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General hedgehogs in general topology

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It was proved by several authors that each uniform space with σ-discrete base has a point-finite base (see e.g. [RR], [P]); it is shown in [RR], that this fact is an immediate corollary of results in [V]. We will show that the converse fails to be true even if one replaces ω_0 by any higher cardinal. So one can see that the class of spaces with point-finite base that could be regarded as a class of nice spaces is wild enough in fact. Our main theorem is connected with an analysis of the following theorem (see [I], [H]): Let (X,ρ) be a pseudometric space, m be a natural number. If a collection H of subsets of X is a 1-cover of a set $Z\subset X$ and ord $H\leq p$ for some natural $p\leq m$ then there exist $\frac{1}{3m}$-discrete collections $K_1,\dots,K_p$ such that $\bigcup_{i=1}^{p} K_i$ is a $\frac{1}{3m}$-cover of Z and refines H. (For notation see bellow.) It would be very pleasant and surprising if Lebesgue number of $\bigcup K_i$ did not depend on an order of H. Using Ramsey theorem we shall show that it is not the case.

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Notation:

Let K be a cardinal. A collection of sets is said to be K-disjoint iff it can be decomposed into K disjoint subfamilies (or less, of course).

Let (X,ρ) be a pseudometric space, ε be a positive real number. A collection $\mathcal{G}$ of subsets of X is said to be ε-cover of Z, $Z\subset X$, if for each $x\in Z$ there is G in $\mathcal{G}$ such that $B_\varepsilon(x)\subset G$. $\mathcal{G}$ is said to be ε-discrete if $\mathrm{dist}(G,H)>\varepsilon$ for each two di tinct members G, H of G.

Given a collection G of sets, the order of G is defined by

ord $G = \sup \{ \text{card } A \mid A \subset G, \bigcap A \neq \emptyset \}$.

The closed interval $[0,1]$ will be denoted by I.

1. Notation:

Let α be a cardinal. By $F_u(\alpha)$ ($F_p(\alpha)$, p is an integer, resp.) we denote the set of all mappings $f : \alpha \longrightarrow I$ such that $\operatorname{coz} f = \{ a \in \alpha \mid f(a) \neq 0 \}$ is a finite set ($|\operatorname{coz} f| \leq p$, resp.) $F_u(\alpha)$ and $F_p(\alpha)$ will denote uniform spaces with the uniformity induced by $l_\infty(\alpha)$, as well.

2. Remark:

$F_p(p)$ is a cube of dimension p. $F_1(\alpha)$ is an α-hedgehog and as such very important (see [F1], [F2]). The great dimension $\Delta\alpha$ (see [I]) of $F_p(\alpha)$ is equal to p. Finally, let us observe that W. Kulpa used spaces $F_p(\alpha)$ to construct universal spaces in [K].

3. Notation:

Given a set B and a cardinal p, $[B]^p = \{ A \subset B \mid |A| = p \}$. For cardinal numbers a, b, K and a positive integer p a symbol $a \xrightarrow[K]{p} b$ denotes that for any mapping $r : [b]^p \longrightarrow K$ there is a set $A \subset b$ of cardinality a such that r is constant on $[A]^p$.

4. Construction:

For $x \in \alpha$, put $\tilde{x} = \{ f \in F_u(\alpha) \mid f(x) \neq 0 \}$. Put $P(\alpha) = \{ \tilde{x} \mid x \in \alpha \} \cup B_1(0)$. $P(\alpha)$ is a $\frac{1}{2}$-cover of $F_u(\alpha)$. $P(\alpha)$ restricted to $F_p(\alpha)$ is a cover of the order $p+1$.

5. Lemma:

1) If $2p \xrightarrow[K]{p} \alpha$ then $P(\alpha)$ on $F_p(\alpha)$ has no K-disjoint ε-refinement whenever $\varepsilon > \frac{1}{p}$.

... If $2p\frac{p}{k}\to\alpha$ for all integers p, then the cover $P(\alpha)$ of $F_u(\alpha)$ has no K-disjoint uniform refinement.

Proof:

1) For a set $V = \{v_1, v_2, \dots, v_p\} \subset \alpha$, $v_1 < v_2 < \dots < v_p$, define a function $\varphi_V : \alpha \to I$ by:

i) $\varphi_V(x) = 0$ for $x \in \alpha - V$

ii) a) if $p=2k$ put $\varphi_V(v_i) = \frac{2i}{p}$ for $1 \leq i \leq k$

$\varphi_V(v_i) = \frac{2p-2i}{p}$ for $k \leq i \leq 2k$

b) if $p=2k+1$ put $\varphi_V(v_i) = \frac{2i-1}{p}$ for $1 \leq i \leq k+1$

$\varphi_V(v_i) = \frac{2p+1-2i}{p}$ for $k+1 \leq i \leq 2k+1$.

For subsets V, W of α where $V = \{v_1, \dots, v_p\}$, $W = \{v_2, \dots, v_{p+1}\}$, $v_1 < v_2 < \dots < v_{p+1}$ it is easy to prove that $B_\varepsilon(\varphi_V) \cap B_\varepsilon(\varphi_W) \neq \emptyset$.

