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GAUSSIAN CURVATURE BASED TANGENTIAL REDISTRIBUTION OF POINTS ON EVOLVING SURFACES*

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Abstract. There exist two main methods for computing a surface evolution, level-set method and Lagrangian method. Redistribution of points is a crucial element in a Lagrangian approach. In this paper we present a point redistribution that compress quads in the areas with a high Gaussian curvature. Numerical method is presented for a mean curvature flow of a surface approximated by quads.

Key words. surface evolution, point redistribution, finite volume method, mean curvature flow

AMS subject classifications. 53C44, 65M08, 65M50

1. Introduction. An important part in computing a surface evolution is a redistribution of points on a surface. An improper distribution of points could lead to an unstable numerical method.

There are several papers dedicated to this problem. One of the approaches is using the so-called Laplacian smoothing method [1]. A method that we build on is controlling the so-called local area density [4]. In a paper [5] a method for a redistribution of points on a curve by a curvature was presented. We generalize this method for evolving surfaces. In a case of surfaces we redistribute points by Gaussian curvature, since saddle point can have a zero mean curvature but it has non-zero Gaussian curvature.

2. Surface evolution models. Let us have an open parametric surface $E = \{\mathbf{x}(t,u,v) | t \in [0,T),\ (u,v) \in \Omega = [0,1]\times[0,1]\}$ evolving in a time t by the following partial differential equation

$$\frac{\partial \mathbf{x}}{\partial t}(t,u,v) = \beta(t,u,v)\mathbf{N}(t,u,v) + \mathbf{V_T}(t,u,v),\ t \in (0,T),\ (u,v) \in \Omega \setminus \partial\Omega, \tag{2.1}$$

where $\mathbf{N}(t,u,v)$ is a unit normal vector and $T > 0$. The vector $\mathbf{V_T}$ represents the evolution in a tangential direction along the surface. An evolution in the tangential direction does not change the image of the surface. In a quad (quadrilateral) approximation of a surface it only change the size and the shape of the quads.

In our numerical experiments we are focusing only on two special cases

$$\beta(t,u,v)\mathbf{N}(t,u,v) = \Delta_{\mathbf{x}}\mathbf{x}(t,u,v), \tag{2.2}$$

$$\beta(t,u,v)\mathbf{N}(t,u,v) = \Delta_{\mathbf{x}}\mathbf{x}(t,u,v) + \mathbf{N}(t,u,v), \tag{2.3}$$

where $\Delta_{\mathbf{x}}\mathbf{x}(t,u,v)$ is a Laplace-Beltrami operator applied on the position vector of the parametrized surface E. This is known to be the mean curvature vector of the

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surface. We want to emphasize that $\Delta_{\mathbf{x}}\mathbf{x}$ is normal to the surface E and does not depend on the parametrization $\mathbf{x}$. It depends only on the shape of E.

We want to have a surface evolution that does not change the boundary curve but can change the distribution of points along the boundary. For this reason we have the following boundary conditions

$$\frac{\partial \mathbf{x}}{\partial t}(t,u,v) = \mathbf{V_T}(t,u,v) \quad t \in (0,T),\ (u,v) \in \partial\Omega, \tag{2.4}$$

where the vector $\mathbf{V_T}$ lies in a tangent direction of a boundary curve. Let us also have the initial condition

$$\mathbf{x}(0,u,v) = \mathbf{x}_0(u,v),\ (u,v) \in \Omega \setminus \partial\Omega. \tag{2.5}$$

3. The tangential redistribution. The variable that we want to control by the tangential redistribution is the local area density

$$g(t,u,v) = ||\partial_u \mathbf{x}(t,u,v) \times \partial_v \mathbf{x}(t,u,v)||. \tag{3.1}$$

It can be understood as the area of the parallelogram with sides $\partial_u \mathbf{x}(t,u,v)$ and $\partial_v \mathbf{x}(t,u,v)$. In a quad approximation of a surface the area density g is proportional to the area of the quads.

In the rest of this section we derive a formula for $\mathbf{V_T}$ in the equation (2.1) that provide us a desired area density.

3.1. Change of the area density in time. For the derivative of the area density it applies [4]

$$\partial_t g = g\Delta_{\mathbf{x}}\mathbf{x} \cdot \beta\mathbf{N} + g\nabla_{\mathbf{x}} \cdot \mathbf{V_T}. \tag{3.2}$$

If we want the area density to converge to a prescribed local area density $c(t,u,v)$, one of the possibilities is to find an area density that satisfies the following ODE

$$\partial_t \left(\frac{g}{A}\right) = \left(\frac{c}{A} - \frac{g}{A}\right)\omega, \tag{3.3}$$

where A is the area of the surface and ω is a parameter controlling the rate at which g converges to c.

