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DIMENSIONS AND PARTITIONS OF UNITY

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A characterization of covering dimension by means of partitions of unity is given. Namely, the dimension can be derived from values of a certain simple quantitative characteristic of the partitions of unity which is closely related to the Lebesgue number of uniform covers.

Let us begin with the so called large and small uniform (covering) dimensions Δd and δd of uniform spaces. Recall their definition. First, if $\mathcal{C}$ is a collection of sets, the order of $\mathcal{C}$ is the least cardinal number, which will be denoted by $ord\,\mathcal{C}$, such that $card\,\mathcal{a} \leq ord\,\mathcal{C}$ whenever $\mathcal{a} \subset \mathcal{C}$, $\bigcap \mathcal{a} \neq \emptyset$. If X is a uniform space then $\Delta d\,X \leq n$, $\delta d\,X \leq n$, means that each uniform, each finite uniform respectively, cover of X can be refined by a uniform cover whose order is at most $n+1$. Then $\Delta d\,X$, and similarly $\delta d\,X$, is the smallest integer n for which the inequality holds. If it holds for no n, we set $\Delta d\,X = \infty$. If $X = \emptyset$ then $\Delta d\,X = \delta d\,X = -1$. We shall use the following almost evident fact: for $\delta d\,X \leq n$, the refining cover may either be supposed finite or not. Notice that the definition of δd is quite similar to the definition of the topological covering dimension dim: open covers are replaced by uniform covers.

If φ is a pseudometric on a set X, $\varepsilon > 0$ then we put $S_{\varphi,\varepsilon} = \{(x,y) \in X \times X \mid \varphi(x,y) < \varepsilon\}$; thus $S_{\varphi,\varepsilon}[H]$

is the ε -neighbourhood of H .

A partition of unity on a topological space X is a family $\{f_a\}$ of continuous non-negative real-valued functions on X such that $\sum_a f_a x = 1$ for each x . Given such a partition of unity, put

$$\lambda\{f_a\} = \inf_{x \in X} \{\sup_a f_a x\} .$$

We say that the partition is subordinated to a cover $\mathcal{G}$, if the family $\{coz\, f_a\}$ refines $\mathcal{G}$; here $coz\, f$ denotes the cozero set of the function f , i.e. $coz\, f = \{x \in X \mid fx \neq 0\}$. A partition $\{f_a\}$ of unity is said to be finite if the family $\{f_a\}$ is finite. Let X be a uniform space. A partition $\{f_a\}$ of unity on X is said to be uniform if each function f_a is uniformly continuous; it is said to be equiuniform if $\{f_a\}$ is a uniformly equicontinuous family of functions.

Theorem 1. Let X be a uniform space, Ω be the collection of all uniform covers of X . For $\mathcal{G}$ in Ω put $c(\mathcal{G}) = \sup \lambda\{f_a\}$ where the supremum is taken over all equiuniform partitions $\{f_a\}$ of unity on X which are subordinated to $\mathcal{G}$. Then

$$\inf_{\mathcal{G} \in \Omega} c(\mathcal{G}) = \frac{1}{\Delta d\, X + 1} .$$

Remark. The formula above is to be understood with the usual conventions like $1/\infty = 0$ etc. If X is void then some care of a convenient definition of all needed symbols is still in place.

Proof. I. Suppose $\Delta d\, X \leq n < \infty$. Let $\mathcal{G} \in \Omega$.

Choose a family $\{U_a\}$ from Ω such that it refines $\mathcal{G}$ and $\operatorname{ord}\{U_a\} \leqq n+1$. Let a cover $\{H_a\}$ be a strict shrinking of $\{U_a\}$, we may suppose that $S_{\varphi,1}[H_a] \subset U_a$ for each a for some uniformly continuous pseudometric φ on X . Put, for each x in X and each a ,

$$g_a x = \min\{1, \operatorname{dist}_\varphi(x, X \setminus U_a)\} .$$

Since $|g_a x - g_a y| \leqq \varphi(x,y)$, $\{g_a\}$ is a uniformly equicontinuous family of non-negative functions. Put $gx = \sum_a g_a x$. Evidently $g_a x = 1$ for $x \in H_a$, $\operatorname{coz} g_a \subset U_a$; thus, for each x , $g_a x > 0$ for $n+1$ indices a at most. Therefore $1 \leqq gx \leqq n+1$, $|gx - gy| \leqq (2n+2)\varphi(x,y)$ which implies that g and $1/g$ are bounded and uniformly continuous. A simple calculation shows that the family $\{f_a\}$ where $f_a = g_a / g$ is uniformly equicontinuous; moreover $\sum_a f_a x = 1$. As $\operatorname{coz} f_a = \operatorname{coz} g_a \subset U_a$ this partition of unity is subordinated to $\mathcal{G}$. For each x , $f_a x > 0$ for $n+1$ indices a at most, $\sum_a f_a x = 1$, thus $f_a x \geqq \frac{1}{n+1}$ for one a at least. As x was arbitrary, $\lambda\{f_a\} \geqq \frac{1}{n+1}$. Thus we have proved $c(\mathcal{G}) \geqq \frac{1}{n+1}$ for each cover $\mathcal{G}$, hence $\inf_{\mathcal{G} \in \Omega} c(\mathcal{G}) \geqq \frac{1}{n+1}$ as well.

