1975-1976

Jiří Vilímovský
Reflections on distal spaces

In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1976. pp. 69–72.

Persistent URL: http://dml.cz/dmlcz/703145

SEMINAR UNIFORM SPACES 1975 - 76

Reflections on distal spaces

Jiří Vilímovský

Embedding preserving modifications in the category of separated uniform spaces are studied. It is shown that the only such reflections on distal spaces are just cardinal reflections.

We shall work in the category of separated uniform spaces and uniformly continuous mappings. A reflective subcategory $\mathcal{R}$ with the reflector r will be called modification if r preserves the underlying sets (i.e $\mathcal{R}$ contains all precompact spaces). Of course, every modification is closed under arbitrary subspaces. The modification r will be called embedding preserving if rX is a subspace of rY whenever X is a subspace of Y. A modification r will be called coarser than s whenever for all X rX has coarser uniformity than sX.

For a modification r and uniform space X we shall denote $r_e X$ the supremum of all uniformities sX (in the order "coarser than"), where s is a set preserving functor which agrees with r on all injective uniform spaces. This defines for any modification r another modification r_e and the following easy statement holds:

Proposition 1: For any uniform space X and modification r the value $r_e X$ can be constructed in the following way: We embed X into some injective space Y and $r_e X$ will be the relative uniformity from rY.
A modification r is embedding preserving if and only if $r = r_e$.

For a cardinal number $\aleph_\alpha$ the cardinal reflection p^α is a modification onto the class of all X such that every uniform cover of X is refined by a uniform cover of cardinality less than $\aleph_\alpha$. A space is called distal if it has a basis of finite-dimensional covers. The class of all distal spaces is reflective and the corresponding reflector D is an embedding preserving modification.

Under generalized continuum hypotheses (GCH) all cardinal reflections are embedding preserving. This result is not known without GCH, but if we restrict ourselves to distal spaces (more generally to spaces with point-finite base) the result is an immediate consequence of the theorem of Vidossich [3]. Moreover one can easily see that for any cardinal number $\aleph_\alpha$, $p^\alpha X$ is distal, whenever X is distal, hence $p^\alpha D = Dp^\alpha$. For details on distal spaces see [1].

Proposition 2 Let $\beta \leq \alpha$ be ordinal numbers, I compact interval, then

$$p^\beta(I \times \aleph_\alpha) = I \times p^\beta \aleph_\alpha$$

(The uniformity on $\aleph_\alpha$ is taken uniformly discrete.)

Proof: Obviously $p^\beta(I \times \aleph_\alpha)$ is finer than $I \times p^\beta \aleph_\alpha$. Conversely take $\mathcal{U} = \{U_i;\ i \in A\}$ uniform cover of $I \times \aleph_\alpha$ of cardinality less than $\aleph_\beta$. (Such covers form a basis of $p^\beta(I \times \aleph_\alpha)$, because $I \times \aleph_\alpha$ has a basis of point finite covers [3].) We can find a finite uniform cover $\{P_1, P_2, \ldots, P_n\}$ of I such that the cover

$$\{P_i \times \{\xi\};\ i \leq n,\ \xi \in \aleph_\alpha\}$$

refines $\mathcal{U}$. For any $J = \{i_1, i_2, \ldots, i_n\} \subset A$ we denote

$$G(J) = \{\xi\ ; P_j \times \{\xi\} \subset U_{i_j},\ j = 1,2,\ldots,n\}$$

The cover

$$\{P_i \times G(J)\}_{i,J}$$

is a uniform cover of $I \times p^\beta \aleph_\alpha$ refining $\mathcal{U}$.

Proposition 3: Let r be a modification, $\aleph_\alpha$ a cardinal number (with uniformly discrete uniformity),

$r\aleph_\alpha \neq \aleph_\alpha$. Then there exists $\beta \leq \alpha$ such that $r\aleph_\alpha = p^\beta\aleph_\alpha$. (Also known to Pelant and Reiterman.)

