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NUMERICAL STUDY ON THE BLOW-UP RATE TO A QUASILINEAR PARABOLIC EQUATION *

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Abstract. In this paper, we consider the blow-up solutions for a quasilinear parabolic partial differential equation $u_t = u^2(u_{xx}+u)$. We numerically investigate the blow-up rates of these solutions by using a numerical method which is recently proposed by the authors [3].

Key words. blow-up rate, type II blow-up, numerical estimate, scale invariance, rescaling algorithm, curvature flow

AMS subject classifications. 35B44, 35K59, 65M99

1. Introduction. In this paper we consider the following quasilinear parabolic partial differential equation:

$$u_t = u^2(u_{xx} + u), \quad x \in (-a, a) \subset \mathbb{R}, t > 0. \tag{1.1}$$

This equation describes the motion of curves by their curvature ([2, 5, 6]). For this equation it was shown that there exist finite time blow-up solutions of so-called Type II under the following initial and boundary conditions

$$\begin{cases} u(t, \pm a) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in [-a, a] \end{cases} \tag{1.2}$$

where $a > \pi/2$ ([1, 8]). Here, we call Type I the blow-up solutions with the blow-up rate $(T-t)^{-1/2}$ which is determined by the spatially uniform blow-up solution of the equation (1.1). The non Type I blow-up solutions are called Type II. In [2] Anada & Ishiwata proved that there exists a solution which blows up in a finite time T with the blow-up rate

$$\left(\frac{1}{(T-t)} \log\log \frac{1}{(T-t)} \right)^{\frac{1}{2}}. \tag{1.3}$$

In [2], they posed several assumptions on the initial function u_0:
(K1) $u_0(x) > 0, \quad x \in (-a, a)$,
(K2) there exists $A > 0$ such that $(u_0(x))^2 + (u_{0x}(x))^2 < A^2, \quad x \in (-a, a)$,
(K3) $u_0(-x) = u_0(x)$,
(K4) $(u_{0x})(x) < 0, \quad x \in (0, a)$,
(K5) $Z\left(\frac{d}{dx}\left[u_0(u_{0x} + u_0)\right], (-a, a)\right) \le 3$.

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Here, $Z(f(x), I)$ is the zero number of f on the interval $I \subset \mathbb{R}$, namely, the number of zeros of f in the interval I and $u_{0x} = \frac{d}{dx}u_0$. To our best knowledge, it is known nothing about the blow-up rates of the Type II blow-up solutions whose initial data do not satisfy the assumptions (K1)–(K5).

There are many evolution equations which has the scaling invariance:

(*) there exist α, β such that if $u(t, \cdot)$ solves the equation then for any $\lambda > 0$, $u^\lambda(t, \cdot) = \lambda^\alpha u(\lambda^\beta t, \cdot)$ solves the same equation.

The equation (1.1) satisfies the scaling invariance (*) in the case of $\beta = 2\alpha$. Recently, we proposed a numerical method to estimate the blow-up rates of the blow-up solutions for evolution equations which possess this scaling invariance ([3]). In this paper, we numerically investigate the blow-up rates of the solutions of the problem (1.1) and (1.2). For this purpose, we adopt our numerical method to this problem.

The organization of the paper is as follows: in section 2, we explain our numerical method, in section 3 we exhibit several numerical examples, in the last section we will give a conclusion at this moment and several remarks.

2. Algorithm using the scaling invariance. We consider the problems which satisfy the following conditions:

- The solution u blows up in a finite time, say T, namely, there exists $T < \infty$ such that $\lim_{t \to T} \|u(t)\| = \infty$.
- The equation has a scaling invariance (*)

Here and hereafter, we use the notation $u(t)$ considering the solution u of the problem as a function from time interval to a normed space X with a norm $\| \cdot \|$. For example, the equation

$$\frac{du}{dt} = u^p \tag{2.1}$$

with a positive initial data satisfies the assumptions above. In fact, the equation possesses blow-up solutions and for a solution $u(t)$ of this equation and for any $\lambda > 0$ if we set

$$u^\lambda(t) = \lambda^\alpha u(\lambda^\beta t), \quad \alpha = \frac{\beta}{p-1}$$

then u^λ is also a solution of (2.1). For this problem $X = (\mathbb{R}, |\cdot|)$. As we already noted, our problem (1.1) also satisfies the assumptions above. For (1.1) we choose $X = L^\infty(-a, a)$.