By rearranging the equation (3.3) and by substituting the equation (3.2) into it we get

$$\nabla_{\mathbf{x}} \cdot \mathbf{V_T} = \Delta_{\mathbf{x}}\mathbf{x} \cdot \beta\mathbf{N} - \frac{1}{A}\iint_E \Delta_{\mathbf{x}}\mathbf{x} \cdot \beta\mathbf{N}\, \mathrm{d}\mathbf{x} + \left(\frac{c}{g} - 1\right)\omega. \tag{3.4}$$

The equation (3.4) does not have a unique solution. By taking a vector field $\mathbf{V_T}$ that is a gradient of some potential φ and a Neumann boundary condition we obtain a PDR that has an infinity many solutions that differs only by a constant. By giving a Dirichlet boundary condition in one arbitrary point we ensure uniqueness of the solution. Since we are only interested in the gradient of φ it does not matter in which point we prescribe the Dirichlet BC. The equation for the potential with the boundary conditions is

$$\begin{aligned} \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}}\varphi(\cdot,u,v) &= \Delta_{\mathbf{x}}\varphi(\cdot,u,v) = \\ &\Delta_{\mathbf{x}}\mathbf{x} \cdot \beta\mathbf{N} - \frac{1}{A}\iint_E \Delta_{\mathbf{x}}\mathbf{x} \cdot \beta\mathbf{N}\, \mathrm{d}\mathbf{x} + \left(\frac{c}{g} - 1\right)\omega, \quad (u,v) \in \Omega \setminus \partial\Omega, \end{aligned} \tag{3.5}$$

$$\nabla_{\mathbf{x}}\varphi(\cdot,u,v)\cdot\, \mathbf{n}\ (\cdot,u,v) = 0, \quad (u,v) \in \partial\Omega \setminus \{(0,0)\}, \tag{3.6}$$

$$\varphi(\cdot,u,v) = 0, \quad (u,v) = (0,0). \tag{3.7}$$

A Neumann boundary condition provides a tangential vector field which has a zero projection to the normal of the boundary. This ensures that points on the boundary are moving only in the direction of the tangential vector of the boundary curve.

Then equations (2.1)-(2.4) acquires the form

$$\frac{\partial \mathbf{x}}{\partial t}(t,u,v) = \Delta_{\mathbf{x}}\mathbf{x}(t,u,v) + \nabla_{\mathbf{x}}\varphi(t,u,v), \quad t \in (0,T),\ (u,v) \in \Omega \setminus \partial\Omega \tag{3.8}$$

$$\frac{\partial \mathbf{x}}{\partial t}(t,u,v) = \nabla_{\mathbf{x}}\varphi(t,u,v), \quad t \in (0,T),\ (u,v) \in \partial\Omega \tag{3.9}$$

4. Choice of the function c. The choice of the function c is crucial for the distribution of points on the surface. An appropriate choice of c can provide a quad approximation of the surface with large quads in the areas with a small Gaussian curvature $G(t,u,v)$ and vice versa. There are two properties that the function c has to satisfy,

$$c(t,u,v) > 0, \qquad \iint_\Omega c(t,u,v)\ \mathrm{dudv} = A. \tag{4.1}$$

The first property has to be satisfied since the size of the quad cannot be negative. Numerical interpretation of the second property is that the sum of the quad sizes has to be equal to the area of the surface.

If we choose c to be inverse proportional to the Gaussian curvature, we obtain a surface approximation with smaller quads in areas of high Gaussian curvature. There are multiple options for how to choose this dependence. First let us define an auxiliary function $\hat{c}$ that has the form

$$\hat{c}(t,u,v) = \Big(p\min\Big(|G(t,u,v)|/\widetilde{G}, 1\Big) + 1\Big)^{-1}, \tag{4.2}$$

where $\widetilde{G}$ is a chosen value that is restricting the maximal value of a function nad p is the chosen parameter. Then the function c is the function $\hat{c}$ normalized

$$c(t,u,v) = A\frac{\hat{c}(t,u,v)}{\iint_\Omega \hat{c}(t,u,v)\ \mathrm{dudv}}. \tag{4.3}$$

This normalization ensures that the second property (4.1) is fulfilled.

5. Surface and PDEs approximation. Let us divide the surface E in the t-th time step into quadrilaterals. Let us denote the vertices of the quads of the surface $\mathbf{x}_{t,i},\ i \in \{1,...,N\}$. Let us have a Q_i quads that has a vertex $\mathbf{x}_{t,i}$. Then let us denote 4 vertices of the q-th quad of $\mathbf{x}_{t,i}$ by $\mathbf{x}_{t,i}^{q,j}$, $j \in \{0,1,2,3\}$, $q \in Q_i$. The vertex $\mathbf{x}_{t,i}^{q,0} = \mathbf{x}_{t,i}$ and other vertices are numbered in an anticlockwise direction. For the vertex $\mathbf{x}_{t,i}^{q,3}$ holds $\mathbf{x}_{t,i}^{q,3} = \mathbf{x}_{t,i}^{q+1,1}$, where $q+1$ is as a $\mathrm{mod}(q+1, Q_i)$. Let us have a function $k(i,q,j)$ that takes the local indexes of a vertex and return its global index. For a better understanding see Fig. 7.1.

Let us interpolate values of $\mathbf{x}$ on quads using a bilinear interpolation

$$\mathbf{x}_{t,i}^{q}(\phi,\rho) = (1-\phi)(1-\rho)\mathbf{x}_{t,i}^{q,0} + \phi(1-\rho)\mathbf{x}_{t,i}^{q,1} + (1-\phi)\rho\mathbf{x}_{t,i}^{q,3} + \phi\rho\mathbf{x}_{i}^{q,2}. \tag{5.1}$$

Every function defined on the surface E is also approximated using bilinear interpolation

$$f_{t,i}^{q}(\phi,\rho) = (1-\phi)(1-\rho)f_{t,i}^{q,0} + \phi(1-\rho)f_{t,i}^{q,1} + (1-\phi)\rho f_{t,i}^{q,3} + \phi\rho f_{t,i}^{q,2}, \tag{5.2}$$

where $f_{t,i}^{q,j}$ is the value of the function f in the vertex $\mathbf{x}_{t,i}^{q,j}$.

