Zdeněk Frolík

Three technical tools in uniform spaces

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Three technical tools in uniform spaces

by Zdeněk Frolík

These are refinements (just a way of looking on concrete functors), partitions of unity, and distal spaces. Perhaps the most important one, the atoms in uniform spaces, is still being developed by several members of the seminar, and therefore we omit general discussion. Just in 1.4 one of the possible applications is mentioned. For convenience of the reader a short outline is described:

§ 1. Refinements

1.1. If $\mathcal{K}$ is a concrete category, then a refinement of $\mathcal{K}$ is an object-onto concrete embedding

$$\mathcal{K} \hookrightarrow \mathcal{L}$$

into a concrete category $\mathcal{L}$. Thus objects of $\mathcal{K}$ coincide with the objects of $\mathcal{L}$, and

$$\mathcal{K}(X,Y) \subset \mathcal{L}(X, Y)$$

for each X and Y.

Our basic examples at this moment are the classical refinements:

$$U \hookrightarrow p \hookrightarrow t \hookrightarrow set^U$$

Here $p(X,Y)$ is the set of all proximally continuous (abbr. proximal) mappings of X into Y, $t(X,Y)$ is the set of all continuous mappings of X into Y, and $set^U(X,Y)$ is the set of all mappings of X into Y.

In general the class of all refinements of $\mathcal{K}$ is ordered by inclusion, $\mathcal{K}$ is the smallest refinement, and $set^{\mathcal{K}}$ is the largest refinement, where $set^{\mathcal{K}}(X,Y)$ is the set of all mappings of X into Y. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two refinements then the meet (infimum) is $\mathcal{L}_1 \wedge \mathcal{L}_2 \quad (X,Y) = \mathcal{L}_1(X,Y) \cap \mathcal{L}_2(X,Y)$, and the join (supremum) is the smallest category $\mathcal{L}_1 \vee \mathcal{L}_2$ such that $\mathcal{L}_1 \vee \mathcal{L}_2(X,Y) \supset \mathcal{L}_1(X,Y) \cup \mathcal{L}_2(X,Y)$.

If F is a concrete covariant functor of $\mathcal{K}$ into any category $\mathcal{S}$, then F generates a refinement s of $\mathcal{K}$ as follows:

$$s(X,Y) = \mathcal{S}(FX,FX)$$

On the other hand any refinement $\mathcal{K} \hookrightarrow \mathcal{L}$ of $\mathcal{K}$ can be obtained in this way. Indeed, let $\sim_{\mathcal{L}}$ consist of all identity bijections in $\mathcal{L}$ which are isomorphisms in $\mathcal{L}$, and consider the factor

$$\mathcal{G} = \mathcal{L}/\sim_{\mathcal{L}}$$

which has for the objects the equivalence classes under the equivalence $\sim_{\mathcal{L}}$, and if $\sim_{\mathcal{L}} X$ is the equivalence class containing X, then

$$\mathcal{G}(\sim_{\mathcal{L}} X, \sim_{\mathcal{L}} Y) = \mathcal{L}(X,Y).$$

Now $\{X \longrightarrow \sim_{\mathcal{L}} X\}$ defines a functor F: $\mathcal{K} \rightarrow \mathcal{L}/\sim_{\mathcal{L}}$ which generates $\mathcal{L}$ in the sense explained above.

It follows that the investigation of refinements of $\mathcal{K}$ is equivalent to the investigation of concrete functors from $\mathcal{K}$. In the examples above we may define t(X,Y) without mentioning topology, and then we get the category of uniformizable topological spaces from t by factorization $t/\sim_t$, or we define the underlying functor t from uniform spaces into the category Top of topological spaces, and define

$$t(X,Y) = \text{Top}\ (tX,tY).$$

Of course

$$t/\sim_t \cong t[U] \hookrightarrow \text{Top}$$

Similarly, we can define proximal mappings p(X,Y) without mentioning the category Prox of proximity spaces, and Prox is then isomorphic with $p/\sim_p$, or we can start with the functor p: U $\longrightarrow$ Prox, and define

$$p(X,Y) = \text{Prox}\ (pX,pY) .$$

The main advantage of refinements is:

A refinement can be defined by many functors, and any functor defining a refinement $\mathcal{K} \hookrightarrow \mathcal{L}$ of $\mathcal{K}$ factors through $\mathcal{L} \rightarrow \mathcal{L}/\sim_{\mathcal{L}}$, and extends to $\mathcal{L}$.

For each X let $p_c X$ be the reflection of U on precompact uniform spaces (the largest precompact uniformity contained in X). Then

$p(X,Y) = U(X,p_c Y) = U(p_c X,p_c Y)$

Thus p is generated by the functor $p_c: U \longrightarrow U$.

For each X let $t_f X$ be the finest uniformity which is topological equivalent to X. Then

$$t(X,Y) = U(t_f X,Y) = U(t_f X,t_f Y)$$

and hence t is generated by $t_f:U \longrightarrow U$.

Intuitively, the "structure" of the objects of $\mathcal{L}/\sim_{\mathcal{L}}$ is "less rich" than the structure of the objects of $\mathcal{K}$. The objects of $\mathcal{L}/\sim_{\mathcal{L}}$ are called $\mathcal{L}$ - spaces. As we noticed the category of $\mathcal{L}$-spaces over $\mathcal{K}$ can be realized in many natural ways.