Suppose, $\mathcal{S}$ is a K-disjoint ε-refinement of $P(\alpha)$ on $F_p(\alpha)$, i.e. $\mathcal{S} = \bigcup\{\mathcal{S}_\iota \mid \iota \in K\}$, $\mathcal{S}_\iota$ s is disjoint family for each $\iota \in K$. Choose a mapping $r : [\alpha]^p \to K$ such that for each $V \in [\alpha]^p$, there is (necessarily unique) $S_V \in \mathcal{S}_{r(V)}$ containing $B_\varepsilon(\varphi_V)$.

By the hypothesis, there is $M = \{v_1, v_2, \dots, v_{2p}\} \subset \alpha$ such that r is constant on $[M]^p$. Put $r([M]^p) = \iota_0$. For $j=0, \dots, p$ put $V_j = \{v_{j+1}, \dots, v_{j+p}\}$. As noted above, $B_\varepsilon(\varphi_{V_j}) \cap B_\varepsilon(\varphi_{V_{j+1}}) \neq \emptyset$ for $j=0, \dots, p-1$. $\mathcal{S}_{\iota_0}$ is a disjoint family, so $S_{V_0} = S_{V_1} = \dots = S_{V_p}$ hence there is $\tilde{S} \in \mathcal{S}$ containing both $B_\varepsilon(\varphi_{V_0})$ and $B_\varepsilon(\varphi_{V_p})$ but $\mathrm{coz}\,\varphi_{V_0} \cap \mathrm{coz}\,\varphi_{V_p} = \emptyset$ and $B_1(0)$ contains no φ_{V_j} hence there is no member of $P(\alpha)$ containing $\tilde{S}$ – a contradiction.

2) If $\mathcal{S}$ is a K-disjoint uniform refinement of $P(\alpha)$ then $\mathcal{S}$ is an ε-cover for some $\varepsilon > 0$. Find now p such that $\varepsilon > \frac{1}{p}$

and restrict $P(\alpha)$ to $F_p(\alpha)$. 1) yields a contradiction.

6.Remark:

Ramsey theorem [R] asserts: for each integers k,m,n there an integer R(k,m,n) such that: $k \xrightarrow[m]{n} R(k,m,n)$ (so $\omega_o \xrightarrow[om]{n} \omega_o$ for each integers m,n).

Erdős-Rado theorem asserts: $K^{+} \xrightarrow[K]{n+1} (2(K,n))^{+}$ for each integ n and cardinal K (2(K,n) is defined by induction: 2(K,0) = 2(K,n+1) = $2^{2(K,n)}$). These two theorems assure that for each integer p and each cardinal K there is α such that $2p \xrightarrow[K]{p}$

7.Corollary:

There is no real number $\varepsilon > 0$ such that $P(\omega_o)$ on F_p(R(2p, is refined by (p+1)-disjoint ε-cover for each integer p.

Proof:

Use Lemma 5 and Ramsey theorem in Remark 6.

8.Remark:

Maybe, someone would like to replace $F_p(R(2p,p+1,p))$ by $F_p($ it can be done, of course, but the assertion is then weaker th that in Corollary 7.

9.Theorem:

For each cardinal K, there is a cardinal α such that $F_u(\alpha)$ has no K-disjoint base although it has a point-finite base.

Proof:

Use Lemma 5 and Erdős-Rado theorem in Remark 6.

10.Concluding remarks:

1) I do not know whether Ramsey type theorems are the best too for finding counterexamples as above. Is it possible that even $F_u(\omega_1)$ has no σ-discrete base ?

2) Let (X,U) be a uniform space. The collection $D_\sigma(U)$ of a

σ-disjoint covers from U forms a base of a uniformity.

Proof:

Clearly, if $P, q \in D_\sigma(U)$ then $P \wedge q \in D_\sigma(U)$. Let $P \in D_\sigma(U)$. So $P = \bigcup\{P_n | n \in \omega_0\}$, P_n is a disjoint family for each n. Find $q \in U$ such that $q^{*} < P$. For $n \in \omega_0$, $P \in P_n$ and $J \subset n$, define $T(n,J,P) = \bigcup\{Q \in q | (n = \min\{m | \exists P' \in P_m : st(Q,q) \subset P'\}) \& \& (J = \{j \in n | \exists P' \in P_j : Q \subset P'\}) \& (st(Q,q) \subset P)\}$. Put $R(n,J) = \{T(n,J,P) | P \in P_n\}$, $R = \bigcup\{R(n,J) | J \subset n \in \omega_0\}$. Clearly, $R \in D_\sigma(U)$. It remains to prove that $R^{*} < P$.

Take $x \in X$. Find $P' \in P_{n'}$ such that $st(x,q) \subset P'$ and n' is minimal such n. Suppose, $x \in T(n,J,P)$, i.e. there is $Q' \in q$ such that $x \in Q'$ and $st(Q',q) \subset P$ hence $n \geq n'$. Q' must be contained in P'. By definition of $T(n,J,P)$, $T(n,J,P)$ must be a subset of P' as well.

Hence the rule $(X,U) \to (X, D_\sigma(U))$ defines a functor from UNIF to UNIF. Let us denote it by s_1. Obviously, s_1 is a modification. By Theorem 9, s_1 is not identical on point-finite uniform spaces. Nevertheless, if all cardinals are non-measurable then s_1 preserves Cauchy filters on point-finite uniformities (see [RR]). There is a problem how s_1 behaves if there exists a measurable cardinal.

3) Under Generalized Continuum Hypothesis, for each cardinal K and each uniform space (X,U), the collection of all α-disjoint uniform covers $(\alpha \in K)$ forms a base of uniformity.

I do not know what situation occurs without assuming GCH.

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