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A NOTE ON THE UNIQUENESS AND STRUCTURE OF SOLUTIONS TO THE DIRICHLET PROBLEM FOR SOME ELLIPTIC SYSTEMS*

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Abstract. In this note, we consider some elliptic systems on a smooth domain of R^n. By using the maximum principle, we can get a more general and complete results of the identical property of positive solution pair, and thus classify the structure of all positive solutions depending on the nonlinearities easily.

Key words. elliptic system, uniqueness, solutions structure

AMS subject classifications. 35J57, 35B09, 35J91

1. Introduction. In this paper, we consider the smooth positive solutions of the following elliptic system

$$\begin{cases} \Delta u + \phi(x) u^p v^q = 0 \\ \Delta v + \phi(x) u^q v^p = 0 \end{cases} \tag{1.1}$$

in Ω with boundary condition

$$(u, v) = (0, 0) \;\text{ on }\; \partial\Omega, \tag{1.2}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, n \geq 3, p > 0, q > 0, \Omega \subset \mathbf{R}^n$ is a domain with the maximum principle holds true, and ϕ is a positive and continuous function in Ω. If $n = 3, \phi \equiv 1$ and $(p, q) = (2, 3)$, system (1.1) arises from the stationary Schrödinger system with critical exponent for Bose-Einstein condensate. We refer the readers to [5], [8], [11], [12], and the references therein. If $\phi = \frac{1}{1+|x|^2}$ then system (1.1) is called a Matukuma-type system. Recently, if $\phi \equiv 1, 1 \leq p, q \leq \frac{n+2}{n-2}$ and $p + q = \frac{n+2}{n-2}$, i.e., the critical exponent case, Li-Ma[10] used the Hardy-Littlewood-Sobolev inequality to prove that any $L^{\frac{2n}{n-2}}(R^n) \times L^{\frac{2n}{n-2}}(R^n)$ positive solution (u, v) to system (1.1) is radial symmetric. Furthermore, they also showed that any $L^{\frac{2n}{n-2}}(R^n) \times L^{\frac{2n}{n-2}}(R^n)$ radial symmetric solution (u, v) is unique and $u \equiv v$. In this note, we consider the general case, $p > 0, q > 0$, and, by using the maximum principle to get a more general and complete result.

Our first theorem is the following.

Theorem 1.1. *Let $\phi > 0$ in Ω, and $p, q > 0$. Then if $q \geq p$ then any positive smooth solution (u, v) of (1.1)-(1.2) satisfies $u \equiv v$.*

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By Theorem 1.1 we easily obtain the following results about the symmetry, existence and uniqueness results.

COROLLARY 1.2. *Let $\phi \equiv 1$. Then the following properties are valid.*

(i) *If $0 < p \le q \le \frac{n+2}{n-2}, 1 \le p+q < \frac{n+2}{n-2}$ and Ω is bounded, then (1.1)-(1.2) possesses one and only one positive solution (u, v) and $u \equiv v$ in Ω. In addition, if Ω is symmetric with respect to some $x_i -$ axis, then (u, v) is symmetric with respect to $x_i -$ axis.*

(ii) *If $0 < p \le q,\ p+q = \frac{n+2}{n-2}$ and $\Omega = \mathbf{R}^n$, then every positive solution (u, v) of (1.1) satisfies $u \equiv v$, and it is radial symmetric with one parameter family of functions*

$$\phi_\lambda(x) = \Big(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x - x_0|^2}\Big)^{\frac{n-2}{2}},$$

where $\lambda > 0$ is a parameter and $x_0 \in \mathbf{R}^n$.

(iii) *If $p+q \ge \frac{n+2}{n-2}$ and $\Omega \ne \mathbf{R}^n$ is a star-shape domain, then (1.1)-(1.2) does not have any positive solution.*

Remark 1.3

(A) If $q < p$ and $\phi \equiv 1$, then equations (1.1)-(1.2) may possess infinite many positive solutions. The details can be found in [2]. For example, if $q = p-1,\ 1 < 2p-1 < \frac{n+2}{n-2}$ and Ω is bounded, then for any $\lambda > 0$, $(\lambda v, v)$ is a positive solution of equations (1.1)-(1.2), where v is the positive solutionof the following equation

$$\left\{ \begin{array}{l} \Delta u + \lambda^{p-1} u^{2p-1} = 0 \ \text{ in } \ \Omega \\ v = 0 \ \text{ on } \partial\Omega. \end{array} \right. \tag{1.3}$$

(B) We note that Corollary 1.2-(ii) was proved by Li-Ma [10] if $1 \le p < q \le \frac{n+2}{n-2},\ p+q = \frac{n+2}{n-2}$ and (u, v) is in $L^{\frac{2n}{n-2}}(R^n) \times L^{\frac{2n}{n-2}}(R^n)$. In this case, system (1.1) will reduce to one equation. Then, if Ω is a ball or R^n, and under the respective condition in the parts (i) and (ii) of Corollary 1.2, by using the method of moving plane, we can finally get that all positive solutions of (1.1)-(1.2) are radially symmetry. By the way, from the Pohozaev identity, we can also get the non-existence result of part (iii) in Corollary 1.2.