1.2. Fine and coarse objects.

Let $\mathcal{K} \hookrightarrow \mathcal{L}$ be a refinement, and let $\mathfrak{X}$ be a class of objects. An object Y is called $\mathcal{L}$ -fine w.r.t. $\mathfrak{X}$ if

$$\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$$

for each X in $\mathfrak{X}$. An object X is called $\mathcal{L}$-coarse w.r.t. $\mathfrak{X}$ if the equality holds for each Y in $\mathfrak{X}$. If $\mathfrak{X}$ is the class of all objects, then we say simply $\mathcal{L}$-fine and $\mathcal{L}$-coarse. Finally, $\mathcal{L}$ -bi-extremal objects w.r.t. $\mathfrak{X}$ are the objects which are $\mathcal{L}$-fine w.r.t. $\mathfrak{X}$ as well as $\mathcal{L}$ -coarse w.r.t. $\mathfrak{X}$.

One checks immediately that the classes of the fine w.r.t. $\mathfrak{X}$ objects are closed under inductive generation in $\mathfrak{X}$, and the class of all coarse w.r.t. $\mathfrak{X}$ objects is closed under projective generation in $\mathfrak{X}$. This implies:

Theorem 1. If $\mathcal{L}$ is a refinement of the category of uniform spaces, or proximity spaces or topological spaces, then the category of $\mathcal{L}$-fine w.r.t. $\mathfrak{X}$ spaces is coreflective, and the category of all $\mathcal{L}$ -

coarse w.r.t. $\mathfrak{X}$ spaces is reflective, and the reflection is just a modification (the reflection maps are identity mappings), and the coreflection is a comodification (it should be remarked that in the categories listed any coreflection on is always a modification).

The coreflection onto $\mathcal{L}$-fine w.r.t. $\mathfrak{X}$ spaces, if it exists, is denoted by $\mathcal{L}_{f,\mathfrak{X}}$, and the reflection on $\mathcal{L}$-coarse w.r.t. $\mathfrak{X}$ spaces by $\mathcal{L}_{c,\mathfrak{X}}$. If $\mathfrak{X}$ is the class of all objects, we simply write $\mathcal{L}_f$ and $\mathcal{L}_c$.

Examples (p,t,Set). Consider the refinements p,t and set of U. One checks immediately that $\text{set}_f X$ is the finest uniformity on X (the diagonal is a uniform vicinity of the diagonal), $\text{Set}_c X$ is the coarsest uniformity on X ($X \times X$ is the only uniform vicinity of the diagonal). And we have

$$\text{Set}(X,Y) = U(\text{Set}_f X,Y) = U(X,\text{Set}_c Y)$$

As concerns t, t-fine uniform spaces are called fine in Isbell [2], universal in Bourbaki, and topologically fine in Čech [1]. It is easy to check that $t_f X$ is the finest uniformity which induces the same topology as X does. One can say that the uniformly continuous pseudometrics on $t_f X$ are just the continuous pseudometrics on X. This is trivial, as well as other descriptions of $t_f X$. We have noticed

$$t(X,Y) = U(t_f X,Y)$$

On the other hand, $t_c X$ is set-coarse for each X, i.e. $t_c X = \text{Set}_c X$. This is proved easily as follows: if $x, y \in X$, $x \neq y$, consider the subspace Y of R consisting of all $\frac{1}{n}$, $n = 1,2,\ldots$, and let $f: Y \to X$ assign x to the points $\frac{1}{2n}$, and y to the points $\frac{1}{2n+1}$. Since Y is discrete, $f: Y \to X$ is continuous, and if f is uniformly continuous then necessarily the points x and y are proximal.

As concerns p, the p-coarse objects, called proximally coarse uniformities in Čech [11], are just precompact uniform spaces. It holds

$$p(X,Y) = U(X,p_cY),$$

but it is not true that

$$p(X,Y) = U(pX,Y)$$

The first example is due to M. Katětov, and independently by C. Dowker. The simplest way (Poljakov, and Čech [11]) is based on the following two results which will be needed in the sequel:

Lemma 1. If $p_c(X\times X) = p_c(Y\times Y)$ then $X = Y$.

Proof. The proximal neighborhoods of the diagonal are just the uniform vicinities of the diagonal.

Lemma 2. If $p_cX = X$, then $p_c(X\times Y) = p_cX\times p_cY$ for each Y.

Proof. Check finite covers of $X\times Y$ if $p_cX = Y$. Now, if $p_cX \neq X$, then

$$p_c(X\times X) \neq p_cX\times p_cX,$$

and

$$p_c(p_cX\times X) = p_c(X\times p_cX) = p_cX\times p_cX$$

$$\inf \{p_cX\times X,\ X\times p_cX\} = X\times X$$

Thus $p_cX\times X$ and $X\times p_cX$ are proximally equivalent, however, the join is strictly finer.

