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ANALYTICAL SOLUTION OF ROTATIONALLY SYMMETRIC STOKES FLOW NEAR CORNERS

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Abstract

We present analytical solution of the Stokes problem in rotationally symmetric domains. This is then used to find the asymptotic behaviour of the solution in the vicinity of corners, also for Navier-Stokes equations. We apply this to construct very precise numerical finite element solution.

1. Introduction

In this paper we analyze the singularities arising in rotationally symmetric tubes with nonsmooth walls, e.g. with forward and/or backward steps, or jumps in diameter. The goal of the paper is to contribute to the asymptotic behaviour of the Stokes flow near 'corners' of the rotationally symmetric tubes. We follow up the methodology used in the paper [5] for the Stokes flow in 2D domains.

We start with the general stream function-vorticity formulation. Then by means of the cylindrical coordinates together with rotational symmetry we derive equations for vorticity and stream function in z, ρ geometry (z axial, ρ radial coordinate) as e.g. in a domain in Fig. 1, or Fig. 2.

Then we perform the transformation to polar coordinates r, ϑ, where the point P in Fig. 1 is the pole. So we get the equations for both stream function and vorticity in polar coordinates r, ϑ. Continuing as in [5] we derive the analytical solution for the singularity near the corner P.

Let us note that the asymptotic behaviour applies also to Navier-Stokes equations. The results will be applied to the flow in a tube with forward and/or backward steps.

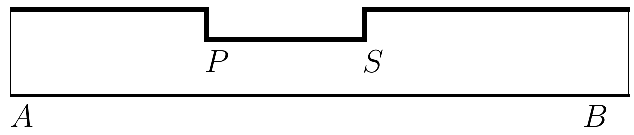

Figure 1: Example of the solution domain in cylindrical z, ρ geometry.

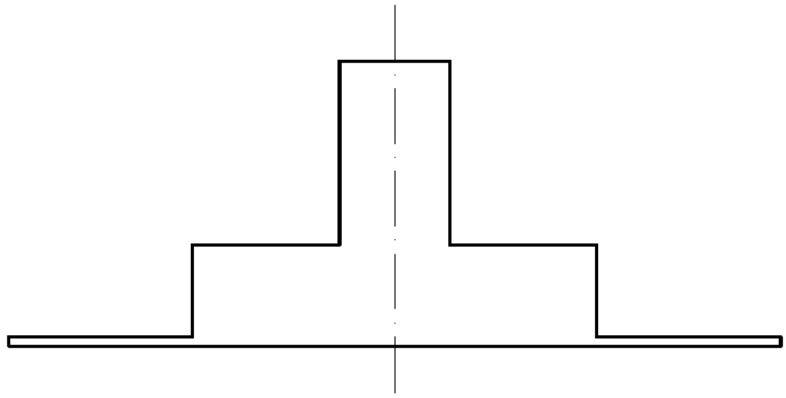

Figure 2: The hydrostatic cell (rotationally symmetric).

2. Stream function-vorticity formulation of the Stokes flow in cylindrical geometry

We start with the general 3D stationary Stokes flow in stream function-vorticity formulation, see e.g. Peyret, Taylor [8],

$$\mathbf{\Omega} = \nabla \times \mathbf{V}, \tag{1}$$

$$-\nu \nabla^2 \mathbf{\Omega} = \frac{1}{\rho} \nabla \times \mathbf{f}, \tag{2}$$

$$\mathbf{V} = \nabla \times \mathbf{\Psi}, \tag{3}$$

$$\nabla^2 \mathbf{\Psi} + \mathbf{\Omega} = \mathbf{0}, \tag{4}$$

where $\mathbf{V}$ is the vector of velocity, $\mathbf{\Omega}$ is the vector of vorticity, $\mathbf{\Psi}$ is the stream function vector, ν the kinematic viscosity, ρ is the density, and $\mathbf{f}$ is the external force.

In the paper we study the Stokes flow in the rotationally symmetric tubes, like e.g. the hydrostatic cell, see Fig. 2, or tube on Fig. 1 with line AB as the axis of symmetry.