Let us have a finite volume $V_{t,i}$ composed of quads defined by the centers of the original quads and the centers of their edges. Let us denote the edges of $V_{t,i}$ on the q-th quad by

$$e_{t,i}^{q,1} = \{\mathbf{x}_{t,i}^{q}(1/2,\rho); \rho \in (0,1/2)\}, \quad e_{t,i}^{q,3} = \{\mathbf{x}_{t,i}^{q}(\phi,1/2); \phi \in (0,1/2)\} \tag{5.3}$$

At last let us integrate the equation (3.5) over the finite volume

$$\begin{aligned} \iint_{V_{t,i}} \Delta_{\mathbf{x}}\varphi \, \mathrm{d}\mathbf{x} = & \iint_{V_{t,i}} \Delta_{\mathbf{x}}\mathbf{x} \cdot \beta\mathbf{N} \, \mathrm{d}\mathbf{x} \\ & - \iint_{V_{t,i}} \frac{1}{A} \iint_{E} \Delta_{\mathbf{x}}\mathbf{x} \cdot \beta\mathbf{N} \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{x} + \iint_{V_{t,i}} \left(\frac{c}{g} - 1\right) \omega \, \mathrm{d}\mathbf{x} \end{aligned} \tag{5.4}$$

and also the equation (3.8)

$$\iint_{V_{t,i}} \frac{\partial \mathbf{x}}{\partial t} \, \mathrm{d}\mathbf{x} = \iint_{V_{t,i}} \Delta_{\mathbf{x}}\mathbf{x} \, \mathrm{d}\mathbf{x} + \iint_{V_{t,i}} \nabla_{\mathbf{x}}\varphi \, \mathrm{d}\mathbf{x} \tag{5.5}$$

For the boundary condition (3.9) it holds that the derivative of a potential φ on the right side in the normal direction is zero. That means yhe direction of the gradient of φ is the tangential direction to the boundary curve. For this reason we have 1D finite volumes on the boundary. They are defined by the points $\left(\mathbf{x}_{t,i}^{1,0} + \mathbf{x}_{t,i}\right)/2$, $\mathbf{x}_{t,i}$, $\left(\mathbf{x}_{t,i}^{Q_i,3} + \mathbf{x}_{t,i}\right)/2$ and after integrating (3.9) on this finite volume we get

$$\int_{V_{t,i}} \frac{\partial \mathbf{x}}{\partial t} \, \mathrm{d}\mathbf{x} = \int_{V_{t,i}} \nabla_{\mathbf{x}}\varphi \, \mathrm{d}\mathbf{x}. \tag{5.6}$$

The integral equations (5.4)-(5.6) form a basis for the finite volume method which leads for (5.5)-(5.6) to a system of equations in a matrix form

$$T_t^{+}\mathbf{X}_{t+1} + T_t^{-}\mathbf{X}_t = B_t\mathbf{X}_{t+1} + A_t\mathbf{X}_{t+1}, \tag{5.7}$$

$$\mathbf{X}_{t+1} = [\mathbf{x}_{t+1,1}, \mathbf{x}_{t+1,2}, \ldots, \mathbf{x}_{t+1,N}]^T, \quad \mathbf{X}_t = [\mathbf{x}_{t,1}, \mathbf{x}_{t,2}, \ldots, \mathbf{x}_{t,N}]^T \quad . \tag{5.8}$$

The matrices T_{t+1}^{+}, T_t^{-} are related to the time derivative, the matrix B_t is related to the evolution in the normal direction and the matrix A_t is related to the evolution in the tangential direction.

For the equation (5.4), it leads to a system of equations

$$D_t\Phi_t = \mathbf{b}_t, \quad \Phi_t = [\varphi_{t,1}, \varphi_{t,2}, \ldots, \varphi_{t,N}]^T . \tag{5.9}$$

The matrix D_t is related to the Laplace-Beltrami operator and $\mathbf{b}_t$ is related to the right hand side of the equation (5.4).

6. The computational algorithm. The algorithm to numerically solve the equations (2.1)-(2.4) (or (3.8)-(3.9)) is as follows.

Let us have a known initial condition $\mathbf{X}_0$ and a number of time steps M.

For$(t = 0; t < M; t++)$

- compute the matrices T_t^{+}, T_t^{-}, B_t

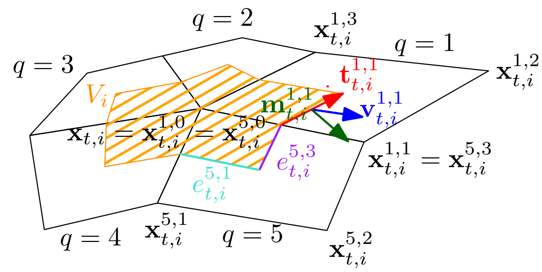

FIG. 7.1. *A sketch of the finite volume V_i composed of five quads. Local notation for quad number 1 and 5 are labeled. For the quad 1, vectors $\mathbf{m}_{t,i}^{1,1}$, $\mathbf{t}_{t,i}^{1,1}$, $\mathbf{v}_{t,i}^{1,1}$ are labeled. For the quad 5, edges $e_{t,i}^{5,1}$, $e_{t,i}^{5,3}$ are labeled.*

- use these matrices to explicitly compute $(\beta\mathbf{N})_{t,i}$ that is used in $\mathbf{b}_t$
- compute the matrix D_t and $\mathbf{b}_t$
- find Φ_t by solving $D_t\Phi_t = \mathbf{b}_t$
- use Φ_t to compute the matrix A_t
- find $\mathbf{X}_{t+1}$ by solving (5.7)

7. The finite volume method. In this section we present the coefficients of the matrices from the previous section derived by the finite volume method. A detailed derivation of the coefficients can be found in a forthcoming paper [2].