Obviously the class of set-fine spaces is hereditary, and finitely productive. On the other hand, the classes of topologically fine and proximally fine spaces are neither finitely productive nor hereditary. In the case of t_f we have $t_f(X\times Y) = t_fX\times t_fY$ just when one of the spaces has finite Hausdorff reflection, or $X\times Y$ is pseudocompact. If X is metrizable and Hausdorff then no dense proper subspace of $t_f X$ is topologicaly fine. As concerns the proximally fine spaces the productivity has been a problem for some time. M. Hušek has constructed two countable topological fine spaces such that the product is not proximally fine, and proved that the product is proximally fine iff all finite subproducts are. It should be noted that G. Tashjian (to appear in Fund.Math.) independently proved a weaker theorem with finite replaced by countable; her proof is an immediate consequence of her lemma on "completely additive" disjoint families of sets in products. We must say that the first non-trivial example of proximally fine spaces were metrizable spaces; one can find a proof in V. Pohlová's second note. Since every uniform space is a subspace of a product of pseudometrizable spaces, it follows that every space is a subspace of a proximally fine space. On the other hand, not many spaces admit an embedding into topologically fine spaces (subspaces of topologically fine spaces are called subfine by Isbell; see his book).

Return to the general case

$$\mathcal{K} \hookrightarrow \mathcal{L} .$$

Following J. Vilímovský, a refinement $\mathcal{L}$ of $\mathcal{K}$ is called fine-maximal if

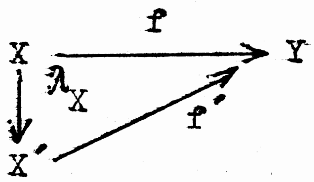
$$\mathcal{L}(X,Y) = \mathcal{K}(\mathcal{K}_f X, Y),$$

and coarse maximal if

$$\mathcal{L}(X,Y) = \mathcal{K}(X, \mathcal{K}_c Y)$$

for each X and Y. J. Vilímovský showed that $\mathcal{K} \hookrightarrow \mathcal{L}$ is fine-maximal iff the subcategory $\mathcal{K}$ of $\mathcal{L}$ is reflective in $\mathcal{K}$ (and then $\mathcal{L}_f$ is the reflection), and similarly for coarse-maximal refinements. Reflectivity is defined in the usual way:

For each object X of $\mathcal{L}$ there exists an $\mathcal{L}$-morphism $\lambda_X : X \to X'$, X' being an object of $\mathcal{K}$, such that for each morphism $f \in \mathcal{L}(X,Y)$, Y being an object of $\mathcal{K}$, there exists exactly one $f': X' \longrightarrow Y$ such that



Obviously, if $\mathcal{L}$ is fine-maximal then we can put $X' = \mathcal{L}_f X$. Conversely, since $\mathcal{K}$ and $\mathcal{L}$ have the same objects we see immediately that λ_X has an inverse which is in $\mathcal{K}$ etc.

The terms fine-maximal and coarse-maximal were used because usually $\mathcal{K}$ is fixed and various $\mathcal{L}$ are studied, and then " $\mathcal{L}$ is fine-maximal" is more convenient to me than " $\mathcal{K}$ is reflective in $\mathcal{L}$ ".

For some categories it is true that no "non-trivial" full subcategory is both reflective and coreflective. For some categories of uniform spaces, in particular for all uniform spaces and all Hausdorff this theorem is proved by M. Hušek, first note.

A similar property of a category is: $\mathcal{K}$ is both reflective and coreflective in no non-trivial refinement of $\mathcal{K}$. This is a theorem of Hušek and Vilímovský (Spring 1974) for $\mathcal{K}$ being the category of Hausdorff uniform spaces (see Hušek and Vilímovský), and Hušek (Spring 1975) for the category of all uniform spaces. We shall return to these theorems in 1.4.

In conclusion we present a trivial, however, useful, characterization of fine-maximal refinements.

Theorem. Assume that $\mathcal{K}$ is the category of uniform spaces, or more generally, a concrete category in which projective generation can be always performed. Then a refinement $\mathcal{L}$ of $\mathcal{K}$ is fine-maximal (i.e.

is reflective in $\mathcal{L}$) if and only if the following condition is satisfied:

if $\{f_\alpha : X \to X_\alpha\}$ is a projectively-generating family in $\mathcal{K}$, then $\{f_\alpha : X \to X_\alpha\}$ is a projectively-generating family in $\mathcal{L}$.

Equivalently: If $X \hookrightarrow Y$ in $\mathcal{K}$, then so in $\mathcal{L}$, and in $X = \Pi\{X_a\}$ in $\mathcal{K}$, then so in $\mathcal{L}$.

Certainly this generalizes as follows:

For a given X,

$U(\mathcal{L}_f X, Y) = \mathcal{K}(X,Y)$ for each Y iff the condition in Theorem is satisfied for this $X = \mathcal{L}_f X$, and all f_α.

1.3. Simply fine and simply coarse objects.

Consider a refinement $\mathcal{K} \hookrightarrow \mathcal{L}$. An object X is called simply $\mathcal{L}$-fine if X is the finest space in its equivalence class $\sim_{\mathcal{L}} X$. Clearly each $\mathcal{L}$-fine space is simply $\mathcal{L}$-fine, and the converse is true for $U \hookrightarrow p$ (this is easy from 1.2, see Ex.(c) below) but it is not true in general (some examples are given in V. Kůrková and Pták-Kosina and the converse is open for the cardinal reflections even if a partial result is given in Rödl.