We first transform the equations (1)–(4) to cylindrical coordinates z, ρ, φ and use the rotational symmetry $\left(\frac{\partial}{\partial \varphi}(\cdot) = 0 \right)$. In what follows we use the following formulas (see Batchelor [1])

$$\nabla^2 \mathbf{F} = \left(\nabla^2 F_z, \nabla^2 F_\rho - \frac{F_\rho}{\rho^2}, \nabla^2 F_\varphi - \frac{F_\varphi}{\rho^2} \right), \tag{5}$$

$$\nabla \times \mathbf{F} = \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho F_\varphi \right), -\frac{\partial F_\varphi}{\partial z}, \frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right), \tag{6}$$

$$\nabla^2 g = \frac{\partial^2 g}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial g}{\partial \rho} \right). \tag{7}$$

Denoting

$$\mathbf{V} = \left(V_z, V_\rho, 0 \right),$$

we get, by (1) and (6),

$$\mathbf{\Omega} = \left(0, 0, \frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho} \right).$$

Now we denote the scalar vorticity

$$\omega = \frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}.$$

Similarly we denote the scalar stream function $\psi = \psi_\varphi$ and, by (3) and (6), we get

$$\mathbf{V} = \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \psi \right), -\frac{\partial \psi}{\partial z}, 0 \right),$$

so that the velocity components are

$$V_z = \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho} \psi, \tag{8}$$

$$V_\rho = -\frac{\partial \psi}{\partial z}.$$

By (4) and (5),

$$\omega = -\left(\nabla^2 \psi - \frac{\psi}{\rho^2} \right),$$

so that, by (7)

$$\boxed{-\omega = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} - \frac{\psi}{\rho^2}.} \tag{9}$$

In the paper we assume the external forces $\mathbf{f} = 0$, so that, by (2) and (5),

$$\nu \left(\nabla^2 \omega - \frac{\omega}{\rho^2} \right) = 0,$$

which, together with (7) gives

$$\boxed{\nu \left(\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \omega}{\partial \rho} - \frac{\omega}{\rho^2} \right) = 0.} \tag{10}$$

The equations (9) and (10) together with (8) describe the flow in a rotationally symmetric domain. If we add appropriate boundary conditions, like e.g. prescribed velocity at the inflow, zero velocity on the wall, symmetry condition on the axis of symmetry, and 'do nothing' condition at the outflow, then the flow is uniquely determined.

In the paper we are interested in the behaviour of the solution near the singular points, like e.g. the points P, Q in Fig. 1. This will be the subject of the next section.

3. Stream function and vorticity near the singular points

In order to investigate the behaviour of the flow in the vicinity of the singular point P, we transform the equations (9) and (10) to polar coordinates r, ϑ with pole in the point $P = [z_0, \rho_0]$. Without loss of generality we take $z_0 = 0$. So the transformation is

$$\begin{aligned} z &= r\cos\vartheta, \\ \rho &= \rho_0 + r\sin\vartheta. \end{aligned} \tag{11}$$

The stream function $\psi(z,\rho)$ after transformation will be denoted, for a moment, as $\psi^*(r,\vartheta)$ i.e.

$$\psi^*(r,\vartheta) = \psi(z,\rho) = \psi(r\cos\vartheta, \rho_0 + r\sin\vartheta).$$

Then, using the chain rule, equation (9) gives the equality

$$\frac{\partial^2\psi^*}{\partial r^2} + \frac{1}{r}\frac{\partial\psi^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi^*}{\partial\vartheta^2} + \frac{1}{\rho}\left(\frac{\partial\psi^*}{\partial r} + \frac{1}{r}\frac{\partial\psi^*}{\partial\vartheta}\right) - \frac{\psi^*}{\rho^2} = -\omega^*. \tag{12}$$

As we are interested in the behaviour of the solution in a small neighborhood of the point P, we assume

$$r \ll \rho. \tag{13}$$

Then we may neglect the terms with the coefficients $\frac{1}{\rho}$ and $\frac{1}{\rho^2}$ in (12) and we get the equation for the stream function in polar coordinates (stars deleted)

$$\boxed{\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\vartheta^2} = -\omega.} \tag{14}$$

The same transformation (11) is done for the vorticity ω in (10)

$$\omega^*(r,\vartheta) = \omega(z,\rho) = \omega(r\cos\vartheta, \rho_0 + r\sin\vartheta).$$

Again, using the chain rule, the equation (10) gives the equality (positive constant ν is omitted)

$$\frac{\partial^2\omega^*}{\partial r^2} + \frac{1}{r}\frac{\partial\omega^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega^*}{\partial\vartheta^2} + \frac{1}{\rho}\left(\frac{\partial\omega^*}{\partial r} + \frac{1}{r}\frac{\partial\omega^*}{\partial\vartheta}\right) - \frac{\omega^*}{\rho^2} = 0.$$

