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HYDROLOGICAL APPLICATIONS OF A MODEL-BASED APPROACH TO FUZZY SET MEMBERSHIP FUNCTIONS

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Abstract: Since the common approach to defining membership functions of fuzzy numbers is rather subjective, another, more objective method is proposed. It is applicable in situations where two models, say M_1 and M_2 , share the same uncertain input parameter p . Model M_1 is used to assess the fuzziness of p , whereas the goal is to assess the fuzziness of the p -dependent output of model M_2 . Simple examples are presented to illustrate the proposed approach.

Keywords: fuzzy set, membership function, uncertainty quantification

MSC: 03E72, 03E75

1. Introduction

This contribution deals with uncertain parameters represented by fuzzy sets, namely with a model-dependent definition of membership functions.

The membership function determines the membership grade of the elements of the corresponding fuzzy set [3], [4], [6], [7]. Unlike classical set theory, where the characteristic function range is limited to the bivalent set $\{0, 1\}$, the membership function range is an interval; without loss of generality, we can limit ourselves to $[0, 1]$, the commonly used range.

For fuzzy numbers, triangular or trapezoidal membership functions are widely used, for instance; see Figure 1. They are directly defined by the analyst on the basis of his or her judgment. Inevitably, strong subjective factors influence the definition. A more objective approach to the definition of a membership function is possible in situations where P , a set of uncertain input parameters, appears in two associated models, say M_1 and M_2 , where the output of the model M_1 is measured and, through solving an inverse problem, enables the identification of the input parameters value. The goal is to assess the uncertainty of the output of the model M_2 via fuzzified input parameters P whose membership function is defined by means of the response of the model M_1 .

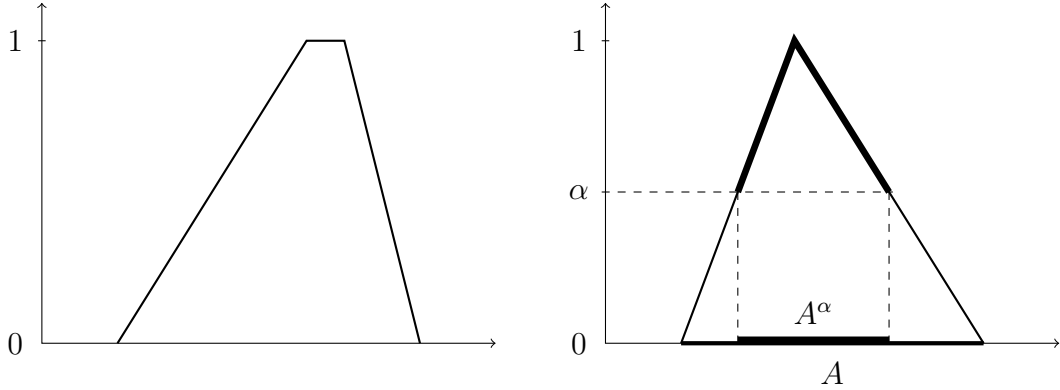


Figure 1: Left: A trapezoidal membership function. Right: A triangular membership function and an α -level set A^α .

Let us consider a space $S = \mathbb{R}^n$, where \mathbb{R} stands for the field of real numbers and n is a natural number. Let μ_A be a continuous membership function defined on S and such that its support (that is, the closure of $\{a \in S \mid \mu_A(a) > 0\}$) is equal to a compact convex subset A of S . Next, we define the α -cuts of A (α -level sets) as

$$A^\alpha = \{a \in A \mid \mu_A(a) \geq \alpha\}, \quad \text{where } \alpha \in [0, 1].$$

Let us note that $A^0 \equiv A$. We assume that A^α is convex for any $\alpha \in [0, 1]$.

Figure 1 depicts two (nonsymmetric) membership functions where A and A^α are closed intervals. We also observe, see Figure 1 (right), that by knowing A^α for any $\alpha \in [0, 1]$, we can reconstruct μ_A . That is,

$$\mu_A(a) = \max\{\alpha \mid a \in A^\alpha\} \quad (1)$$

for any $a \in A \subset S$.