7.1. The approximation of the time derivative. Let us assume a constant time derivative on the finite volume and let us approximate the time derivative by a finite difference. Then the first integral in the equation (5.5) becomes

$$m(V_{t,i})\frac{\mathbf{x}_{t+1,i} - \mathbf{x}_{t,i}}{\tau}, \tag{7.1}$$

where

$$\begin{aligned} m(V_{t,i}) = \sum_{q=1}^{Q_i} &\left\| \frac{\mathbf{x}_{t,i}^{q,1} - \mathbf{x}_{t,i}^{q,0}}{2} \times \left(\frac{\mathbf{x}_{t,i}^{q,0} + \mathbf{x}_{t,i}^{q,1} + \mathbf{x}_{t,i}^{q,2} + \mathbf{x}_{t,i}^{q,3}}{4} - \mathbf{x}_{t,i}^{q,0} \right) \right\| \Big/ 2 \\ &+ \left\| \frac{\mathbf{x}_{t,i}^{q,3} - \mathbf{x}_{t,i}^{q,0}}{2} \times \left(\frac{\mathbf{x}_{t,i}^{q,0} + \mathbf{x}_{t,i}^{q,1} + \mathbf{x}_{t,i}^{q,2} + \mathbf{x}_{t,i}^{q,3}}{4} - \mathbf{x}_{t,i}^{q,0} \right) \right\| \Big/ 2 \end{aligned} \tag{7.2}$$

and τ is the time step. Then the only non-zero coefficients of the matrices T_t^+ and T_t^- are the diagonal coefficients

$$T_{t,i,i}^+ = m(V_{t,i})/\tau, \quad T_{t,i,i}^- = -m(V_{t,i})/\tau. \tag{7.3}$$

7.2. The finite volume approximation of the Laplace-Beltrami operator. We are applying Laplace-Beltrami operator to the vector function $\mathbf{x}(t,u,v)$ and to a scalar function $\varphi(t,u,v)$. Using a bilinear approximation we can derive the following vectors on the edges $e_{t,i}^{q,1}$, $e_{t,i}^{q,3}$ (see Fig. 7.1)

$$\begin{aligned} \mathbf{t}_{t,i}^{q,1} &= -\frac{1}{2}\mathbf{x}_{t,i}^{q,0} - \frac{1}{2}\mathbf{x}_{t,i}^{q,1} + \frac{1}{2}\mathbf{x}_{t,i}^{q,3} + \frac{1}{2}\mathbf{x}_{t,i}^{q,2}, \\ \mathbf{t}_{t,i}^{q,3} &= -\frac{1}{2}\mathbf{x}_{t,i}^{q,0} + \frac{1}{2}\mathbf{x}_{t,i}^{q,1} - \frac{1}{2}\mathbf{x}_{t,i}^{q,3} + \frac{1}{2}\mathbf{x}_{t,i}^{q,2}. \end{aligned} \tag{7.4}$$

$$\mathbf{v}_{t,i}^{q,1} = -\frac{3}{4}\mathbf{x}_{t,i}^{q,0} + \frac{3}{4}\mathbf{x}_{t,i}^{q,1} - \frac{1}{4}\mathbf{x}_{t,i}^{q,3} + \frac{1}{4}\mathbf{x}_{t,i}^{q,2},$$
$$\mathbf{v}_{t,i}^{q,3} = -\frac{3}{4}\mathbf{x}_{t,i}^{q,0} - \frac{1}{4}\mathbf{x}_{t,i}^{q,1} + \frac{3}{4}\mathbf{x}_{t,i}^{q,3} + \frac{1}{4}\mathbf{x}_{t,i}^{q,2}, \tag{7.5}$$

$$\mathbf{m}_{t,i}^{q,1} = \mathbf{v}_{t,i}^{q,1} - \frac{\mathbf{v}_{t,i}^{q,1}\cdot\mathbf{t}_{t,i}^{q,1}}{\mathbf{t}_{t,i}^{q,1}\cdot\mathbf{t}_{t,i}^{q,1}}\mathbf{t}_{t,i}^{q,1}, \quad \mathbf{m}_{t,i}^{q,3} = \mathbf{v}_{t,i}^{q,3} - \frac{\mathbf{v}_{t,i}^{q,3}\cdot\mathbf{t}_{t,i}^{q,3}}{\mathbf{t}_{t,i}^{q,3}\cdot\mathbf{t}_{t,i}^{q,3}}\mathbf{t}_{t,i}^{q,3}. \tag{7.6}$$