Note two simple results which are used without any reference:

(1) X is $\mathcal{L}$-fine iff X is simply $\mathcal{L}$-fine, and the reduced product (in $\mathcal{K}$) of any two $\mathcal{L}$-morphisms with domain X is an $\mathcal{L}$-morphism.

(2) Assume that for each X there exists a simply $\mathcal{L}$-fine space X' in $\sim_{\mathcal{L}} X$. Then $X' = \mathcal{L}_f X$ for each X iff $\{X \to X'\}$ is functorial.

Similar results hold for "coarse".

Definition. An object X is called simply $\mathcal{L}$-biextremal if X is simultaneously simply $\mathcal{L}$-fine and simply $\mathcal{L}$-coarse.

Clearly the bi-extremal objects are simply bi-extremal, and X is simply bi-extremal iff the equivalence class $\sim_{\mathcal{L}} X$ is a singleton (if the refinement is given by a functor F, then this means FX = FY implies X = Y). In P. Pták it is shown that there exists a distally coarse, simply distally fine space which is not distally fine (for the definition of distal spaces see § 3). We just check the classical refinements.

Examples. (a) $U \hookrightarrow set^U$. Since the refinement is both fine-maximal and coarse-maximal, simply fine is fine and simply coarse is coarse, and hence the singletons are the only non-void bi-extremal spaces.

(b) $U \hookrightarrow t$. Since the refinement is fine-coarse, simply fine spaces are fine. We already noticed that topological coarse spaces are just the set-coarse spaces. On the other hand, the simply topologically coarse uniform spaces are just the spaces X such that the topology is locally compact and the uniformity is the relativization of the unique uniformity of the one-point compactification of X. The simply topologically bi-extremal spaces were characterized by E. Hewitt as follows: if X_1 and X_2 are two distinct sets in X, then the closure of one of them is compact. This statement is equivalent to saying: all the compactifications of the induced topological space are equivalent, and this is equivalent to: the proximity of X is topologically simply bi-extremal (we have in mind the refinement $Prox \hookrightarrow t$).

(c) $U \hookrightarrow p$. Simply coarse objects are coarse (this is obvious), and simply fine uniform spaces are fine, which follows from the formula $p(pX \times Y) = pX \times pY$ which was recalled in 1.2. Thus simply bi-extremal objects are bi-extremal. A trivial example: simply topologically bi-extremal spaces are proximally bi-extremal. Another example: precompact metrizable spaces. Proximally bi-extremal spaces have been studied by Isbell. A nice characterization: every proximally continuous image is precompact. Some properties may appear in some papers by A. Hager.

1.4. Functors preserving the structure.

If $\mathcal{K} \hookrightarrow \mathcal{L}$ is a refinement denote by Inv $(\mathcal{L})$ the class of all functors $F: \mathcal{K} \to \mathcal{K}$ such that F preserves the $\mathcal{L}$-structure, i.e.

$$FX \in \sim_{\mathcal{L}} X$$

for each X.

If $\mathcal{L}$ is given by a functor L of $\mathcal{K}$ into something then $F \in$ Inv $(\mathcal{L})$ iff $L \circ F = L$.

A functor $F: \mathcal{K} \to \mathcal{K}$ is positive (negative) if FX is coarser (finer) than X for each X. Denote by $\mathrm{Inv}_+(\mathcal{L})$ and $\mathrm{Inv}_-(\mathcal{L})$ the positive or the negative functors in Inv $(\mathcal{L})$.

Clearly Inv $(\mathcal{L})$ is closed under composition, and contains the identity functor.

Also $\mathrm{Inv}_+(\mathcal{L})$ is fine-directed, and $\mathrm{Inv}_-(\mathcal{L})$ is coarse-directed. Indeed $F_1 \circ F_2$ is coarser (finer) than both F_1 and F_2 if F_i are positive (negative).

The finest element of $\mathrm{Inv}_+(\mathcal{L})$, if it exists, is denoted by $\mathcal{L}_+$. Similarly we define $\mathcal{L}_-$.

Since $\mathcal{L}_+$ and $\mathcal{L}_-$ are idempotent, $\mathcal{L}_+$ is a reflection since $\mathcal{L}_+$ is positive, and $\mathcal{L}_-$ is a coreflection since $\mathcal{L}_-$ is negative.

If $\mathcal{L}_f$ preserves the structure, i.e. if $\mathcal{L}$ is fine-maximal, then $\mathcal{L}_- = \mathcal{L}_f$ and similarly for $\mathcal{L}_+$.

Examples. (a) $U \hookrightarrow set^U$. $Set_f = Set_-$, $set_c = set_+$.

(b) $U \hookrightarrow p$. Since p_c preserves the structure, $p_+ = p_c$. Next p_- is the identity functor. In addition, any functor in Inv(p) is positive. Indeed, assume $F \in Inv(p)$. Given any space X we have

$$pF(X \times X) = p(X \times X).$$

Always the identity

$$F(X \times X) \longrightarrow FX \times FX$$

is uniformly continuous, and hence

$$p(X \times X) \longrightarrow p(FX \times FX)$$

is uniformly continuous, and hence by a result of Poljakov and the author, the identity

$$X \longrightarrow FX$$

is uniformly continuous.

