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Reflections not preserving completeness

J. Pelant

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Introduction: A problem whether each uniform space has a point-finite or equivalently, uniformly locally finite base of uniform covers, was solved negatively in [P]. This problem was first raised in [S] and further questions related to this problem have been discovered since 1960 (see e.g. [V_1], [I], p. 142). One of these questions is the following (see [I], [RR]):

Let X be a complete uniform space with a base of non-measurable covers. p^1 denotes a separable modification. Is it true that p^1X must be complete? If X has a point-finite base then the answer is yes (see [RR]) and so an e: tence of a counterexample again implies an existence of a uniform space having no point-finite base. We are going to give such a counterexample. We will show even more: Let m be an ordinal number. There is a complete uniform space X, card X = $(2^{\omega_m})^+$, such that p^mX is not complete.

(p^mX is defined as a uniformity formed by all pseudometrics induced by X such that any uniformly discrete set has cardinality $< \omega_m$.)

It implies that if r is a modification preserving completeness then rX = X for any uniformly discrete space X.

Further modifications, formed by covers with a point-character less than some cardinal m, will be discussed.

I wish to thank Z. Frolík who turned my attention to the above problems.

Construction: Let m be a cardinal number. $H_n = \{ \frac{i}{2^n} \mid i$ is an integer, $0 \leq i \leq 2^n \}$ for $n = 0,1,2,\ldots$, $H = \bigcup_{n=0}^{\infty} H_n$. Put $M(m) = \{f: H \rightarrow \exp m \times \exp m \mid (pr_1 f(p) \supset pr_2 f(p)$ for each $p \in H$) and $(pr_2 f(p) \supset pr_1 f(q)$ for each $p > q) \}$. (exp m denotes a set of all nonempty subsets of m , although it is not usual.)

We define a pseudometric uniformity on M(m). Put $\mathcal{K}(m,2^n) = \{f: H_n - \{0\} \rightarrow \exp m \times \exp m \mid$ there is $F \in M(m)$ such that $F/H_n - \{0\} = f \}$. For $U \in \mathcal{K}(m,2^n)$ define $\tilde{U} = \{f \in M(m) \mid pr_2 f(p) \supset pr_1 U(p) \supset pr_2 U(p) \supset pr_1 f(p - \frac{1}{2^n})$ for each $p \in H_n - \{0\}\}$. Define $\mathcal{U}_n = \{\tilde{U}\}_{U \in \mathcal{K}(m,2^n)}$, $\{\mathcal{U}_n\}_{n=0}^{\infty}$ is a base of a uniform space U(m) on an underlying set M(m) .

Remark: Clearly, U(m) is complete and non-Hausdorff (the last property does not represent any problem with respect to our aim). In the sequel, only $U(\omega_1)$ will be examined, as a procedure for other cardinals is quite similar.

Observation: For a uniform cover $\mathcal{S} = \{S_a\}_{a \in A}$ of $U(\omega_1)$ which is refined by $\mathcal{U}_n$ ($\mathcal{U}_j$ does not refine $\mathcal{S}$ for $j < n$), define $v_{\mathcal{S}} : \mathcal{K}(\omega_1,2^n) \rightarrow \exp A$ by $v(K) = \{a \in A \mid \tilde{K} \subset S_a \}$. Put $\tilde{v}_{\mathcal{S}}(f) = \bigcup \{v_{\mathcal{S}}(K) \mid f \in \tilde{K}$ and $K \in \mathcal{K}(\omega_1,2^n)\}$ for $f \in M(\omega_1)$. Clearly, $S_a \supset \tilde{v}_{\mathcal{S}}^{-1}(\{C \subset A \mid a \in C\})$ and a point-character of $\mathcal{S}$ is not greater than $\sup\{\text{card}\, \tilde{v}_{\mathcal{S}}(f) \mid f \in M(\omega_1)\}$. A well-known result of [V_2] can be reformulated: there is a base $\mathcal{B}$ of countable uniform covers of $U(\omega_1)$ (or any other uniform space) such that each member of $\mathcal{B}$ is of the form

$\{\tilde{v}^{-1}(\{C \subset \omega_0 \mid j \in C\})_{j \in \omega_0}$ where v is a mapping from $\mathcal{K}(\omega_1, 2^n)$ into $\exp \omega_0$ such that:

(1) card $\tilde{v}(f) < \omega_0$ for each $f \in M(\omega_0)$,

$v(K) \neq \emptyset$ for each $K \in \mathcal{K}(\omega_1, 2^n)$,

$v(K) \supset v(L)$ if $\tilde{L} \supset \tilde{K}$ for each $K, L \in \mathcal{K}(\omega_1, 2^n)$.

Notation: Let $V \in \mathcal{K}(\omega_1, 2^n)$ such that card $pr_j V(p) = \omega_1$ for each $p \in H_n - \{0\}$, $j = 1,2$. Let $\{X_i\}_{i=1}^{2^m}$, $\{Y_i\}_{i=1}^{2^m}$ be sequences of countable subsets of ω_1.

