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ON DOMAIN DECOMPOSITION METHODS FOR OPTIMAL CONTROL PROBLEMS

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Abstract

In this note, we introduce a new approach to study overlapping domain decomposition methods for optimal control systems governed by partial differential equations. The model considered in our paper is systems governed by wave equations. Our technique could be used for several other equations as well.

1. Introduction

The research about using domain decomposition methods to resolve optimal control problems started with the pioneering work of A. Bensoussan, R. Glowinski and P.L. Lions [8] in the 70's and B. Depres and J.D. Benamou in the early 90's [2, 1, 7, 6, 5, 4, 4, 3]. Since then, this research line has become very active with several works of J. E. Lagnese and G. Leugering [13, 11, 10, 9, 12]. However, most of the works on domain decomposition methods for optimal control of systems governed by partial differential equations are devoted to nonoverlapping algorithms, though overlapping algorithms are proved to be more stable and much faster [14]. One of the reasons is that there was no convergence proof of the overlapping algorithms. In the series of papers [17, 16, 18, 15], we develop a new technique to study the convergence of overlapping algorithms. The technique is proved to be applicable for the convergence study of domain decomposition algorithms for several kinds of partial differential equations. Within the frame of developing our new technique for different convergence problems, this note is devoted to the application of the technique to study an overlapping domain decomposition for optimal control systems governed by wave equations, which was studied in [1] but only for the nonoverlapping case. Our technique has the potential of being a new tool to extend many of the previous studies from nonoverlapping to overlapping algorithms. For the sake of simplicity, we only consider a decomposition with two subdomains, however, our technique could be extended to the multisubdomains case without any difficulty.

2. Model description and definition of the domain decomposition algorithm

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Similarly as in [1], we consider the following wave equation defined on $(0,T)\times\Omega$

$$\begin{cases} \partial_{tt}y(t,x) - \Delta y(t,x) = f(t,x) + v(t,x) \text{ on } (0,T)\times\Omega, \\ y(0,x) = y_0(x); \qquad \partial_t y(0,x) = y_1(x) \text{ on } \Omega, \\ y(t,x) = g(t,x) \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{1}$$

where $y_0, y_1 \in L^2(\Omega)$, $g \in L^2((0,T)\times\partial\Omega)$.

Let U be a convex subset of $L^2((0,T)\times\Omega)$ and define the function

$$J(v,y) = \frac{1}{2}\int_{(0,T)\times\Omega} (\gamma|y(x)|^2 + \alpha|v(t,x)|^2)dxdt, \tag{2}$$

where α and γ are positive constants.

We consider the following optimization problem

$$\min_{v\in U} J(v, y(v)). \tag{3}$$

Following [1], we need to solve

$$\begin{cases} \partial_{tt}p\,(t,x) - \Delta p(t,x) = y(t,x) \text{ on } (0,T)\times\Omega, \\ p\,(T,x) = 0; \qquad \partial_t p(T,x) = 0 \text{ on } \Omega, \\ p\,(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \\ \int_{(0,T)\times\Omega}(p+\alpha v)(w-v)dxdt \geq 0 \;\forall w\in U. \end{cases} \tag{4}$$

We now design an overlapping domain decomposition method to resolve the system (1) and (4). Divide the domain Ω into two overlapping subdomains Ω_1 and Ω_2 in the following sense

$$\Omega = \Omega_1 \cup \Omega_2,$$

$$(\partial\Omega_1 \backslash \partial\Omega) \cap (\partial\Omega_2 \backslash \partial\Omega) = \emptyset.$$