(c) $U \hookrightarrow t$. Since t_f preserves t, $t_- = t_f$. Next $t_+ = p_c$, and in addition, each functor $F \in Inv(t)$ is between t_f and $t_+ = p_c$. If X is compact, then FX = X because the compact topology is induced by exactly one uniformity. It follows that FX is finer than X if X is precompact.
Finally, for any X, FX is finer than Fp_cX, and this is finer than p_cX.

The plus-functors and the minus-functors are of interest in the case when the coarse and fine functors do not preserve the structure, and this is certainly the most interesting case. Notice that t determines p($t_+ = p$). At this moment the most interesting results concern coz and hyper-coz refinement.

Remark. It might be of some interest to know which refinements of U are given by a functor of U into itself. Also it would be interesting to know examples of alternating functors in Inv($\mathcal{L}$): those which are not between $\mathcal{L}_-$ and $\mathcal{L}_+$.

The theorem of Hušek-Vilímovský that the category of Hausdorff uniform spaces has no non-trivial both fine-maximal and coarse-maximal refinement follows immediately from the fact that p_- is identity, and from the observation of Vilímovský that p_c is the coarsest concrete functor on Hausdorff uniform spaces.

A similar theorem for all uniform spaces (proved by Hušek) is an immediate consequence of the fact that every uniform space is a quotient of a sum of atoms (in uniform spaces) which are not proximally set-fine. (This follows from examples in Čech: Topological spaces, or J. Isbell's book - recalled in Hušek's first note , Pelant-Reiterman description of atoms, and the following simple observation:

if a concrete functor is positive on a class of spaces, then it is positive on the inductive progeny of $\mathcal{K}$.

§ 2. Partitions of unity

2.1. Generalities. A partition of unity on a set X is a family $\{f_a | a \in A\}$ of non-negative functions such that $\Sigma \{f_a x\} = 1$ for each x in X. Now if X has a structure, then one is interested just in the partitions of unity which are closely related to that structure. It is convenient to look on the partitions as maps into various spaces.

Fix a set of indices A. The set R^A can be viewed as the product of topological linear spaces. For each $p \geqq 1$ we define for $x = \{x_a\} \in R^A$

$$(\| x \|_p)^p = \Sigma \{|x_a|^p\} .$$

For each $p \geqq 1$ the set of all x with $\| x \|_p < \infty$ is a Banach space (linear structure is inherited from R^A, the norm is $\| \; \|_p$) denoted by $\ell_p(A)$. For $p = \infty$ we take

$$\| x \|_\infty = \sup \{|x_a|\}$$

and get $\ell_\infty(A)$. We are interested (this is supported by theorems) just in the cases $p = 1$ and $p = \infty$. Clearly the identity embedding

$$\ell_1(A) \longrightarrow \ell_\infty(A)$$

is a linear continuous mapping, hence a uniformly continuous mapping. The closure of $\ell_1(A)$ in $\ell_\infty(A)$ is usually denoted by $c_o(A)$. The elements of $c_o(A)$ are those $x = \{x_a\}$ for which the set $\{a \,|\, |x_a| > \varepsilon\}$ is finite for each $\varepsilon > 0$. If we consider A as a discrete locally compact topological space then we may say that $c_o(A)$ are the continuous functions which have the limit 0 at the infinity.

It should be remarked that $\ell_1(A)$ is finite-dimensional iff $\ell_1(A)$ is isomorphic to $\ell_\infty(A)$ (as topological vector spaces), or iff the sets $\ell_1(A)$ and $\ell_\infty(A)$ coincide.

Recall that the dual of $\ell_1(A)$ is $\ell_\infty(A)$ and the pairing is $\langle x,y\rangle = \Sigma \{x_a, y_a\}$. Hence the weak topology on $\ell_1(A)$ (the resulting space is denoted $\ell_1 w$) is projectively generated by all $\{x \longrightarrow \langle x,y\rangle\} : \ell_1 w \longrightarrow R$, $y \in \ell_\infty(A)$.

The dual of $c_o(A)$ is $\ell_1(A)$, the pairing is again $\langle x,y\rangle$, and it follows easily that the weak topology on c_o is just the point-wise convergence, i.e. $c_{ow}(A)$ is a subspace of R^A.

Coming back to partitions, if $\{f_a\}$ is a partition, then $\{f_a x\}$ is an element of the unit sphere S in $\ell_1(A)$, moreover, it is an element of S^+ which is the intersection of S with the positive cone. To avoid any misunderstanding, recall that

$$S(A) = \{x \mid \|x\|_1 = 1\}$$

$$S^+(A) = \{x \mid x \in S,\ x_a \geqq 0 \text{ for each } a\}.$$

We have noticed that each partition $\{f_a\}$ defines a mapping

$$x \longrightarrow \{f_a x\} : X \longrightarrow S^+ .$$

Conversely, if $f: X \longrightarrow S^+$ is any mapping then $\{(fx)_a\}$ is a partition of unity.