The same idea applied to a finite sequence $\{\alpha_i\}_{i=1}^n \subset [0, 1]$ is used in numerical algorithms to approximate the membership function of a model output.

To this end, let us consider Φ , a quantity of interest whose value at a is continuously determined by an a -dependent mathematical or computational model. That is, we can view Φ as a (possibly rather complex) map from A to \mathbb{R} . If A is fuzzy, then $R_\Phi = \{y \in \mathbb{R} \mid \exists a \in A \ y = \Phi(a)\}$, the range of $\Phi|_A$, is also fuzzy and its membership function can be inferred by Zadeh's extension principle, see [3], [4], [7], for instance. The principle says that μ_{R_Φ} , the membership function of the fuzzy set R_Φ , can be obtained by applying the following rule

$$\mu_{R_\Phi}(y) = \max_{\{a \in A \mid y = \Phi(a)\}} \mu_A(a) \quad (2)$$

at each $y \in R_\Phi$.

Since R_Φ is an interval, it can be easier to obtain μ_Φ not directly from (2), but from (1) where A^α is replaced by R_Φ^α , the α -cut of R_Φ that coincides with the range of $\Phi|_{A^\alpha}$.

By virtue of the convexity and compactness assumptions,

$$R_{\Phi}^{\alpha} = \left[\min_{a \in A^{\alpha}} \Phi(a), \max_{a \in A^{\alpha}} \Phi(a) \right]; \quad (3)$$

see [4], for example.

Let us note that supremum appears in (1) and (2) in general if the assumptions on A and μ_A are weakened.

2. Model-driven membership function

Let us assume that a model M_1 is represented by $\psi(a, \cdot)$, a real continuous function dependent on a parameter $a \in B \subset S$. Moreover, let a be uncertain, let the output $\psi(a, \cdot)$ be measured at points $\{x_i\}_{i=1}^k$, and let the respective recorded values be denoted by $\{r_i\}_{i=1}^k$.

Next, let us identify the weighted least squares minimizer

$$a_{\min} = \arg \min_{a \in B} \omega(a), \quad \text{where } \omega(a) = \sum_{i=1}^k w_i (r_i - \psi(a, x_i))^2 \quad (4)$$

and w_i are positive weights. It is assumed that $\omega(a_{\min}) > 0$. The quantity ω will help to define the membership function describing the fuzziness of the input of the quantity of interest Φ that is determined by a model M_2 .

In [2], examples of membership functions are given, but more general options exist for the definition of the membership function. Take $0 < c_1, c_2, c_3, c_3$ odd, and

$$\mu_1(b) = 1 + c_1 \left(1 - \left(\frac{\omega(b)}{\omega(b_{\min})} \right)^{c_2} \right)^{c_3}, \quad \mu_2(b) = 1 + c_1 \left(\left(\frac{\omega(b_{\min})}{\omega(b)} \right)^{c_2} - 1 \right)^{c_3}, \quad (5)$$

for instance. We observe that $\omega(b)/\omega(b_{\min}) \geq 1$.

For a fixed c_1, c_2, c_3 and $i \in \{1, 2\}$, the fuzzy set A is then defined by

$$A = \{a \in B \mid \mu_i(a) \in [0, 1]\}. \quad (6)$$

A natural choice might be $c_1 = 1, c_2 = 1/2$ or $c_2 = 1$, and $c_3 = 1$.

Once the ω -based fuzzy set A and its membership function μ_A are established, the membership function $\mu_{R_{\Phi}}$ associated with the quantity of interest Φ is determined by Zadeh's extension principle; see Section 1.

3. Examples

Let us illustrate the above theory by simple examples.