The overlapping domain decomposition algorithm with Robin transmission condition now reads for $i \in \{1,2\}$

$$\begin{cases} \partial_{tt}y_i^{n+1} - \Delta y_i^{n+1} = f(t,x) + v_i^{n+1}(t,x) \text{ on } (0,T)\times\Omega_i, \\ y_i^{n+1}(0,x) = y_0(x), \qquad \partial_t y_i^{n+1}(0,x) = y_1(x) \text{ on } \Omega_i, \\ y_i^{n+1}(t,x) = g(t,x) \text{ on } (0,T)\times\partial\Omega_i, \end{cases}$$

$$\begin{cases} \partial_{tt} p_i^{n+1} - \Delta p_i^{n+1} = \gamma y_i^n(t,x) \text{ on } (0,T)\times\Omega_i, \\ p_i^{n+1}(T,x) = 0, \quad \partial_t p_i^{n+1}(T,x) = 0 \text{ on } \Omega_i, \\ p_i^{n+1}(t,x) = 0 \text{ on } (0,T)\times\partial\Omega_i, \end{cases}$$

$$\int_{(0,T)\times\Omega_i} (p_i^{n+1} + \alpha v_i^{n+1})(w_i - v_i^{n+1}) dxdt \geq 0,$$

with the transmission condition on $\partial\Omega_i \backslash \partial\Omega$

$$\partial_{\nu_i} y_i^{n+1} + r_i p_i^{n+1} = \partial_{\nu_i} y_{3-i}^n + r_i p_{3-i}^n,$$
$$\partial_{\nu_i} p_i^{n+1} + r_i y_i^{n+1} = \partial_{\nu_i} p_{3-i}^n + r_i y_{3-i}^n,$$

where ν_i is the outward normal outward unit normal vector of Ω_i on the boundary $\partial\Omega_i \backslash \partial\Omega$ and r_i is a positive constant. At step 0, we choose an initial guess (y_i^0, p_i^0) in $C^2([0,T]\times\overline{\Omega})$. We can see that the algorithm is well-posed and $(y_i^n, p_i^n, v_i^n) \in L^2(0,T,H^2(\Omega_i)) \times L^2(0,T,H^2(\Omega_i)) \times L^2(0,T,H^2(\Omega_i))$.

3. Convergence of the algorithm

For $i \in \{1,2\}$ we define

$$\tilde{y}_i^{n+1} = y_i^{n+1} - y,$$
$$\tilde{p}_i^{n+1} = p_i^{n+1} - p,$$
$$\tilde{v}_i^{n+1} = v_i^{n+1} - v,$$

and get the following systems

$$\begin{cases} \partial_{tt} \tilde{y}_i^{n+1} - \Delta \tilde{y}_i^{n+1} = \tilde{v}_i^{n+1}(t,x) \text{ on } (0,T)\times\Omega_i, \\ \tilde{y}_i^{n+1}(0,x) = 0, \quad \partial_t \tilde{y}_i^{n+1}(0,x) = 0 \text{ on } \Omega_i, \\ \tilde{y}_i^{n+1}(t,x) = 0 \text{ on } (0,T)\times\partial\Omega_i, \end{cases}$$

$$\begin{cases} \partial_{tt} \tilde{p}_i^{n+1} - \Delta \tilde{p}_i^{n+1} = \gamma \tilde{y}_i^n(t,x) \text{ on } (0,T)\times\Omega_i, \\ \tilde{p}_i^{n+1}(T,x) = 0, \quad \partial_t \tilde{p}_i^{n+1}(T,x) = 0 \text{ on } \Omega_i, \\ \tilde{p}_i^{n+1}(t,x) = 0 \text{ on } (0,T)\times\partial\Omega_i, \end{cases}$$

with the transmission condition on $\partial\Omega_i \backslash \partial\Omega$

$$\partial_{\nu_i} \tilde{y}_i^{n+1} + r_i \tilde{p}_i^{n+1} = \partial_{\nu_i} \tilde{y}_{3-i}^n + r_i \tilde{p}_{3-i}^n,$$
$$\partial_{\nu_i} \tilde{p}_i^{n+1} + r_i \tilde{y}_i^{n+1} = \partial_{\nu_i} \tilde{p}_{3-i}^n + r_i \tilde{y}_{3-i}^n.$$

We suppose that for any $n \in \mathbb{N}$, $\tilde{v}_i^n$ is extended by 0 in (T, ∞) and still denote by $\tilde{y}_i^{n+1}$ the solution of

$$\begin{cases} \partial_{tt}\tilde{y}_i^{n+1} - \Delta\tilde{y}_i^{n+1} = \tilde{v}_i^{n+1}(t,x) \text{ on } (0,\infty)\times\Omega_i, \\ \tilde{y}_i^{n+1}(0,x) = 0, \qquad \partial_t\tilde{y}_i^{n+1}(0,x) = 0 \text{ on } \Omega_i, \\ \tilde{y}_i^{n+1}(t,x) = 0 \text{ on } (0,\infty)\times\partial\Omega_i. \end{cases}$$