$V - \{X_i, Y_i\}_{i=1}^{2^m}$ denotes a member of $\mathcal{K}(\omega_1, 2^n)$ defined as follows:

$$pr_1((V - \{X_i, Y_i\})(\frac{t}{2^m})) = pr_1 V(\frac{t}{2^m}) - (\bigcup_{i=t}^{2^m} X_i \cup \bigcup_{i=t+1}^{2^m} Y_i),$$

$$pr_2((V - \{X_i, Y_i\})(\frac{t}{2^m})) = pr_2 V(\frac{t}{2^m}) - (\bigcup_{i=t}^{2^m} X_i \cup \bigcup_{i=t}^{2^m} Y_i).$$

Lemma: For each $f \in M(\omega_1)$ such that card $pr_j f(p) = \omega_1$ for each $p \in H$ and $j = 1,2$, and for each mapping $v: \mathcal{K}(\omega_1, 2^n) \longrightarrow \exp \omega_1$ satisfying (1) from Observation, the following formula holds (all sets X_i, Y_i are countable):

$$\exists p \in \omega_0 \ \forall X_{2^m} \ \exists Y_{2^m} \supset X_{2^m} \ \forall X_{2^m-1} \ \exists Y_{2^m-1} \supset X_{2^m-1} \ldots$$
$$\ldots \forall X_1 \ \exists Y_1 \supset X_1 : v(f^{(n)} - \{X_i, Y_i\}_{i=1}^{2^m}) \ni p,$$

where $f^{(n)} = f/H_n - \{0\}$.

Proof: Suppose the contrary, i.e.

(2) $\forall p \in \omega_0 \ \exists X_{2n} \ \forall Y_{2n} \ \exists X_{2n-1} \ \forall Y_{2n-1} \supset X_{2n-1} \ldots \exists X_1 \ \forall Y_1 \supset X_1 :$

$$v(f^{(n)} - \{X_i, Y_i\}) \not\ni p .$$

For each $p \in \omega_0$ choose X^p_{2n} according to (2). Put $Y_{2n} = \bigcup \{X^p_{2n} \mid p \in \omega_0\}$. Choose X^p_{2n-1} according to (2) for each $p \in \omega_0$. Put $Y_{2n-1} = \bigcup \{X^p_{2n-1} \mid p \in \omega_0\}$ ... . Choose X^p_1 for each $p \in \omega_0$. Put $Y_1 = \bigcup \{X^p_1 \mid p \in \omega_0\}$. Define an element $V \in \mathcal{K}(\omega_1, 2^n)$ by $pr_1 V(j) = pr_1 f(j) - \bigcup \{X^p_t \mid p \in \omega_0,\ 1 \geq \frac{t}{2^n} \geq j\}$, $pr_2 V(j) = pr_1 V(j) \cap pr_2 f(j)$ for each $j \in H$. Then $\widetilde{V} \supset \widetilde{f^{(n)} - \{X^p_i, Y_i\}}$ for each $p \in \omega_0$ hence $v(V) \subset \bigcap \{v(f^{(n)} - \{X^p_i, Y_i\}^{2^n}_{i=1}) \mid p \in \omega_0\} = \emptyset$ which is a contradiction ($v(V)$ must be nonempty).

Remark: Lemma remains valid if we replace ω_1 by any other regular uncountable cardinal ω_α and ω_0 by any $\omega_\delta < \omega_\alpha$.

Fact: For each $v: \mathcal{K}(\omega_1, 2^{n_v}) \longrightarrow \exp \omega_0$ from Observation, denote p from Lemma by p_v. Define a collection $\mathcal{F} = \{\bigcup \{\widetilde{K} \mid v(K) \ni p_v,\ K \in \mathcal{K}(\omega_1, 2^{n_v})\}\}_v$. It follows from Lemma that $\mathcal{F}$ is a filter (it is easy to show that for any $v_1, \ldots, v_j$ there is $K \in \mathcal{K}(\omega_1, 2^n)$, $n = \max_{i=1,\ldots,j} n_{v_i}$ such that $\widetilde{v}_i(g) \ni p_{v_i}$ for each $g \in \widetilde{K}$, $i = 1, \ldots, j$, j is an integer. Evidently, $\mathcal{F}$ is a Cauchy filter in $p^1 U(\omega_1)$. Lemma implies that this filter cannot converge in the induced topology, QED.

Corollary: p^1 does not preserve completeness.

Notation: Let α be an ordinal number. Let X be a uniform space. $b^{\alpha}X$ denotes a uniformity formed by all pseudometrics induced by X which have a point-character less than ω_{α} .

Note: 1) b^{α} is a modification for each α .

2) $b^{0}X$ is formed by all point-finite covers of X .

Corollary: b^{0} does not preserve completeness.

Proof: The statement follows from Rice-Reynolds theorem: X is complete. If X is non-measurable, then $p^{1}X$ is complete iff $b^{0}X$ is complete (see [RR]).

As mentioned above, one can prove by the same way

Theorem: Let α be an ordinal number. Then $p^{\alpha}(U(\alpha^{+}))$ is not complete.

Concluding remark: We would like to mention one interesting fact: using a method from [RR] and Theorem B (see below) derived by B. Balcar from Příkrý's results, one can prove that it is consistent with ZFC to suppose that there is no ordinal number α such that b^{α} preserves completeness. We are not going to put it down as there is a possibility to prove the last statement using only usual axioms of ZFC: Observation and Lemma can be easily restated in such a way that a theorem for b^{α} can be derived.

Theorem B : It is consistent with ZFC to suppose that the following assertion holds:

Let α be a regular cardinal. Let $\mathcal{F}$ be a uniform ultrafilter (i.e. card $F = \beta$ for each $F \in \mathcal{F}$) on a cardinal $\beta \geq \alpha$.

Then there is a partition $\{R_\iota\}_{\iota \in \alpha}$ of α such that a filter defined by $\tilde{\mathcal{F}} = \{\{\iota \in \alpha \mid F \cap R_\iota \neq \emptyset\}\}_{F \in \mathcal{F}}$ is a uniform ultrafilter on α .

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