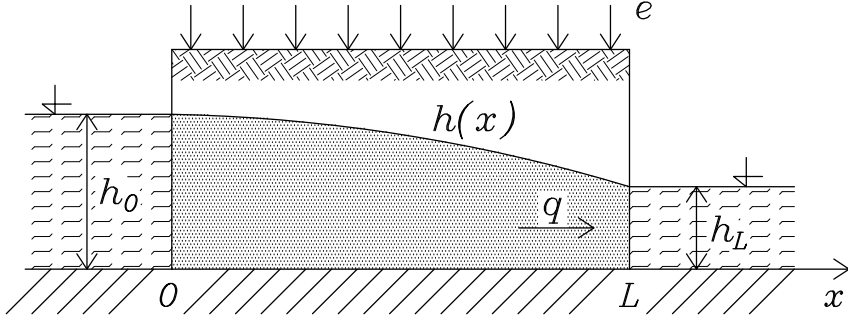


Figure 2: A permeable embankment separates two reservoirs and is subjected to infiltration or evaporation. The groundwater level function h is given by (8).

3.1. Two water levels separated by a permeable embankment

Figure 2 shows a cross section of an embankment separating two reservoirs. The embankment is L units wide and made of a permeable material. The water levels in reservoirs, namely h_0 and h_L , are different. We can assume that $h_0 > h_L$.

Due to head of water (difference of water levels), groundwater flow and also seepage through the embankment exist. The groundwater level is modeled by a smooth function h defined on the interval $[0, L]$. To add an external factor, let us introduce a constant e representing evaporation ($e > 0$) or infiltration ($e < 0$); see Figure 2, where $e < 0$.

A simple but commonly used approximation h of the true groundwater level in the embankment is based on Dupuit's postulates and solves

$$\frac{d}{dx} \left(-Kh(x) \frac{dh}{dx}(x) \right) + e = 0, \quad h(0) = h_0, \quad h(L) = h_L, \quad (7)$$

where $0 < K \in \mathbb{R}$ is the saturated hydraulic conductivity; see [5]. Since (7) is equivalent to

$$\frac{d^2}{dx^2} h^2(x) = 2 \frac{e}{K},$$

one can easily check that

$$h_{e,K}^2(x) = \frac{e}{K} x^2 + \left(\frac{h_L^2 - h_0^2}{L} - \frac{e}{K} L \right) x + h_0^2 \quad (8)$$

is the squared solution to (7).

We will assess seepage q (per unit length) and evaporation rate e in two steps.

3.1.1. Seepage

Seepage through the embankment at $x = L$ and consistent with (8) is (see [5]) given by

$$\Phi(K) \equiv q(L) = -\frac{eL}{2} + K \frac{h_0^2 - h_L^2}{2L}, \quad (9)$$

where Φ indicates that $q \equiv q(L)$ is the quantity of interest whose membership function $\hat{\mu}$ will be inferred.

We can apply (8) to obtain K . To this end, let us drill two vertical boreholes into the embankment at $x_1 = L/3$ and $x_2 = 2L/3$ and assess the groundwater level h there. We obtain r_1 and r_2 , respectively. Since we do not know e in (8) and since it is easier to measure infiltration rate e_{in} than evaporation rate e_{ev} , we measure e_{in} during rainfall and use $e = e_{\text{in}}$ in (8). We assume that e_{in} is measured accurately, that is, known exactly, but the values r_1 and r_2 are burdened with errors.

Let us define

$$\omega(e_{\text{in}}, K) = \sum_{i=1}^2 (r_i - h_{e_{\text{in}}, K}(x_i))^2, \quad \mu_1(K) = 2 - \sqrt{\frac{\omega(e_{\text{in}}, K)}{\omega(e_{\text{in}}, K_{\text{min}})}}, \quad (10)$$

where K_{min} is identified by the least squares method; see (4) where $h_{e_{\text{in}}, K}(x_i)$ plays the role of $\psi(a, x_i)$. As a consequence, K is fuzzified and a fuzzy interval $A = \{K \in \mathbb{R} \mid \mu_1(K) \in [0, 1]\}$, see (6), is considered for the saturated hydraulic conductivity.