Due to assumption (13) we get the equation for the vorticity in polar coordinates (stars deleted)

$$\boxed{\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}+\frac{1}{r^2}\frac{\partial^2\omega}{\partial\vartheta^2}=0.} \tag{15}$$

Let us note that the velocity components u_r, u_ϑ are related to the stream function as follows

$$u_r=\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}, \qquad u_\vartheta=-\frac{\partial\psi}{\partial r}. \tag{16}$$

If we compare equations (14), resp. (15) that apply to rotationally symmetric flow, with the equations (4), resp. (5) in [5] that apply to plain flow, we see that they are identical. In other words, we proved the following assertion.

Assertion 1. Under the assumption (13), the asymptotic behaviour of the Stokes flow near the corners of the rotationally symmetric tube is the same as that of the Stokes flow in a 2D channel.

4. Analytical solution for singularities

In the paper [5] we used the separation of variables

$$\psi\left(r,\vartheta\right)=P\left(r\right)F\left(\vartheta\right), \tag{17}$$

$$\omega\left(r,\vartheta\right)=R\left(r\right)G\left(\vartheta\right) \tag{18}$$

in order to find the singular part of the solution of the equations (14) and (15) in the neighborhood of the point P (see Fig. 1). There it was done for the channel flow. Now, due to the identical equations, cf. Assertion 1, we may proceed in the same way also in the case of rotationally symmetric flow.

Namely, we consider fluid flow in the rotationally symmetric region with boundary corner of nonconvex internal angle α, cf. Fig. 1. We assume a rigid boundary and nonslip boundary conditions, so that the boundary conditions for the stream function are

$$\psi\left(r,0\right)=0, \qquad \psi\left(r,\alpha\right)=0, \tag{19}$$

$$\frac{\partial\psi}{\partial\vartheta}\left(r,0\right)=0, \qquad \frac{\partial\psi}{\partial\vartheta}\left(r,\alpha\right)=0. \tag{20}$$

As proved in [5], the singular part of the stream function ψ is

$$\psi\left(r,\vartheta\right)=r^{\gamma+1}F\left(\vartheta\right), \tag{21}$$

where γ is the solution of the algebraic equation

$$\gamma^2\sin^2\alpha-\alpha\sin^2\gamma=0. \tag{22}$$

In the case of domains in Figs. 1 and 2 the angle is

$$\alpha=\frac{3}{2}\pi. \tag{23}$$

So that, by (22),

$$\gamma = 0.5444837, \tag{24}$$

and we get for the stream function the asymptotic behaviour near the angle $\frac{3\pi}{2}$:

$$\psi(r, \vartheta) = r^{1.54448} \cdot F(\vartheta), \tag{25}$$

where the function F does not depend on r. Consequently, for the velocity components, by (16) we have the asymptotics

$$\begin{aligned} u_r &= r^{\gamma}\ F_1(\vartheta) = r^{0.54448}\ F_1(\vartheta), \\ u_{\vartheta} &= r^{\gamma}\ F_2(\vartheta) = r^{0.54448}\ F_2(\vartheta), \end{aligned} \tag{26}$$

where the functions $F_1(\vartheta), F_2(\vartheta)$ are independent of r.

For pressure, we derived in [5] the asymptotic behaviour

$$p \approx r^{\gamma-1}\Phi_p(\vartheta) \approx r^{-0.45552}\Phi_p(\vartheta), \tag{27}$$

where the function $\Phi_p(\vartheta)$ is independent of r.

Let us note that for 2D channel flow, the same asymptotics were also found by a different technique in Kondratiev [6] and in Ladeveze and Peyret [7]. For rotationally symmetric flow the technique based on Kondratiev was used in [2]. Further, we note that the asymptotics (26) and (27) apply also to the Navier-Stokes equations, see e.g. [2].

5. Application to finite element solution of Navier-Stokes equations

In [4] and [5] we described the way how to make use of the asymptotics of the solution near the singular points. Together with the a priori error estimates we suggested and applied the algorithm for designing the finite element mesh in the neighbourhood of the singular point. Due to the Assertion 1, the results obtained in [4] may be applied to axisymmetric flows, using the 2D domain as a cross section of the axisymmetric tube.

For evaluating the achieved accuracy of the approximate solution, we use the a posteriori error estimator, see e.g. [3].