If $f_{t,i}^{q,j}$ is one of the coordinates of $\mathbf{x}_{t,i}^{q,j}$ then the q-th quad contributes to the coefficients $B_{t,i,k(i,q,j)}$ by the values

$$B_{t,i,k(i,q,0)} \mathrel{+}= \frac{m(e_{t,i}^{q,1})}{||\mathbf{m}_{t,i}^{q,1}||}\left(-\frac{3}{4}+\frac{1}{2}a_i^{q,1}\right) + \frac{m(e_{t,i}^{q,3})}{||\mathbf{m}_{t,i}^{q,3}||}\left(-\frac{3}{4}+\frac{1}{2}a_i^{q,3}\right),$$
$$B_{t,i,k(i,q,1)} \mathrel{+}= \frac{m(e_{t,i}^{q,1})}{||\mathbf{m}_{t,i}^{q,1}||}\left(\frac{3}{4}+\frac{1}{2}a_i^{q,1}\right) + \frac{m(e_{t,i}^{q,3})}{||\mathbf{m}_{t,i}^{q,3}||}\left(-\frac{1}{4}-\frac{1}{2}a_i^{q,3}\right),$$
$$B_{t,i,k(i,q,2)} \mathrel{+}= \frac{m(e_{t,i}^{q,1})}{||\mathbf{m}_{t,i}^{q,1}||}\left(\frac{1}{4}-\frac{1}{2}a_i^{q,1}\right) + \frac{m(e_{t,i}^{q,3})}{||\mathbf{m}_{t,i}^{q,3}||}\left(\frac{1}{4}-\frac{1}{2}a_i^{q,3}\right),$$
$$B_{t,i,k(i,q,3)} \mathrel{+}= \frac{m(e_{t,i}^{q,1})}{||\mathbf{m}_{t,i}^{q,1}||}\left(-\frac{1}{4}-\frac{1}{2}a_i^{q,1}\right) + \frac{m(e_{t,i}^{q,3})}{||\mathbf{m}_{t,i}^{q,3}||}\left(\frac{3}{4}+\frac{1}{2}a_i^{q,3}\right). \tag{7.7}$$

where

$$a_{t,i}^{q,1} = \frac{\mathbf{v}_{t,i}^{q,1}\cdot\mathbf{t}_{t,i}^{q,1}}{\mathbf{t}_{t,i}^{q,1}\cdot\mathbf{t}_{t,i}^{q,1}}, \quad a_{t,i}^{q,3} = \frac{\mathbf{v}_{t,i}^{q,3}\cdot\mathbf{t}_{t,i}^{q,3}}{\mathbf{t}_{t,i}^{q,3}\cdot\mathbf{t}_{t,i}^{q,3}}. \tag{7.8}$$

If $f_{t,i}^{q,j} = \varphi_{t,i}^{q,j}$ then the q-th quad contributes to the coefficients $\Phi_{t,i,k(i,q,j)}$ by the same values.

7.3. The approximation of the right hand side in the equation (5.4). Let us approximate the first integral in the equation (5.4) by assuming $\beta\mathbf{N}$ and $\left(\frac{c}{g}-1\right)\omega$ are constant on a finite volume. Let us denote this constant value on the finite volume $V_{t,i}$ by $(\beta\mathbf{N})_{t,i}$ and $\left(\frac{c_{t,i}}{g_{t,i}}-1\right)\omega$. This vector can be approximated by explicitly computing the movement in the normal direction

$$(\beta\mathbf{N})_{t,i} = \left((B_{t,i} - T_{t,i}^{-})\cdot\mathbf{X}_t/T_{t,i,i}^{+} - \mathbf{x}_{t,i}\right)/\tau. \tag{7.9}$$

Then the right hand side has the form

$$\mathbf{b}_{t,i} = (\beta\mathbf{N})_{t,i}\cdot\iint_{V_{t,i}}\Delta_{\mathbf{x}}\mathbf{x}\,\mathrm{d}\mathbf{x}$$
$$- m(V_{t,i})\frac{1}{A}(\beta\mathbf{N})_{t,i}\cdot\sum_{j=1}^{N}\iint_{V_j}\Delta_{\mathbf{x}}\mathbf{x}\,\mathrm{d}\mathbf{x} + m(V_{t,i})\left(\frac{c_{t,i}}{g_{t,i}}-1\right)\omega \tag{7.10}$$

and $\iint_{V_{t,i}}\Delta_{\mathbf{x}}\mathbf{x}\,\mathrm{d}\mathbf{x}$ is approximated as in section (7.2).

The local area density $g_{t,i}$ in (7.10) is dependent on the parametrization $\mathbf{x}(u,v)$. In the numerical approximation of the surface we do not have any so we take such

$\mathbf{x}(u,v)$ that projects a rectangle dudv with size 1 onto the quarter of quad. So $g_{t,i}$ is approximated by

$$g_{t,i} = m(V_{t,i})/Q_i \tag{7.11}$$

and $c_{t,i}$ is approximated by

$$c_{t,i} = 1\Big/\left(p\min\left(|G_{t,i}|/\widetilde{G},1\right)+1\right)\Bigg/\sum_{j=1}^{N} Q_j\Big/\left(p\min\left(|G_{t,j}|/\widetilde{G},1\right)+1\right). \tag{7.12}$$