We observe that $\hat{\mu}$ is a shifted “multiple” of μ_1 in the sense that each α -cut of the fuzzy interval determined by $\hat{\mu}$ is obtained as the $(h_0^2 - h_L^2)/(2L)$ multiple of A^α shifted by $-e_{\text{in}}L/2$; see (9). Consequently, there is no need to solve the minimization and maximization problems (3) to obtain α -cuts of the fuzzy quantity $q = \Phi(K)$ in this extremely simple example.

For $L = 10$, $h_0 = 4$, $h_L = 3$, $e_{\text{in}} = -3 \times 10^{-7}$, $r_1 = 4.41$, $r_2 = 4.09$, we obtain $\hat{\mu}$ as depicted in Figure 3 (left).

3.1.2. Evaporation

Let us pay attention to evaporation, a new quantity of interest. To evaluate the evaporation rate e_{ev} during a dry-weather period, we again assess h at x_1 and x_2 with the respective outputs \tilde{r}_1 and \tilde{r}_2 . Like in (10), we define

$$\tilde{\omega}(e_{\text{ev}}, K) = \sum_{i=1}^2 (\tilde{r}_i - h_{e_{\text{ev}}, K}(x_i))^2 \quad (11)$$

but, unlike (10), $e_{\text{ev}} \equiv e$ is not known. For each fixed K , an inverse problem can be solved, that is, the evaporation rate can be found that minimizes (11). However, since K is fuzzy, we have to consider $K \in A^\alpha$, where A^α are the α -cuts determined by μ_1 through r_i and e_{in} ; see (10). The model M_1 remains unchanged, but the model M_2 becomes the K -dependent inverse problem now.

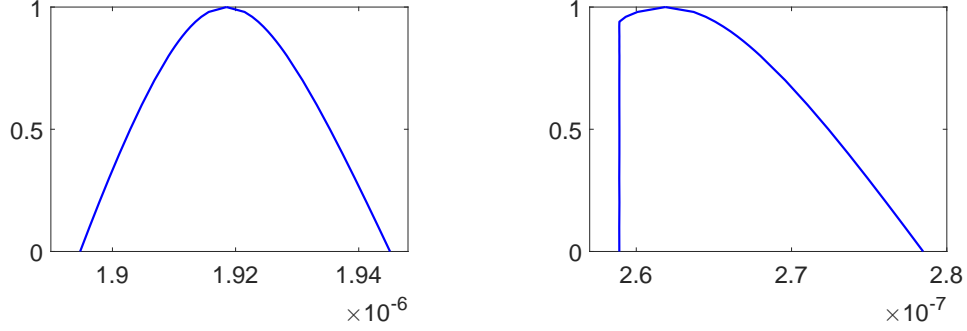


Figure 3: Left: The membership function $\hat{\mu}$ of q . Right: The membership function $\tilde{\mu}$ of e_{ev} . In both graphs, the vertical axis represents α and the horizontal axis represents the quantity of interest q and e_{ev} , respectively.

To get $R_{e_{ev}}^\alpha = [e_{ev,\min}^\alpha, e_{ev,\max}^\alpha]$, a parallel to (3), we solve

$$K_{ev,\min}^\alpha = \arg \min_{K \in A^\alpha} \min_{e_{ev} \in I_e} \tilde{\omega}(e_{ev}, K) \text{ and } K_{ev,\max}^\alpha = \arg \max_{K \in A^\alpha} \min_{e_{ev} \in I_e} \tilde{\omega}(e_{ev}, K), \quad (12)$$

where I_e is a chosen sufficiently large interval bounding the search. Then

$$e_{ev,\min}^\alpha = \arg \min_{e_{ev} \in I_e} \tilde{\omega}(e_{ev}, K_{ev,\min}^\alpha) \text{ and } e_{ev,\max}^\alpha = \arg \min_{e_{ev} \in I_e} \tilde{\omega}(e_{ev}, K_{ev,\max}^\alpha). \quad (13)$$

Since only a finite number of levels α is used in calculations, there is no need to solve (13) in practice. The values $e_{ev,\min}^\alpha$ and $e_{ev,\max}^\alpha$ are stored in the course of solving (12).