Definition. A partition $\{f_a\}$ of unity is called E-uniformly continuous, or simply an E-partition, where E is one of the spaces ℓ_1, c_o, ℓ_{1w}, c_{ow} if the map $\{f_a\}: X \longrightarrow E$ is uniformly continuous. Instead of c_o partition we say ℓ_∞-partition (the uniformities induced on ℓ_1 by c_o and ℓ_∞ coincide). We are used to say weak ℓ_1-partition instead of ℓ_{1w}-partition, and weak c_o-partition or weak ℓ_∞-partition instead of c_{ow}-partition.

Notation: S_1, S_{1w}, S_o, S_{ow} denotes the uniform subspace S of, respectively, ℓ_1 or ℓ_{1w}, or c_o, or c_{ow}.

The distinction of various structures on S is pointless in topology because of the following easy and well-known result:

Lemma. The uniform spaces S_1, S_o and S_{ow} are topologically equivalent (i.e. $t_f S_1 = t_f S_{1w} = t_f S_o = t_f S_{ow}$)

Since

$$\begin{array}{ccc} \ell_1 & \longrightarrow & c_o \\ \downarrow & & \downarrow \\ \ell_{1w} & \longrightarrow & c_{ow} \end{array}$$

it is enough to show that the identity $S_{ow} \rightarrow \ell_1$ is continuous, that means, if $x^b \rightarrow x$ in S_{ow} (pointwise!), then $\| x^b - x \|_1 \rightarrow 0$. However, this is very easy.

2.2. Partitions subordinated to a cover. If $f: X \rightarrow R$, then we define

$$\text{coz } f = \{x \mid x \in X,\ fx \neq 0\} .$$

With every mapping $f = \{f_a\}: X \rightarrow \ell_\infty(A)$ such that $fx \neq 0$ for each x, there is associated the cover $\{\text{coz } f_a\}$ of X, and the Lebesgue function $\lambda(f): X \rightarrow R$ defined as follows:

$$\lambda(f)x = \sup\{|f_a x| \mid a \in A\} = \| fx \|_\infty ,$$

and the Lebesgue number

$$\inf\{\lambda(f)x \mid x \in X\} = \inf\{\| fx \|_\infty\} .$$

Observe that the Lebesgue number of f is the Lebesgue number of the cover $\{\text{coz } x_a \cap f[X]\}$ of $f[X]$ in the metric space $f[X] \subset \ell_\infty$. Hence:

Proposition. If $f: X \rightarrow \ell_\infty(A)$ is uniformly continuous and the Lebesgue number is positive, then $\{\text{coz } f_a\}$ is a uniform cover of X.

Proof. If the Lebesgue number is $r > 0$, then the r-balls in $f[X] \subset \ell_\infty(A)$ refine $\{\text{coz } x_a \cap f[X]\}$.

Corollary. If $\{f_a\}$ is an ℓ_∞-partition of X such that $\{\text{coz } f_a\}$ is point-bounded, then $\{\text{coz } f_a\}$ is a uniform cover.

Proof. If each point is at most in k coz f_a's then the Lebesgue number is $\geqq {}^1/_k$.

Remark. A short examination of the proof of Corollary may lead to a conjecture that the Lebesgue numbers of the ℓ_∞-partitions characterize (in an obvious way) the uniform dimension of the space. J. Hejcman proved the conjecture, and this was the first result of the seminar.

Lemma. If $f: X \longrightarrow \ell_\infty(A)$ is uniformly continuous then the Lebesgue function $\lambda(f)$ is uniformly continuous.

Proof. Since $\lambda(f)(x) = \| fx \|_\infty$, and since $\| \cdot \|_\infty$ is uniformly continuous on ℓ_∞ , the composite $\lambda(f)$ is uniformly continuous. (As always, $\| x \| - \| y \| \leqq$ $\leqq \| x - y \|$.)

For further use we recall the following simple result from Isbell:

Theorem 1. If $\mathcal{U}$ is a uniform cover of X, then there exists an ℓ_∞-partition of X such that $\{coz\ f_a\}$ refines $\mathcal{U}$.

Using this result, M. Zahradník proved that Theorem 1 is true for any $p > 1$ (this is simple) and moreover, he proved that Theorem 1 does not hold for ℓ_1 in the following cases: X is an infinite-dimensional Banach space, and $\mathcal{U}$ is the cover by 1-balls (this result seems to be highly non-trivial).

2.3. Examples. We must state the well known Theorem 1. Every uniform space is projectively generated by bounded maps in some $\ell_\infty(A)$; one can take X for A.

This follows immediately from the well known fact that every bounded metric space $\langle X,d \rangle$ is isometric with a bounded subset of $\ell_\infty(X)$. One assigns to each $x \in X$ the function $\{ y \longrightarrow d\langle x,y \rangle \}$.

On the other hand, we shall prove in the next §

that just distal spaces are projectively generated by ℓ_∞-partitions.

Another useful class of spaces is described in the next simple result.

Theorem 2. Let $\mathcal{K}$ be the class of all spaces X with the following property:

for each uniform cover $\mathcal{U}$ is refined by $\{\operatorname{coz} f_a\}$ for some ℓ_1-partition $\{f_a\}$.

Then $\mathcal{K}$ is productive, and hereditary, hence reflective.

We shall see that the class of spaces in Theorem 2 contains many quite "fine" spaces, e.g. sub-metric-t_f.

