ONE FOLKLORISTIC LEMMA ON CARDINAL REFLECTIONS IN UNIF

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Let $(X, \mathcal{U})$ be a uniform space. Let κ be a cardinal number. Let $\{U_a\}_{a \in A}$ $\mathrm{card}\, A < \kappa$, be a $\mathcal{U}$-cover of X. Denote by $\mathcal{V} = \{V_b\}_{b \in B}$ some $\mathcal{U}$-cover which star-refines $\{U_a\}$. For $x \in X$, denote $S(x) = \{a \mid a \in A \,\&\, \mathrm{st}(x, \mathcal{V}) \subset U_a\}$. For $Y \subset X$, denote $I(Y) = \{a \mid a \in A \,\&\, Y \subset U_a\}$. A mapping $\varkappa : X \to A$ such that $\varkappa(x) \in S(x)$ for each $x \in X$ is called a choice mapping.

<u>Lemma:</u> There is a $\mathcal{U}$-cover $\{C_a\}_{a \in A}$ which star-refines $\{U_a\}_{a \in A}$ and is refined by $\mathcal{V}$ if and only if the following assertions hold:

there exist a choice mapping $\varkappa$ and a partition $\{B_a\}_{a \in A}$ of B such that

$$\varkappa\Big(\bigcup_{b \in B_a} V_b\Big) = \bigcup_{b \in B_a} \varkappa(V_b) \subset \bigcap_{b \in B_a} I(V_b) = I\Big(\bigcup_{b \in B_a} V_b\Big) .$$

<u>Proof:</u> The necessity of the condition is obvious. Conversely, if a choice mapping $\varkappa$ and a partition $\{B_a\}_{a \in A}$ of B from Lemma exist, then we define $\{C_a\}_{a \in A}$ by $C_a = \bigcup_{b \in B_a} V_b$. Clearly, $\mathcal{V} \prec \{C_a\}_{a \in A}$. Let $x \in X$.

$$I(\mathrm{st}(x, \{C_a\}_{a \in A})) = \bigcap \{I(C_a) \mid x \in C_a\} \supset \bigcap \varkappa(C_a) \ni \varkappa(x) .$$

<u>Application:</u>

1) Assume that $2^\alpha < \kappa$ for any $\alpha < \kappa$. Each $\mathcal{U}$-cover $\{U_a\}_{a \in A}$, $\mathrm{card}\, A < \kappa$, is star-refined by a $\mathcal{U}$-cover

$\{C_a\}_{a\in A}$ (See [1].)

Proof: Consider $\{V_b\}_{b\in B} \overset{**}{\lessdot} \{U_a\}_{a\in A}$. Define $r: X \longrightarrow A$ by $r(x) = \min\{a \mid st(x, \{V_b\}_{b\in B}) \subset U_a\}$; we may assume that A is an ordinal. For $a \in A$ and $J \subset \subset a$, define $B(a, J) = \{b \mid$ is the minimal number of A such that $st(V_b, \{V_b\}_{b\in B}) \subset U_a$ & $I(V_b) \cap a = J\}$. The assumption implies that $\operatorname{card}\{(a, J) \mid J \subset a \in A\} = \operatorname{card} A$. It follows from the definition of r and $B(a, J)$ that $r(V_b) \subset J \cup \{a\}$ for any $b \in B(a, J)$, hence $\bigcap_{b\in B(a,J)} I(V_b) \supset \supset J \cup \{a\} \supset \bigcup_{b\in B(a,J)} r(V_b)$ and Lemma applies.

2) $(X, \mathcal{U})$ is a uniform space, $\{U_a\}_{a\in A}$ $\operatorname{card} A < K$ is a $\mathcal{U}$-cover. Let $\{W_c\}_{c\in C} = \mathbf{W}$ be a σ-pointwise finite $\mathcal{U}$-cover, that refines $\{U_a\}_{a\in A}$. Then there is a $\mathcal{U}$-cover $\{V_a\}_{a\in A}$ which refines starwise $\{U_a\}_{a\in A}$. See [2].

Proof: Choose any mapping $g: C \longrightarrow A$ such that $W_c \subset U_{g(c)}$. By assumptions, $C = \bigcup_{n=0}^{\infty} C_n$ and each collection $\{W_c \mid c \in C_n\}$ is pointwise finite. Consider $\mathbf{V} = \{V_b\}_{b\in B} \overset{**}{\lessdot} \mathbf{W}$. Assume that C is a well-ordered set, take C_n with the induced ordering. Define $r: X \longrightarrow A$ by $r(x) = g(c)$ where $c \in C$ is the minimal element of C_j such that $st(x, \{V_b\}) \subset W_c$ and there is not any point with this property being an element of C_k for $k < j$. As $\mathbf{W}$ is pointwise finite and $\mathbf{V} \overset{**}{\lessdot} \mathbf{W}$, $r(V_b)$

is finite for any $b \in B$. For $J \in \mathcal{P}_{fin}(A)$ define $B_J =$ $= \{b \in B \mid \varkappa(V_b) = J\}$. As $\operatorname{card} \mathcal{P}_{fin}(A) = \operatorname{card} A$ and $\bigcup_{b \in B_J} \varkappa(V_b) = J \subset \bigcap_{b \in B_J} I(V_b)$, Lemma applies.

References:

[1] Anna Kucia: On coverings of a uniformity, Coll. Math. 27(1973), 73-74.

[2] Giovanni Vidossich: A note on cardinal reflections in the category of uniform spaces, Proc. A.M.S. 23(1969), 55-58.

Added in proof: In Application 1, if $K = \alpha^+$, then it is enough to suppose $2^\beta \leq \alpha$ for all $\beta < \alpha$.