Finally the Gaussian curvature is approximated by [3]

$$\begin{aligned} G_{t,i} = \frac{4}{m(V_{t,i})}\Bigg(2\pi - \sum_{q=1}^{Q_i} &\arccos\left(\frac{(\mathbf{x}_{t,i}^{q,1}-\mathbf{x}_{t,i}^{q,0})\cdot(\mathbf{x}_{t,i}^{q,2}-\mathbf{x}_{t,i}^{q,0})}{||\mathbf{x}_{t,i}^{q,1}-\mathbf{x}_{t,i}^{q,0}||\,||\mathbf{x}_{t,i}^{q,2}-\mathbf{x}_{t,i}^{q,0}||}\right)\\ &+\arccos\left(\frac{(\mathbf{x}_{t,i}^{q,3}-\mathbf{x}_{t,i}^{q,0})\cdot(\mathbf{x}_{t,i}^{q,2}-\mathbf{x}_{t,i}^{q,0})}{||\mathbf{x}_{t,i}^{q,3}-\mathbf{x}_{t,i}^{q,0}||\,||\mathbf{x}_{t,i}^{q,2}-\mathbf{x}_{t,i}^{q,0}||}\right)\\ &+\arccos\left(\frac{(\mathbf{x}_{t,i}^{q,3}-\mathbf{x}_{t,i}^{q,0})\cdot(\mathbf{x}_{t,i}^{q,1}-\mathbf{x}_{t,i}^{q,0})}{||\mathbf{x}_{t,i}^{q,3}-\mathbf{x}_{t,i}^{q,0}||\,||\mathbf{x}_{t,i}^{q,1}-\mathbf{x}_{t,i}^{q,0}||}\right)\Bigg). \end{aligned} \tag{7.13}$$

7.4. The finite volume approximation of the surface gradient. In this section we approximate the integral of the surface gradient in the equation (5.5). Let us approximate the function φ using a bilinear interpolation. Let us define

$$\begin{aligned} \phi_{t,i}^{q,1} &= \frac{3}{8}\varphi_{t,i}^{q,0} + \frac{3}{8}\varphi_{t,i}^{q,1} + \frac{1}{8}\varphi_{t,i}^{q,2} + \frac{1}{8}\varphi_{t,i}^{q,3} - \tilde{\varphi}_{t,i},\\ \phi_{t,i}^{q,3} &= \frac{3}{8}\varphi_{t,i}^{q,0} + \frac{1}{8}\varphi_{t,i}^{q,1} + \frac{1}{8}\varphi_{t,i}^{q,2} + \frac{3}{8}\varphi_{t,i}^{q,3} - \tilde{\varphi}_{t,i}, \end{aligned} \tag{7.14}$$

where

$$\tilde{\varphi}_{t,i} = \frac{1}{Q_i}\sum_{q=1}^{Q_i}\frac{1}{8}\varphi_{t,i}^{q,0} + \frac{2}{8}\varphi_{t,i}^{q,1} + \frac{1}{8}\varphi_{t,i}^{q,2} + \frac{2}{8}\varphi_{t,i}^{q,3}. \tag{7.15}$$

Then the q-th quad contributes to the coefficients $A_{t,i,k(i,q,j)}$ by the values

$$\begin{aligned} A_{t,i,k(i,q,0)} &\mathrel{+}= \phi_{t,i}^{q,1}\frac{m(e_{t,i}^{q,1})}{||\mathbf{m}_{t,i}^{q,1}||}\left(-\frac{3}{4}+\frac{1}{2}a_i^{q,1}\right) + \phi_{t,i}^{q,3}\frac{m(e_{t,i}^{q,3})}{||\mathbf{m}_{t,i}^{q,3}||}\left(-\frac{3}{4}+\frac{1}{2}a_i^{q,3}\right),\\ A_{t,i,k(i,q,1)} &\mathrel{+}= \phi_{t,i}^{q,1}\frac{m(e_{t,i}^{q,1})}{||\mathbf{m}_{t,i}^{q,1}||}\left(\frac{3}{4}+\frac{1}{2}a_i^{q,1}\right) + \phi_{t,i}^{q,3}\frac{m(e_{t,i}^{q,3})}{||\mathbf{m}_{t,i}^{q,3}||}\left(-\frac{1}{4}-\frac{1}{2}a_i^{q,3}\right),\\ A_{t,i,k(i,q,2)} &\mathrel{+}= \phi_{t,i}^{q,1}\frac{m(e_{t,i}^{q,1})}{||\mathbf{m}_{t,i}^{q,1}||}\left(\frac{1}{4}-\frac{1}{2}a_i^{q,1}\right) + \phi_{t,i}^{q,3}\frac{m(e_{t,i}^{q,3})}{||\mathbf{m}_{t,i}^{q,3}||}\left(\frac{1}{4}-\frac{1}{2}a_i^{q,3}\right),\\ A_{t,i,k(i,q,3)} &\mathrel{+}= \phi_{t,i}^{q,1}\frac{m(e_{t,i}^{q,1})}{||\mathbf{m}_{t,i}^{q,1}||}\left(-\frac{1}{4}-\frac{1}{2}a_{t,i}^{q,1}\right) + \phi_{t,i}^{q,3}\frac{m(e_{t,i}^{q,3})}{||\mathbf{m}_{t,i}^{q,3}||}\left(\frac{3}{4}+\frac{1}{2}a_{t,i}^{q,3}\right). \end{aligned} \tag{7.16}$$