In conclusion we indicate several coreflective classes of spaces:

Let E_1 and E_2 be uniformities admitted in 2.1. Let $\mathcal{K}(E_1,E_2)$ be the class of all X such that every E_1-partition is an E_2-partition. It is known that

$\mathcal{K}(\ell_\infty, \ell_1)$ is the class of metric-t_f spaces

$\mathcal{K}(\ell_\infty, \ell_{1w})$ is the class of Alexandrov spaces.

(See the second author's note, or Fourier paper).

As concerns other non-trivial possibilities just some negative results are known to me.

§ 3. Distal spaces

The only reference is the author's "Basic refinements of uniform spaces", Lecture Notes in Mathematics 378. Springer-Verlag, 140-158, where distal spaces are introduced.

3.1. Recall that a family $\{X_a\}$ of subsets of a uniform space X is called uniformly discrete if there exists a uniformly continuous pseudometric ϱ on X such that $\{X_a\}$ is metrically discrete in $\langle X, \varrho \rangle$, or equivalently, there exists a uniform vicinity U of the diagonal such that $U[X_a] \cap X_b = \emptyset$ for each $a \neq b$.

Definition. A mapping $f: X \longrightarrow Y$ is called distal if for each uniformly discrete family $\{Y_a\}$ in Y, the family $\{f^{-1}[Y_a]\}$ is uniformly discrete in X. The set of all distal mappings from X into Y is denoted by D(X,Y).

It is obvious that $U(X,Y) \subset D(X,Y)$ and the class D of all distal mappings froms a category which is a refinement of U. Thus

$$U \hookrightarrow D \hookrightarrow Set^{U}.$$

Given a cardinal $\aleph_\alpha$, one defines α-distal mapping to be the mapping $f: X \longrightarrow Y$ such that the preimage of each uniformly discrete family is uniformly discrete provided that the cardinal of the index set is less than $\aleph_\alpha$.

Again the α-distal morphisms form a refinement D^α of U, and $D^0 = p$ (For $\alpha = 0$ the definition of α-distal maps coincides with a familiar description of proximal maps.) Thus we have

$$U \hookrightarrow D \hookrightarrow D^\alpha \hookrightarrow D^0 = p.$$

3.2. It was proved in the above reference that D is coarse maximal, i.e.

$$D(X,Y) = U(X,D_c Y),$$

and $D_c Y$ was described as follows: the finite-dimensional covers (see 3.3) of Y form a basis for the uniform covers of $D_c Y$. The same procedure gives:

Theorem. For each α

$$D^{\alpha}\langle X,Y\rangle = U(X,D_c^{\alpha} Y),$$

and the finite-dimensional uniform covers of Y with the cardinal less than $\aleph_{\alpha}$ form a basis for the uniform covers of $D_c^{\alpha} Y$.

Kosina and Pták have given an intrinsic characterization of the collection of all uniformly discrete families in a uniform space, i.e. an axiomatic definition of distal spaces. It should be remarked that a nice characterization is due to Williams.

Remark. The zero-dimensional uniform covers of a space X form a uniformity which will be denoted by $D_o X$. Clearly D_o is a reflection. Also each of D_o^{α}, defined in an obvious way, is a reflection, and D_o is the coarsest non-trivial reflection of uniform spaces (if it is non-trivial then the coreflective class must contain a discrete two-point space).

On the other hand, one-dimensional, or more generally, n-dimensional covers, $n > 0$, does not form a base for a uniformity. Of course, the one-dimensional covers form a subbasis for $D_c X$.

3.3. ℓ_∞-partitions generate D_c.

Theorem. If $f: X \to \ell_1(A)$ has bounded range, and if $f: X \to \ell_\infty(A)$ is uniformly continuous then so is $f: D_c X \to \ell_\infty(A)$.

Corollary 1. $D_c X$ is projectively generated by maps in Theorem.

Proof of Corollary. It is enough to show that each finite-dimensional uniform cover is realized by an ℓ_∞-partition. This is however almost evident from 2.2 and 2.3. Given such a cover $\mathcal{U}$, take any ℓ_∞-partition $\{f_U \mid U \in \mathcal{U}\}$ subordinated to $\mathcal{U}$. This partition realizes $\mathcal{U}$ by 2.2.

Proof of Theorem. We may and shall assume that $f \geqq 0$. For each positive real $r > 0$ define

$$f_r = \{(f_a - r)_+\} .$$

Clearly $f_r : X \longrightarrow \ell_\infty(A)$ is uniformly continuous and

$$\| f_r x - fx \| \leqq r$$

for each x. Hence it is enough to show that each

$$f_r : D_c X \longrightarrow \ell_\infty(A)$$

is uniformly continuous, and to this end it is enough to show that the range of f_r is finite-dimensional. For any x at most $[1/r]$ coordinates is non-zero and hence the image is contained in the subspace $B_{[1/r]}$ of all elements of the unit ball of $\ell_1(A)$ which have all but $[1/r]$ coordinates zero. This space has uniform dimension $[1/r]$, see Isbell's book.

Corollary 2. Every bounded set in $\ell_1(A)$ is distally coarse in the uniformity inherited from $\ell_\infty(A)$.