2.1. Rescaling algorithm. We explain our proposed method. First, we fix constants $M > 0$ and λ $(0 < \lambda < 1)$. Second, by using the scaling invariance (*) we repeat to rescale the solution and construct $\{t_m\}$ and $\{\tau_m\}$ as follows:

$$t_0 = 0, \quad t_m = \min\{\, t \mid \ (\lambda)^{-\alpha(m-1)} M = \|u(t)\|\} \ (m = 1, 2, 3, \dots),$$

$$\lambda^{\beta(m-1)} \tau_m = (t_m - t_{m-1}). \tag{2.2}$$

Then we have

$$T = \sum_{k=1}^{\infty} (\lambda^\beta)^{k-1} \tau_k, \quad (T - t_m) = \lambda^{\beta(m-1)} \sum_{l=1}^{\infty} \lambda^{\beta l} \tau_{l+m},$$

$$(T-t_m)^{\frac{\alpha}{\beta}}\|u(t_m)\| = \left(\sum_{l=1}^{\infty}\lambda^{\beta l}\tau_{l+m}\right)^{\frac{\alpha}{\beta}} M =: (f(m))^{\frac{\alpha}{\beta}} M, \tag{2.3}$$

$$(T-t_m) = \lambda^{\beta(m-1)} f(m). \tag{2.4}$$

Using these equations we can prove a relation between the sequence $\{\tau_m\}$ and the blow-up rate of the solution (Theorem 2.1). Third, using appropriate numerical approximation for the problem, we numerically construct the sequence $\{\tau_m\}$ for finite ms and observe the behavior of it with respect to m. Such kind of numerical method is called rescaling algorithm which is originally proposed by Berger & Kohn [7] for the blow-up problem of Fujita type. At last, supposing the behavior of $\{\tau_m\}$ as $m \to \infty$ is same as the observed one we estimate the blow-up rate from the Theorem 2.1.

In [3], we examined the effectiveness of our method by applying our method to several examples where the blow-up rates of the solutions are theoretically known. For all of these examples, we could estimate the blow-up rates correctly, especially, we could estimate not only the blow-up rates of simple power type (Type I) but also the blow-up rates of complex forms with log or log log (Type II).

2.2. Relation between τ_m and the blow-up rate. We proved in [3] the following relation between the sequence $\{\tau_m\}$ and the blow-up rate:

THEOREM 2.1.

1. *if* $\tau_m = O(1)$ *then the blow-up rate is* $O((T-t_m)^{-\frac{\alpha}{\beta}})$,
2. *if* $\tau_m = Cm^k + o(m^k)$ *for some integer* k *then the blow-up rate is* $O((T-t_m)^{-\frac{\alpha}{\beta}}(\log(T-t_m)^{-1})^{k\frac{\alpha}{\beta}})$,
3. *if* $\tau_m = C\log m + o(\log m)$ *then the blow-up rate is* $O((T-t_m)^{-\frac{\alpha}{\beta}}(\log\log(T-t_m)^{-1})^{\frac{\alpha}{\beta}})$,
4. *if* $\tau_m = O(e^{km})$ *then the blow-up rate is* $O((T-t_m)^{r(\lambda)})$, $\quad r(\lambda) = \dfrac{\alpha\log\frac{1}{\lambda}}{k-\beta\log\frac{1}{\lambda}}$.

REMARK 2.1. *Although it is not complete classification for the behavior of* $\{\tau_m\}$, *it is still useful. We note that this relation holds for the exact solution of the problem and the numerically constructed* $\{\tau_m\}$ *inevitably contains errors. We need to observe the behavior of* $\{\tau_m\}$ *for finite* m.

3. Numerical experiments. Now we exhibit several results of numerical experiments where the initial condition does not satisfy at least one of (K1)–(K5). Here we note that we use the standard finite difference scheme for numerical solutions of (1.1) and we set the numerical parameters as $\lambda = 0.5$ and $M = 2$ for all examples below.

In the figures 3.1 and 3.2 we plot the results which breaks the condition (K4) and (K5). In these figures, the horizontal axis is the number of rescaling times m and the vertical axis is $\exp(\tau_m)/m^\alpha$ for several α. In the figure 3.1, we plot the case where $a = \pi$ and $u_0(x) = 0.5(\cos(x/2) + \sin^2 x)$ with $\alpha = 1.1, 1.4, 1.5, 1.6$. Here, we note that we do not need to determine the precise value of α for evaluating the behavior of τ_m. Indeed if we know that