Using the change of variable $t \to T - t$, we still denote by $\tilde{p}_i^{n+1}$ the solution of

$$\begin{cases} \partial_{tt}\tilde{p}_i^{n+1} - \Delta\tilde{p}_i^{n+1} = \gamma\tilde{y}_i^{n}(T-t,x) \text{ on } (0,\infty)\times\Omega_i, \\ \tilde{p}_i^{n}(0,x) = 0, \qquad \partial_t\tilde{p}_i^{n+1}(0,x) = 0 \text{ on } \Omega_i, \\ \tilde{p}_i^{n+1}(t,x) = 0 \text{ on } (0,\infty)\times\partial\Omega_i, \end{cases}$$

with the assumption that $\tilde{y}_i^n(T-t,x) = 0$ for $t > T$. Let H be a positive constant to be chosen later. Define

$$\bar{y}_i^n = \left(\int_0^\infty |\tilde{y}_i^n| \exp\left(-\sqrt{H}t\right) dt\right) g_i^n; \quad \bar{p}_i^n = \left(\int_0^\infty |\tilde{p}_i^n| \exp\left(-\sqrt{H}t\right) dt\right) g_i^n,$$

with $g_i^n \in C^2(\mathbb{R}^N, \mathbb{R})$, $g_i^n > 0$ to be chosen later. For $F : \Omega \to \mathbb{R}$, we define the following norm

$$|||F||| = \left[\int_{\mathrm{supp}(F)} \left|\int_0^\infty |F| \exp\left(-\sqrt{H}t\right) dt\right|^2 dx\right]^{1/2}.$$

Similarly as in [15], a simple calculation leads to

$$-\Delta\bar{y}_i^{n+1} + H\bar{y}_i^{n+1} + \left(-\sum_{\alpha=1}^N \frac{\partial_\alpha g_i^{n+1}}{g_i^{n+1}} + \frac{\nabla g_i^{n+1}}{g_i^{n+1}}\right)\bar{y}_i^{n+1} + \sum_{\alpha=1}^N \frac{2\partial_{x_\alpha} g_i^{n+1}}{g_i}\partial_{x_\alpha}\bar{y}_i^{n+1} \tag{5}$$
$$= \int_0^T v_i^{n+1} \,\mathrm{sign}\,(\tilde{y}_i^{n+1}) \exp\left(-\sqrt{H}t\right) dt \text{ on } \Omega_i,$$

$$-\Delta\bar{p}_i^{n+1} + H\bar{p}_i^{n+1} + \left(-2\sum_{\alpha=1}^N \frac{\partial_{x_\alpha} g_i^{n+1}}{g_i^{n+1}} + \frac{\nabla g_i^{n+1}}{g_i^{n+1}}\right)\bar{p}_i^{n+1} + \sum_{\alpha=1}^N 2\frac{\partial_{x_\alpha} g_i^{n+1}}{g_i^{n}}\partial_{x_\alpha}\bar{p}_i^{n} \tag{6}$$
$$= \gamma\int_0^T y_i^{n}(T-t) \,\mathrm{sign}\,(\tilde{p}_i^{n+1}) \exp\left(-\sqrt{H}t\right) dt \text{ on } \Omega_i.$$

