper we need to study a considerably weaker property, introduced by Hodel in [H] where a space $(X, \textcircled{T})$ is called a <u>β-space</u> if there is a sequence $B_n \quad X \longrightarrow \textcircled{T}$ of functions satisfying:

1) $x \in B_n(x)$ for each $x \in X$ and each $n \geq 1$;

2) if $\langle x_n \rangle$ is a sequence in X for which $\bigcap\{B_n(x_n) | n \geq 1\} \neq \emptyset$ then $\langle x_n \rangle$ has a cluster point in X.

The theorem which I want to describe is

I. Theorem: Let X be a GO space. If X is hereditarily a β-space (= each subspace of X is a β-space) then X is paracompact.

Proof: Suppose not. Then there is a stationary set S in some κ such that S is hereditarily a β-space. Next, I invoke a lemma which reduces the problem to the first-countable case.

J. Lemma: Suppose S is stationary in κ and is a β-space. Then there is a first-countable subspace T of S which is also stationary in κ.

Proof: Suppose $\langle B_n \rangle$ is the sequence of functions which makes S a β-space, and let S^d be the set of non-isolated points of the space S. Let $T_0 = S^d$. For each $x \in T_0$ let $f_1(x)$ be the first element of S such that $f_1(x) < x$ and $[f_1(x), x] \cap S \subset B_1(x)$. Since f_1 is regressive, the PDL yields a point $y_1 \in S$ such that the set $T_1 = \{x \in T_0 | f_1(x) = y_1\}$ is stationary. Inductively find stationary sets $T_0 \supset T_1 \supset T_2 \supset \ldots$, functions $f_{n+1}: T_n \rightarrow S$ and points $y_n \in S$ such that $T_n = \{x \in T_{n-1} | f_n(x) = y_n\}$. Since $cf(\kappa) > \omega_0$ there is a $z \in S$ having $z > y_n$ for every n. Let $T_n' = T_n \cap [z, \kappa)$. Each T_n' is stationary and if $t_n \in T_n'$ then $z \in [y_n, t_n) \cap S \subset B_n(t_n)$ so that the sequence $\langle t_n \rangle$ must have a cluster point in S. Now define $T = \{x \in S^d | cf(x) = \omega_0\}$. Clearly T is a first-countable subspace of S. I assert that T is stationary. For let $C \in cub(\kappa)$. Because each of the sets T_n' is stationary, there are sequences $\langle t_n \rangle$ and $\langle c_n \rangle$ having $c_n \in C$, $t_n \in T_n'$ and $c_n < t_n < c_{n+1}$ for each n.

From above, the sequence $\langle t_n \rangle$ must have a cluster point in S, say u. But then $u \in T \cap C$ as required. $\square$.

I can now return to the proof of Theorem I. Since S is stationary in κ and hereditarily a β-space, there is a stationary, first-countable subspace T of S which also is hereditarily a β-space. According to the Ulam-Solovay theorem (Theorem B, above), there are two disjoint stationary subspaces U and V of T. Consider the space U, and let $\langle B_n \rangle$ be the sequence of functions making U a β-space. As in Lemma J, there are stationary subsets $U_1 \supset U_2 \supset \ldots$ of U having the property that if $x_n \in U_n$ then the sequence $\langle x_n \rangle$ must cluster in the subspace U. Let $C = \{x \in T \mid$ there are points $x_n \in U_n$ such that $\langle x_n \rangle$ converges to $x\}$. Because T is first-countable, the set C is relatively closed in T. Because of the special properties of the sets U_n, C is cofinal in T and $C \subset U$. But then $C \cap V = \emptyset$, which is impossible since V is also stationary. $\square$.

In closing, let me make a little list of things which you might try to prove using stationary sets and the theorem from [EL]. Let X be a GO space:

1) If every open cover of X has a point-countable refinement, then X is paracompact.
2) If $\pi[B]$ is a Borel set in X whenever B is a Borel subset of the space $X \times X$ (where π is projection onto the first coordinate) then X is hereditarily paracompact.
3) If S is a stationary subset of κ such that S is first-countable and such that, for each ordinal $\lambda < \kappa$ having $cf(\lambda) > \omega_0$, $S \cap [0, \lambda)$ is not stationary in

λ , then S is metrizable.

4) Suppose that X is a D-space, i.e., that whenever $\{U(x) | x \in X\}$ is an open covering of X having $x \in U(x)$ for each x, then there is a closed discrete subset D of X such that $\{U(x) | x \in D\}$ covers X. Then X is paracompact.

5) If X is realcompact then X is paracompact.

The first result is well-known and the fifth is very easy. The third is a theorem of Juhasz, while the second and fourth are unpublished results of mine.

Finally, let me pose a much more difficult question. Theorem C, above, gives a single class of spaces which contains, in some sense, an example of every non-paracompact GO space. Is there an analogous theorem for metrizability? I can prove [BL_2] that if X is a non-metrizable GO space, then some subspace of X is not a p-space [Ar], but that result is not sufficiently concrete.

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