II. Suppose $\Delta d\, X \geqq n$. Choose $\mathcal{H}$ in Ω for which there is no refinement whose order is at most $n+1$. Let $\{f_a\}$ be an arbitrary equiuniform partition of unity which is subordinated to $\mathcal{H}$. Let us estimate

$\lambda\{f_a\}$. Suppose $\lambda\{f_a\} > \frac{1}{n+1}$. Of course, $\lambda\{f_a\} > \frac{1}{n+1} + \varepsilon$ for some positive ε. This means for each $x \in X$ there exists a such that $f_a x > \frac{1}{n+1} + \varepsilon$. Now put, for each a,

$$V_a = \{x \in X \mid f_a x > \frac{1}{n+1}\}, \quad F_a = \{x \in X \mid f_a x > \frac{1}{n+1} + \varepsilon\}.$$

As $\{f_a\}$ is uniformly equicontinuous, the formula

$$\varphi(x, y) = \sup_a |f_a x - f_a y|$$

defines a uniformly continuous pseudometric on X. Since $\bigcup_a F_a = X$ and $V_a \supset S_{\varphi,\varepsilon}[F_a]$ for each a, the family $\{V_a\}$ is a uniform cover of X. It refines $\mathcal{H}$ because $V_a \subset \operatorname{coz} f_a$. Thus $\operatorname{ord}\{V_a\} > n + 1$. Take a point x which belongs to more than $n + 1$ sets V_a. Then $f_a x > \frac{1}{n+1}$ for $n + 2$ indices a at least, hence $\sum_a f_a x > 1$ which is a contradiction. We have proved $c(\mathcal{H}) \leqq \frac{1}{n+1}$, thus $\inf_{\mathcal{G} \in \Omega} c(\mathcal{G}) \leqq \frac{1}{n+1}$ which completes the proof.

Theorem 2. Let X be a uniform space, Ω_0 be the collection of all finite uniform covers of X. For $\mathcal{G}$ in Ω_0 let $c(\mathcal{G})$, $c_0(\mathcal{G})$ be the supremum of $\lambda\{f_a\}$ over all equiuniform, finite uniform respectively, partitions $\{f_a\}$ of unity on X which are subordinated to $\mathcal{G}$. Then

$$\inf_{\mathcal{G} \in \Omega_0} c(\mathcal{G}) = \inf_{\mathcal{G} \in \Omega_0} c_0(\mathcal{G}) = \frac{1}{\delta d\, X + 1}$$

Proof. Repeat the proof of Theorem 1 with Ω_0 instead of Ω and δd instead of Δd ; it may be slightly simplified in some places. The partition of unity obtained in the first part of the proof is always finite. The partition of unity considered in the second part may be either arbitrary equiuniform or arbitrary finite uniform (which is necessarily equiuniform, too).

Notice that the second equality in the formula in Theorem 2 also follows from Theorem 1 applied to the totally bounded modification of X .

Let us present still a similar characterization of topological covering dimension.

Theorem 3. Let X be a normal topological space, Ω be the collection of all locally finite open covers of X , Ω_0 be the collection of all finite open covers of X . For $\mathcal{G}$ in Ω let $c(\mathcal{G})$, $c_0(\mathcal{G})$ be the supremum of $\lambda\{f_a\}$ over all, all finite respectively, partitions $\{f_a\}$ of unity on X which are subordinated to $\mathcal{G}$. Then

$$\inf_{\mathcal{G}\in\Omega} c(\mathcal{G}) = \inf_{\mathcal{G}\in\Omega_0} c(\mathcal{G}) = \inf_{\mathcal{G}\in\Omega_0} c_0(\mathcal{G}) = \frac{1}{\dim X + 1} .$$

Proof. It is again possible to repeat essentially the proof of Theorem 1. For locally finite covers, the well-known Dowker theorem (Every locally finite open cover can be refined by a cover with order at most $\dim X + 1$) must be used. No pseudometric is needed, if H_a were closed the functions g_a may be constructed directly;

$\{V_a\}$ is evidently an open cover.

In another way, Theorem 3 also follows from Theorems 1 and 2 if X is endowed with the fine uniformity. Then $\Delta d\, X = \delta d\, X = \dim X$. It suffices that Ω is only a base of all uniform covers. The equiuniformity of the partitions of unity is easy (prove the continuity of the pseudometric φ defined by $\varphi(x,y) = \sum_a |f_a x - f_a y|$), see also Z. Frolík, this volume, p. 8 .

Notice that Theorem 3 might be still generalized for non-normal spaces if a suitable definition of the covering dimension were used.