Choosing g_i^n such that $\nabla g_i^n - r_i g_i^n = 0$ on $\partial\Omega_i\backslash\Omega$, the transmission condition become

$$\begin{aligned}
\partial_{\nu_i}\bar{y}_i^{n+1} =& \partial_{\nu_i}\left(\int_0^\infty |\tilde{y}_i^n| \exp\left(-\sqrt{H}t\right) dt g_i^n\right) \\
=& \left[\int_0^\infty (\partial_{\nu_i}|\tilde{y}_i^n| + r_i|\tilde{y}_i^n|) \exp\left(-\sqrt{H}t\right) dt\right] g_i^n \\
&+ \int_0^\infty |\tilde{y}_i^n| \exp\left(-\sqrt{H}t\right) dt(\partial_{\nu_i} - r_i) g_i^n \\
=& \frac{1}{\beta_i}\partial_{\nu_i}\bar{y}_i^{n+1} \text{ on } \partial\Omega_i\backslash\partial\Omega,
\end{aligned}$$

by choosing g_i^n and g_{3-i}^n, we can make β_i to be a very large positive constant. Similarly, we also have

$$\beta_i\partial_{\nu_i}\bar{p}_i^{n+1} = \partial_{\nu_i}\bar{p}_{3-i}^n.$$

Let φ_{3-i}^n be a function in $H^1(\Omega\backslash\overline{\Omega_i})$ and φ_i^{n+1} be a function in $H^1(\Omega_i)$ such that $\varphi_i^{n+1} = \varphi_{3-i}^n$ on $\partial\Omega_i\backslash\partial\Omega$ and use them as test functions for (5) and (6)

$$\begin{aligned}
&\int_{\Omega\backslash\Omega_i} \nabla\bar{y}_{3-i}^n \nabla\varphi_{3-i}^n dx + \int_{\Omega\backslash\Omega_i} \sum_{\alpha=1}^N 2\frac{\partial_{x_\alpha} g_{3-i}}{g_{3-i}}\partial_{x_\alpha}\bar{y}_{3-i}^n \varphi_{3-i} dx \\
&+ \int_{\Omega\backslash\Omega_i}\left(\frac{\Delta g_{3-i}}{g_{3-i}} - 2\sum_{\alpha=1}^N \frac{\partial_{x_\alpha} g_{3-i}}{g_{3-i}}\right)\bar{y}_{3-i}^n\varphi_{3-i}^n dx + \int_{\Omega\backslash\Omega_i} H\bar{y}_{3-i}^n\varphi_{3-i}^n dx \\
&- \int_{\Omega\backslash\Omega_i}\int_0^T v_{3-i}^n \operatorname{sign}(\tilde{y}_{3-i}^n)\exp\left(-\sqrt{H}t\right) dt\varphi_{3-i}^n dx \\
=& -\beta_i\Bigg\{\int_{\Omega_i}\nabla\bar{y}_i^{n+1}\nabla\varphi_i^{n+1} dx + \int_{\Omega_i}\sum_{\alpha=1}^N 2\frac{\partial_{x_\alpha} g_i^{n+1}}{g_i^{n+1}}\partial_{x_\alpha}\bar{y}_i^{n+1}\varphi_i^{n+1} dx \\
&+ \int_{\Omega_i}\left(\frac{\Delta g_i^{n+1}}{g_i^{n+1}} - 2\sum_{\alpha=1}^N \frac{\partial_{x_\alpha} g_i^{n+1}}{g_i^{n+1}}\right)\bar{y}_i^{n+1}\varphi_i^{n+1} dx + \int_{\bar{\Omega}_i} H\bar{y}_i^{n+1}\varphi_i^{n+1} dx \\
&- \int_{\Omega_i}\int_0^T v_i^{n+1}\operatorname{sign}(\tilde{y}_i^{n+1})\exp\left(-\sqrt{H}t\right) dt\varphi_i^{n+1} dx\Bigg\}.
\end{aligned} \tag{7}$$

In the above equation choose φ_i^{n+1} to be $\bar{y}_i^{n+1}$. Then there exists a function ρ such that ρ is defined on $\Omega\backslash\Omega_i$ and

$$\begin{aligned}
\|\rho\|_{H^1(\Omega\backslash\Omega_i)} &\leq C_1\|\bar{y}_i^{n+1}\|_{H^1(\Omega_i)}, \\
\|\rho\|_{L^2(\Omega\backslash\Omega_i)} &\leq C_1\|\bar{y}_i^{n+1}\|_{L^2(\Omega_i)},
\end{aligned}$$

where C_1 is a positive constant depending on Ω, Ω_1, and Ω_2. Choose φ_{3-i}^n to be ρ, then for H large enough, (7) implies