For the special case of the boundary finite volumes (5.6) we have the coefficients

$$A_{t,i,k(i,1,1)} = \left(\frac{\varphi_{t,i}^{1,1}+\varphi_{t,i}}{2} - \frac{\varphi_{t,i}^{1,1}+2\varphi_{t,i}+\varphi_{t,i}^{Q_i,3}}{4}\right)\frac{1}{||\mathbf{x}_{t,i}^{1,1}-\mathbf{x}_{t,i}||},$$

$$A_{t,i,k(i,Q_i,3)} = \left(\frac{\varphi_{t,i}^{Q_i,3} + \varphi_{t,i}}{2} - \frac{\varphi_{t,i}^{1,1} + 2\varphi_{t,i} + \varphi_{t,i}^{Q_i,3}}{4} \right) \frac{1}{||\mathbf{x}_{t,i}^{Q_i,3} - \mathbf{x}_{t,i}||}, \tag{7.17}$$
$$A_{t,i,i} = -A_{t,i,k(i,1,1)} - A_{t,i,k(i,Q_i,3)}.$$

8. Numerical experiments. In this section we present three numerical experiments. In first two experiments we present mean curvature flow (2.2) of an open surface with redistribution of points by the Gaussian curvature. In the last experiment we present an evolution of a closed surface by (2.3). A value of interest is the difference between the area density g and the desired area density c. This norm is numerically computed as

$$error_t = \sqrt{\frac{\sum_{i=1}^{N}(g_{t,i} - c_{t,i})^2}{\sum_{i=1}^{N}(g_{t,i})^2}}. \tag{8.1}$$

For all the experiments we used the time step $\tau = 0.1$ and the following parameters, $\omega = 1$, $p = 10$, $\widetilde{G} = 2.4$.

The first experiment has the initial condition in the shape of a cylinder with radius 1 and height 1 with 1225 points; and 250 time steps were computed. The surface in time steps 0, 5, 10, 250 can be seen on figure 8.2. A decreasing $error_t$ for this evolution can be seen on figure 8.1, top. In time 0, there is constant g on the surface and also a constant Gaussian curvature, hence $error_0 = 0$. After some time, points with a higher Gaussian curvature occur near the boundary. The redistribution responds to this and decreases the area of the corresponding quads. After that, the highest Gaussian curvature points move to the center of the cylinder. Then the surface acquires a steady state and the local area density does not change in time although it is not constant on the surface.

The second experiment has an initial condition in the shape of a hyperbolic paraboloid $z = x^2 - y^2$ on a domain $(x, y) \in (-1, 1) \times (-1, 1)$ with 400 points and 250 time steps were computed. The surface in time steps 0, 20, 40, 60 can be seen on figure 8.3. A decreasing $error_t$ for this evolution can be seen on figure 8.1, left. In the beginning the points with a high Gaussian curvature are in the middle of the surface. Thus the quads start to accumulate in this area. After some time, the mean curvature evolution causes a decrease of the Gaussian curvature in this area. This results in an enlarging of quads.

Special case of an evolving surface is presented in the last experiment. The surface is closed, therefore there are no boundary conditions (2.4), (3.6). The initial condition is a dumbbell like surface with 2168 points and 500 time steps were computed. The surface in time steps 0, 20, 60, 300 can be seen on figure 8.4. A decreasing $error_t$ for this evolution can be seen on figure 8.1, right. In the beginning there are points with a higher curvature in the corners and on the edges of the surface. Then the surface starts to smooth out. In the steady state there is a constant Gaussian curvature on the surface, hence constant c and g.

9. Conclusion. We presented a method for a redistribution of points by Gaussian curvature. We have shown 3 experiments presenting the performance of this method. We checked that the local area density converges to the prescribed local are density resulting in refinement of the surface approximation in areas of high Gaussian curvature. The method can be generalized to triangular meshes and a mean curvature dependent redistribution, which can be an objective of our further research.

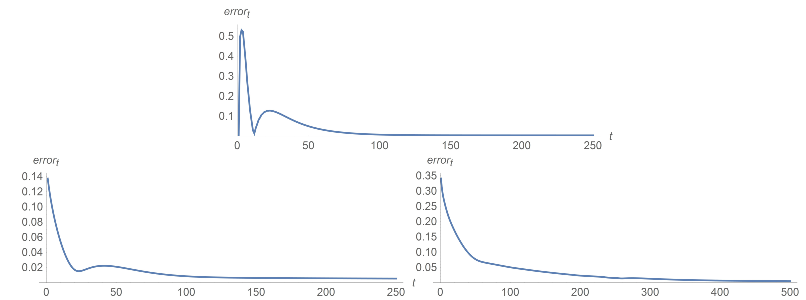

FIG. 8.1. *A graph of* $error_t$ *for the experiments. Top: the evolving cylinder. Bottom left: the evolving hyperbolic paraboloid. Bottom right: the evolving dumbbell like surface.*