3.4. Stone-Weierstrass theorem.

Recall the Stone-Weierstrass theorem for proximity spaces (Čech's book).

Let $U_b(X)$ be the algebra of all bounded uniformly continuous functions on X. If $\mathcal{F} \subset U_b(X)$ projectively generates $p_c X$, then $U_b(X)$ is the smallest uniformly closed algebra containing $\mathcal{F}$.

A similar theorem is true for ℓ_∞-distal ℓ_1-bounded mappings. However, the statement is not beautiful because, perhaps, one should add the following two conditions:

1) Algebra is closed under "permutation" maps of $\ell_1(A)$

2) The composition with each coordinate-function is in the algebra.

It should be remarked that $D_c X$ is projectively generated with the ℓ_∞-partitions if the index set is large enough (the cardinal of X suffices). Use Proposition 2.2 and Corollary.

3.5. Distally fine spaces.

Since $U \hookrightarrow D \hookrightarrow p$, each proximally fine space is distally fine, hence metrizable spaces are distally fine. Also all proximally coarse spaces are distally fine, hence distally bi-extremal. Indeed, if X is proximally coarse, and if $f: X \longrightarrow Y$ is an onto distal mapping, then Y is proximally coarse, and hence f is uniformly continuous.

Since $\mathrm{Inv}(p) = \mathrm{Inv}_+(p)$, necessarily $\mathrm{Inv}(D) = \mathrm{Inv}_+(D)$, and hence D_- is the identity functor.

Recall that a simply bi-extremal space does not need to be distally fine (see Pták's note).

3.6. Cardinal reflections p^α .

In this field the best result is J. Pelant's space which answers several important questions in the way it was hoped for.

Definition. The starting point is the following elementary lemma which appears in Isbell's book.

Lemma. The following properties of a space X are equivalent:

(a) $D_c^\alpha X = D_c X$.

(b) The uniform covers of cardinal less than $\aleph_\alpha$ form a basis for all uniform covers.

Proof. Work with a metric space.

Definition. The distal character of a space X is the smallest ordinal α such that $D^{\alpha}X = DX$, or equivalently, such that every uniform cover is refined by a uniform cover of cardinal less than $\aleph_{\alpha}$. A space is called α-distal if the distal character is at most α.

It is obvious that the class of all α-distal spaces is hereditary and productive, hence there exists a modification functor $X \longrightarrow p_c^{\alpha} X$.

Define now a refinement p^{α} of U by setting

$$p^{\alpha}(X,Y) = U(X, p_c^{\alpha} Y),$$

then α-distal spaces are just the p^{α}-coarse spaces, and p_c^{α} coincides with p_c^{α} defined w.r.t. p^{α}.

Note that

$$U \hookrightarrow p^{\alpha} \hookrightarrow p^{0} = p .$$

As usual it is desirable to have a nice description of $p_c^{\alpha} X$ by means of X. Evidently $p_c^{\alpha} X$ is projectively generated by all $f: X \longrightarrow M \in U$ such that M are α-distal metrizable spaces. This amounts to saying:

$\mathcal{U}$ is a uniform cover of $p_c^{\alpha} X$ iff there exists a sequence $\{\mathcal{U}_n\}$ of uniform covers of X such that $\mathcal{U}_1$ refines $\mathcal{U}$, each $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$, and the cardinal of each $\mathcal{U}_n$ is less than $\aleph_{\alpha}$.

If a uniform cover $\mathcal{U}$ is point-finite then by a theorem of Vidossich, if the cardinal of $\mathcal{U}$ is less than $\aleph_{\alpha}$ then $\mathcal{U}$ is uniform on $p_c^{\alpha} X$. In particular, if $\alpha = 0$ or $\alpha = 1$ then a uniform cover of X is uniform on $p_c^{\alpha} X$ iff it is refined by a uniform cover of X of cardinal less than $\aleph_{\alpha}$. This statement is also true if GCH is assumed (Kucia). Both results are proved by the same procedure in a J. Pelant´s note.

What is very important, J. Pelant exhibited an example (under certain set-theoretical assumptions) where the statement is not true for $\alpha = 2$.

Remark. Recently J. Pelant showed that p_c^1 reflection of a complete metric space does not need to be complete. In fact, he proved much more. The proofs appear in Seminar Uniform Spaces Notes 1974-75. However, the conjecture that no non-trivial reflection preserves completeness is yet open. The problem is difficult because not many reflections are known.

The following question is yet open:

if $X \hookrightarrow Y$ then $p^\alpha X \hookrightarrow p^\alpha Y$ for each α .

Since

$$p^\alpha \hookrightarrow p^0 = p,$$

we get immediately (see 1.4):

$$\mathrm{Inv}(p^\alpha) = \mathrm{Inv}_+(p^\alpha).$$

The following problem seems to be difficult:

is each simply p^α-fine space p^α-fine?

We know that the answer is in affirmative for $\alpha = 0$, however, for $\alpha = 1$ the problem is open, and a possible candidate for a counter-example is studied in the Rödl's note, who checks the "counter-example" in the category of zero-dimensional spaces.

References

[1] Čech E.: Topological spaces, Academia, Prague 1966.

[2] Isbell J.: Uniform spaces, Amer. Math. Soc., Providence, 1964.