$$\begin{aligned}
&\sum_{i=1}^{2} C_2\Bigg\{\frac{1}{2}\int_{\Omega\backslash\Omega_i} |\nabla \bar{y}_{3-i}^n|^2 dx + \frac{H}{2}\int_{\Omega\backslash\Omega_i} |\bar{y}_{3-i}^n|^2 dx \\
&- \int_{\Omega\backslash\Omega_i}\int_0^T v_{3-i}^n \text{sign}\,(\tilde{y}_{3-i}^n)\exp\left(-\sqrt{H}t\right)\, dt\bar{y}_{3-i}^n dx\Bigg\} \\
\geq &\sum_{i=1}^{2} \beta_i\Bigg\{\frac{1}{2}\int_{\Omega_i} |\nabla \bar{y}_i^{n+1}|^2 dx + \frac{H}{2}\int_{\Omega_i} |\bar{y}_i^{n+1}|^2 dx \\
&- \int_{\Omega_i}\int_0^T v_i^{n+1} \text{sign}\,(\tilde{y}_i^{n+1})\exp\left(-\sqrt{H}t\right)\, dt\bar{y}_i^{n+1} dx\Bigg\},
\end{aligned} \tag{8}$$

where C_2 is some constants depending only on the structure of the equation. In a similar way, we have

$$\begin{aligned}
&\sum_{i=1}^{2} C_3\Bigg\{\frac{1}{2}\int_{\Omega\backslash\Omega_i} |\nabla \bar{p}_{3-i}^n|^2 dx + \frac{H}{2}\int_{\Omega\backslash\Omega_i} |\bar{p}_{3-i}^n|^2 dx \\
&- \gamma\int_{\Omega\backslash\Omega_i}\int_0^T y_{3-i}^{n-1} \text{sign}\,(\tilde{p}_{3-i}^n)\exp\left(-\sqrt{H}t\right)\, dt\bar{p}_{3-i}^n dx\Bigg\} \\
\geq &\sum_{i=1}^{2} \beta_i\Bigg\{\frac{1}{2}\int_{\Omega_i} |\nabla \bar{p}_i^{n+1}|^2 dx + \frac{H}{2}\int_{\Omega_i} |\bar{p}_i^{n+1}|^2 dx \\
&- \gamma\int_{\Omega_i}\int_0^T y_i^{n} \text{sign}\,(\tilde{p}_i^{n+1})\exp\left(-\sqrt{H}t\right)\, dt\phi_i^{n+1} dx\Bigg\},
\end{aligned}$$

where ϕ_i^{n+1} plays a similar role as the role of ϕ_i^{n+1} in the estimate of $\bar{y}_i^{n+1}$

$$\begin{aligned}
\|\phi_i^{n+1}\|_{H^1(\Omega\backslash\Omega_i)} &\leq C_1\|\bar{p}_i^{n+1}\|_{H^1(\Omega_i)}, \\
\|\phi_i^{n+1}\|_{L^2(\Omega\backslash\Omega_i)} &\leq C_1\|\bar{p}_i^{n+1}\|_{L^2(\Omega_i)}.
\end{aligned}$$

Similarly as [15], taking β_i and H to be very large, and using the equation (as in [1])

$$\int_{(0,T)\times\Omega_i} (p_i^{n+1} + \alpha v_i^{n+1})(w_i - v_i^{n+1})\, dxdt \geq 0,$$

we get

$$\lim_{n\to\infty} (|||\nabla y_i^n||| + |||y_i^n||| + |||\nabla p_i^n||| + |||p_i^n|||) = 0.$$

Notice that the fact $|||\nabla y_i^n|||$, $|||y_i^n|||$, $|||\nabla p_i^n|||$, $|||p_i^n|||$, $|||v_i^n|||$ are well-defined is also included in the convergence result.

Theorem 3.1 *The algorithm converges in the following sense:*

$$\lim_{n\to\infty}(|||\nabla y_i^n||| + |||y_i^n||| + |||\nabla p_i^n||| + |||p_i^n||| + |||v_i^n|||) = 0.$$

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