Proof: Let β be the smallest ordinal number γ such that $r\aleph_\alpha$ has no uniformly discrete subspace of cardinality $\aleph_\gamma$. Then $\beta \leq \alpha$ and $r\aleph_\alpha = p^\beta r\aleph_\alpha$, hence $p^\beta\aleph_\alpha$ is finer than $r\aleph_\alpha$. Conversely take a partition $\mathcal{U}$ of $\aleph_\alpha$ into $\aleph_\gamma < \aleph_\beta$ elements. $\mathcal{U}$ can be realized by a uniformly continuous surjective mapping onto $\aleph_\gamma$ with uniformly discrete uniformity. $\aleph_\gamma$ is a uniform subspace of $r\aleph_\alpha$, hence so is $r\aleph_\gamma$, and hence $r\aleph_\gamma = \aleph_\gamma$. So the mapping realizing $\mathcal{U}$ remains uniformly continuous from $r\aleph_\alpha$ into $\aleph_\gamma$, hence $\mathcal{U}$ is a uniform cover of $r\aleph_\alpha$.

Proposition 4: Let X be a cartesian product of a compact interval I with a uniformly discrete space $\aleph_\alpha$, r be an embedding preserving modification with $rX \neq X$. Then there is $\beta \leq \alpha$ such that $rX = p^\beta X$.

Proof: Using Proposition 3 we have $r\aleph_\alpha = p^\beta\aleph_\alpha$ for some $\beta \leq \alpha$. rX is finer than $I \times r\aleph_\alpha = I \times p^\beta\aleph_\alpha$ which is equal to $p^\beta X$ by Proposition 2. Assume that rX contains a uniformly discrete subspace of cardinality $\aleph_\beta$, then $r\aleph_\alpha$ being a subspace of rX also contains such a subspace and this is the contradiction with $r\aleph_\alpha = p^\beta\aleph_\alpha$, hence $rX = p^\beta X$.

Recall that the hedgehog over a set A, denoted by H(A) is the set of all $\langle a,x \rangle$, $a \in A$, $0 \leq x \leq 1$, where we consider $\langle a,0 \rangle = \langle b,0 \rangle$ for each a, b in A with the metric $d(\langle a,x \rangle, \langle a,y \rangle) = |x - y|$ and $d(\langle a,x \rangle, \langle b,y \rangle) = x + y$ if $a \neq b$. Subspaces of products of hedgehogs are exactly distal spaces (see [1]). The following lemma appears in [2]:

Lemma: Assume that the uniform spaces X and Y are topologically equivalent. If $x \in X$ and if the uniformities of X and Y coincide on the complement of each neighbourhood of x, then X = Y.

Immediately from this lemma and Proposition 4 we obtain the following:

Proposition 5: Let $H = H(\aleph_\alpha)$ be the hedgehog over $\aleph_\alpha$, r an embedding preserving modification, then $rH = p^\beta H$ for some $\beta \leq \alpha$.

Remark: In the preceding proposition we can omit the words "embedding preserving", because H is an injective space [4], so we can replace r by r_e from Proposition 1.

Using the fact that each distal space is a subspace of a product of hedgehogs one can easily derive the following:

Theorem: Let X be a distal space, r an embedding preserving modification, then there exists a cardinal number $\aleph_\alpha$ such that $rX = p^\alpha X$.

Corollary: If X is a distal space there is only finite number of embedding preserving modifications with distinct values on X.

References

[1] Frolík Z.: Basic refinements of uniform spaces, Lecture Notes in Mathematics 378. Springer-Verlag, 140-158

[2] Frolík Z.: Cozero refinement of uniform spaces, Seminar Uniform Spaces 1975 (informal seminar notes)

[3] Vidossich G.: A note on cardinal reflections in the category of uniform spaces, Proc. A.M.S. 23(1969), 55-58.

[4] Pelant J., Pták P.: Injectivity of polyhedra, this volume.