(C) By using the same ideas and proofs, some classes of systems, e.g., Schrödinger-type system, Matukuma-type system, etc, have also their respective results of Theorem 1.1 and Corollary 1.2. The details can be found in [2]. For example, let $\phi(x) = \frac{1}{1+|x|^2}$, we can also consider the following Matukuma-type system

$$\left\{ \begin{array}{l} \Delta u + \frac{1}{1+|x|^2} u^p v^q = 0 \ \text{ in } \ R^3 \\ \Delta v + \frac{1}{1+|x|^2} u^q v^p = 0 \ \text{ in } \ R^3. \end{array} \right. \tag{1.4}$$

Then, from Theorem 1.1 and by using Theorems 1-2 in [13], we easily obtain the following results

THEOREM 1.3. *Suppose $0 < p \le q$ and $1 < p + q < 5$. Then the following statements are valid.*

(i) *Every positive entire solution (u, v) of equation (1.4) satisfies $u \equiv v$ in R^3, and it is radially symmetric about the origin with $u'(r) < 0\ \forall r > 0$.*

(ii) *Let* $TM(u) = \frac{1}{4\pi}\int_{R^3} \frac{u^p}{1+|x|^2}\, dx$ *be the total mass of* u*. Then equation (1.4) has an unique positive entire solution with finite total mass, and has infinitely many positive entire solutions with infinite total mass.*

In Section 2, based on the maximum principle, we can get the proof of Theorem 1.1. Applying Theorem 1.1 and by using the well-known results of Yamabe problem, we also easily get Corollary 1.2.

2. Identical Property and Proof of Main Results. We prove Theorem 1.1 and Corollary 1.2 in this section.

Proof of Theorem 1.1. Let (u, v) be a positive solution of (1.1)-(1.2), and let $w = u - v$. Then, by (1.1), w satisfies

$$\Delta w = \phi(x)(-u^p v^q + u^q v^p) \text{ in } \Omega, \; w = 0 \text{ on } \partial\Omega. \tag{2.1}$$

We divide the proof into the following steps.

Step 1. If $q > p > 0$ then we want to show $w \equiv 0$, i.e., $u \equiv v$, in Ω.

First, we prove $v(x) \geq u(x)\ \forall x \in \Omega$. Suppose not, then there exists some $x_0 \in \Omega$ such that $w(x_0) = u(x_0) - v(x_0) > 0$. Then there exists some $x_1 \in \text{int}(\Omega)$ such that

$$w(x_1) = \max_{x\in\Omega} w(x) > 0 \text{ and } \Delta w(x_1) \leq 0. \tag{2.2}$$

By $\phi > 0$ and (2.1)-(2.2), we easily obtain

$$0 \geq \Delta w(x_1) = -\phi(x_1)(u^p(x_1)v^q(x_1)(1 - (\frac{u(x_1)}{v(x_1)})^{q-p})) > 0.$$

This contradiction shows $v(x) \geq u(x)\ \forall x \in \Omega$.

Now, suppose $v \not\equiv u$ in Ω. Then by (1.1)-(1.2), we easily deduce that

$$0 = \int_\Omega (v\Delta u - u\Delta v)dx = \int_\Omega -\phi(x)u^p v^{q+1}(1 - (\frac{u}{v})^{q+1-p})dx < 0.$$

This contradiction proves $u \equiv v$ if $q > p > 0$.

Step 2. If $p = q > 0$. then by (2.1) we easily obtain

$$\Delta w = \phi(x)(u^p v^p - u^p v^p) = 0 \text{ in } \Omega \text{ and } w = 0 \text{ on } \partial\Omega.$$

This shows $u \equiv v$ in Ω.

By **Steps 1 and 2** we complete the proof of Theorem 1.1. q.e.d.

Now we are in a position to prove Corollary 1.2.

Proof of Corollary 1.2. Let $\phi \equiv 1$ and (u, v) be a positive solution of (1.1). By our main result, Theorem 1.1, we obtain that $u \equiv v$, and then system (1.1) reduces to the following one equation

$$\begin{aligned} &\Delta u + u^{p+q} = 0 \text{ in } \Omega \\ &u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.3}$$

Then by the well-known Yamabe problem and the prescribing scalar curvature problem, e.g., see [6], [1], [4], [3], [9] and the references therein, we easily obtain the results (i)-(iii) of Corollary 1.2. We complete the proof. q.e.d

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