6. Numerical results

We study flow in the rotationally symmetric domain of the hydrostatic cell from Fig. 2. Similar results were obtained for a two-dimensional flow problem in [4]. Figure 3 shows the shape of the singularity in pressure solution near the bottom corner of the hydrostatic cell. In Fig. 4, we compare the asymptotic behaviour of pressure near the bottom corner of the cell obtained by formula (27) with the solution in the horizontal cut obtained by FEM. Let us note that the finite element mesh was not designed by our algorithm here, and we used a simple local refinement offered by the program GMSH. A more precise FEM solution would need a finer mesh.

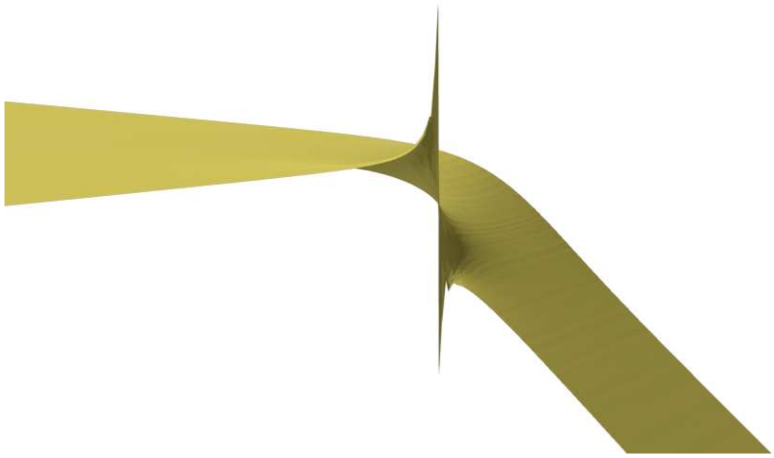

Figure 3: Singularity of pressure in hydrostatic cell.

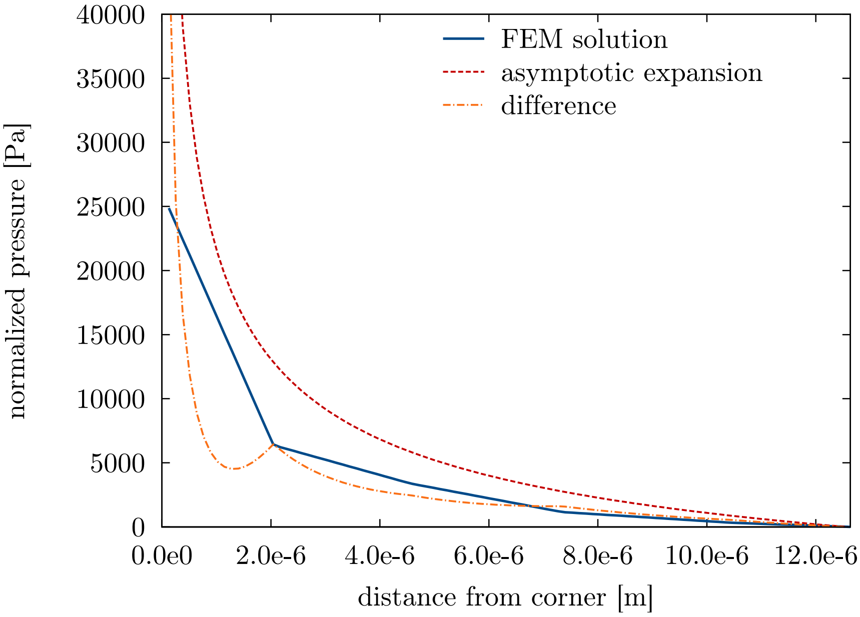

Figure 4: Pressure near the bottom corner: asymptotic versus FEM solution.

7. Conclusion

In the paper, we are interested in Stokes problem with singularities caused by nonconvex corners in rotationally symmetric domains. We proved that the asymptotic behaviour of the Stokes flow near the corners of the rotationally symmetric tube is the same as that of the Stokes flow in a 2D channel. For the Stokes flow we find analytically the principal part of the asymptotics of the solution in the vicinity of corners. This result may be used on one hand to construct the finite element mesh adjusted to singularity. This mesh is then used to find a very precise solution to Stokes but also Navier-Stokes equations. On the other hand, the analytical solution of the Stokes flow near corners may be used to test other methods.

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