For $\tilde{r}_1 = 2.90$ and $\tilde{r}_2 = 2.60$ entering the calculations, the membership function $\tilde{\mu}$ of e_{ev} is depicted in Figure 3 (right).

The graph, which might seem strange at first glance, shows that e_{ev} is represented by a crisp value at the level $\alpha = 1$ because also the 1-cut of A is a singleton set comprising a unique K . If we start to increase the amount of uncertainty in K by decreasing α , we also decrease $e_{ev,\min}^\alpha$ as the solution of the min-min problem (12)-(13). For $\alpha < 0.94$, the condition $K \in A^\alpha$ is no longer an active constraint in the minimization of $\tilde{\omega}$ with respect to e_{ev} and the minimizer $e_{ev,\min}^\alpha$ is no longer dependent on α .

Problem (12) is, in fact, a sort of best- and worst-case scenario problems. Indeed, in the min-min problem, e_{ev} and K “cooperate” to minimize (11), whereas K is an “antagonist” of e_{ev} in the max-min problem (12) in which the minimizer of $\tilde{\omega}$ is sought under the worst conditions that K can produce.

4. Conclusions

The ideas presented in Section 1 are applicable to parameters belonging to other spaces than \mathbb{R} or \mathbb{R}^n . We can, for instance, take $S \subset C([d_1, d_2])$, where $C([d_1, d_2])$ stands for the space of continuous functions on an interval $[d_1, d_2]$, and consider a problem M_1 represented by, say, an ordinary differential equation (ODE) $D_a u = f$ supplemented by initial or boundary conditions, where D_a is an a -dependent differential operator, $a \in S$. Let us assume that inaccurate measurements $\{r_i\}_{i=1}^n$ are associated with $u_a(x_i)$, the ODE solution at $\{x_i\}_{i=1}^n$. Under some assumptions, a function $b_{\min} \in S$ can be identified by the least squares method as in (4). Consequently, the fuzzification of the identified parameter-function can be done as in Section 2.

If a scalar quantity of interest represents the output of an a -dependent Model 2, Zadeh's principle can again be applied to obtain the membership function associated with the quantity of interest. Besides a , Model 2 can depend on other parameters either crisp or fuzzy. In calculations, S is approximated by a set of functions controlled by a finite number of parameters. As a consequence, the approximate problem is formulated in terms of finite dimensional fuzzy sets and their α -cuts. Dealing with the latter can still be a rather hard task because A^α will enter the minimization (maximization) problem (3) as a constraint determined by (5) and the Model 1 output. Such constraint can be (and usually will be) non-linear.

The common concept of membership functions is sometimes awkward. Traditionally, the range of membership functions is limited to (subsets of) $[0, 1]$. This limits flexibility in the grading of fuzzy uncertainty. To make things easier, we can adopt the approach presented in [1] within the framework of info-gap decision theory and use membership functions in an "upside down" form where the amount of uncertainty is minimal at $\alpha = 0$ and increases with increasing α . In this approach, the upper bound of α is not limited to 1, but can be arbitrary large and can even increase in the course of computing. An example can be inferred from μ_1 in (5) as follows

$$\hat{\mu}_A(b) = c_1 \left(\left(\frac{\omega(b)}{\omega(b_{\min})} \right)^{c_2} - 1 \right)^{c_3},$$

where c_1 , c_2 , and c_3 are positive constants.

The α -cuts associated with such "upside down" membership functions are defined by $A^\alpha = \{a \in A \mid \mu_A(a) \leq \alpha\}$.

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