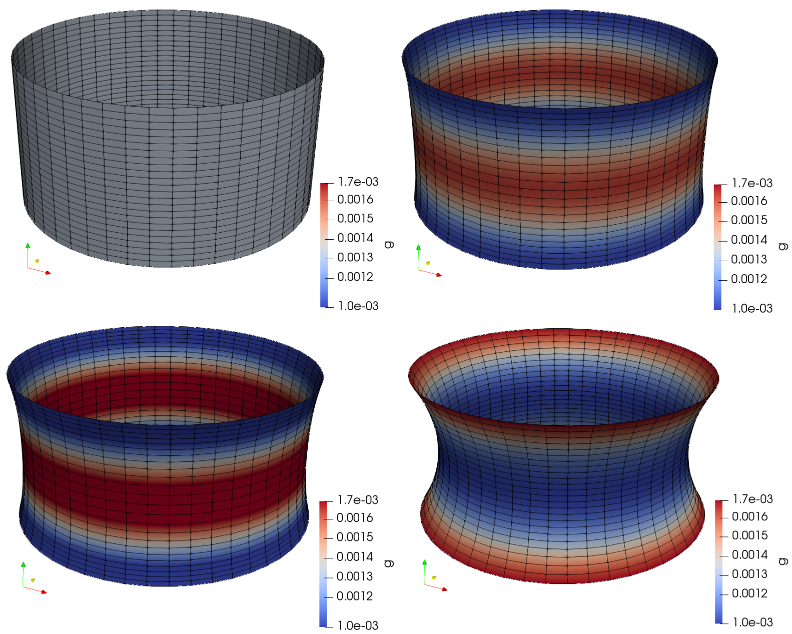

FIG. 8.2. *An evolving surface at time steps 0, 5, 10, 250.*

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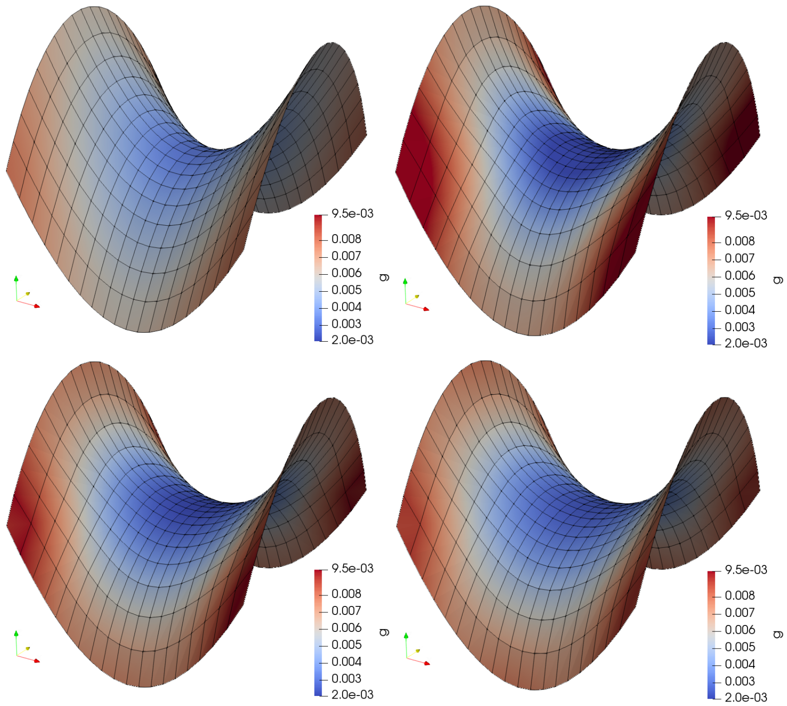

Fig. 8.3. *An evolving paraboloid at time steps 0, 20, 40, 60.*

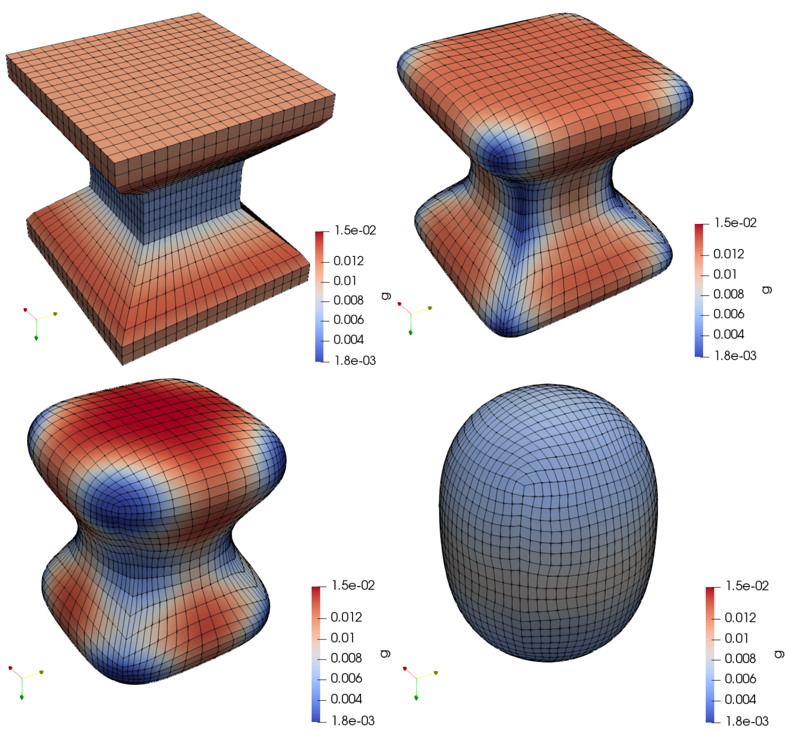

Fig. 8.4. *An evolving dumbbell like surface at time steps 0, 20, 60, 300.*