$$\frac{\exp(\tau_m)}{m^\alpha} \sim 1$$

then we have

$$\tau_m \sim \log m^\alpha = \alpha\log m \sim \log m.$$

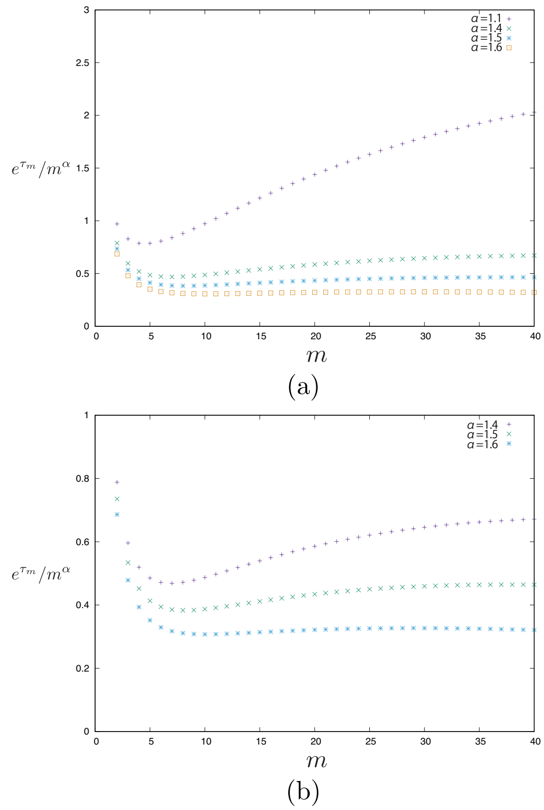


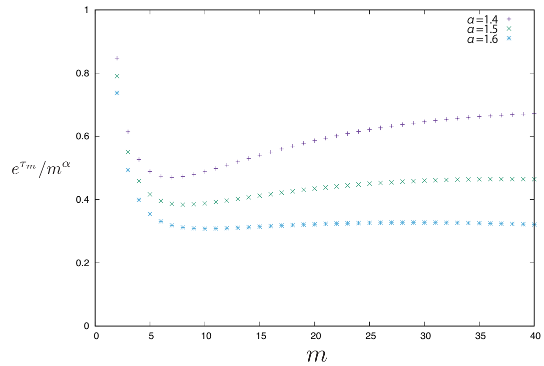
FIG. 3.1. *(a)* $\exp(\tau_m)/m^\alpha\,(\alpha = 1.1, 1.4, 1.5, 1.6)$*, (b) zoomed version of (a) for* $\alpha = 1.4, 1.5, 1.6$*.*

Here $A \sim B$ means $c_1 A \leq B \leq c_2 A$ for some $c_1, c_2 > 0$. Hence we can apply the third part of the Theorem 2.1 and we can conclude the blow-up rate is of $\log\log$ type. From the figure, we can see that in the case where $\alpha = 1.1$, $\exp(\tau_m)/m^\alpha$ increases and from the figure 3.2 (b) we can suppose that $\exp(\tau_m)/m^\alpha$ is bounded from both bellow and above for some value of α between 1.4 and 1.6, namely we can suppose $\tau_m = O(\log m)$.

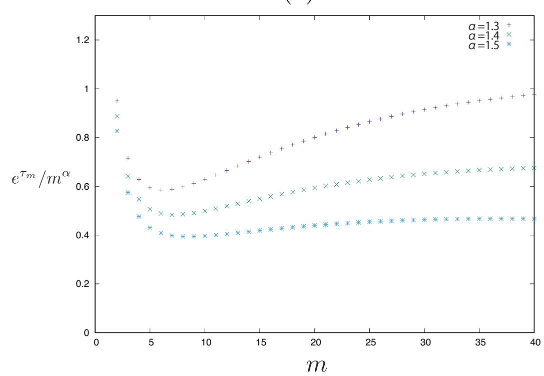
In the figures 3.2(a) and (b), we plot the case where $a = \pi, u_0(x) = 0.5(\cos(x/2) + \sin^2 2x)$, $a = \pi, u_0(x) = 0.5(\cos(x/2) + \sin^2 4x)$ and $a = 2\pi, u_0(x) = \sin\left(\pi\left(\frac{x}{2\pi}\right)^8\right) + 0.01\cos(x/4)$. The initial data (a) and (b) have much undulation than the previous one and initial data (c) has a peak near the boundary. From these figures, we can also observe that $\tau_m = O(\log m)$.

Thus, the blow-up rate of all these numerical solutions are estimated as (1.3).

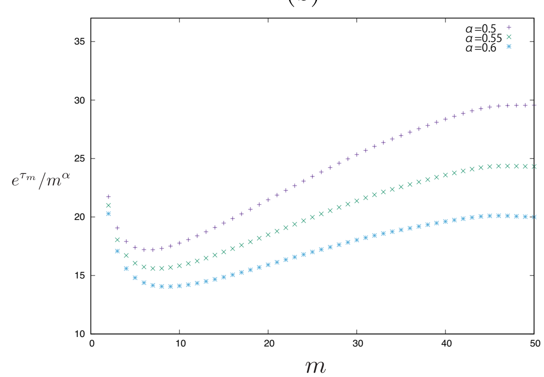
4. Conclusion and remarks. In this paper we numerically estimate the blow-up rate of type II solutions for the problem (1.1) and (1.2) as (1.3). Thus we conclude that the estimate (1.3) may be valid for wider class of initial data. On the other hand it is still open whether the other blow-up rates of type II blow-up solutions exist or not. Moreover, there are no information on the blow-up rate of type II blow-up solutions to $u_t = u^\delta(u_{xx} + u)$ for the case $\delta > 2$ and the higher dimensional problem: $u_t = u^\delta(\Delta u + u)$ with $\delta \geq 2$. These are challenging issues.



(a)



(b)



(c)

FIG. 3.2. *We plot m vs. $\exp(\tau_m)/m^\alpha$. (a) $a = \pi, u_0(x) = 0.5(\cos(x/2) + \sin^2 2x)$ ($\alpha = 1.4, 1.5, 1.6$), (b) $a = \pi, u_0(x) = 0.5(\cos(x/2) + \sin^2 4x)$ ($\alpha = 1.3, 1.4, 1.5$). (c) $a = 2\pi, u_0(x) = \sin\left(\pi\left(\frac{x}{2\pi}\right)^8\right) + 0.01\cos(x/4)$ ($\alpha = 0.5, 0.55, 0.6$).*

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