Ivona Svobodová

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NUMERICAL APPROXIMATION OF THE NON-LINEAR FOURTH-ORDER BOUNDARY-VALUE PROBLEM*

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Abstract

We consider functionals of a potential energy $\mathcal{P}_\psi(u)$ corresponding to *an axisymmetric boundary-value problem.* We are dealing with *a deflection of a thin annular plate* with *Neumann boundary conditions.* Various types of the subsoil of the plate are described by various types of the *nondifferentiable* nonlinear term $\psi(u)$. The aim of the paper is to find a suitable computational algorithm.

1. Introduction

Let us consider an axisymmetric annular elastic thin plate. In addition, the body is in the contact with an elastic unilateral subsoil. Consequently, the reaction of the (a-priori unknown) active part of the subsoil has to be taken into account in the mathematical model.

The fourth-order model is based on the well-known Kirchhoff theory for thin plates, for derivation see [1], [2]. This model can be formulated in terms of a variational equation. The potential $\mathcal{P}$ of this problem is quadratic.

The nonlinear term ψ, which represents the subsoil, is added to the identity in the classical formulation. The form ψ is a combination of the positive and negative parts u^+ and u^- of the deflection $u = u(r)$. The complete potential $\mathcal{P}_\psi$ is the sum of the quadratic potential $\mathcal{P}$ and the potential of the subsoil response.

The aim of the paper are to introduce and to discuss the difficulties with the discrete version of the problem obtained through the finite element method. Finally, we propose one possible way of the numerical realization of the mathematical problem.

2. Setting of the problem

The set $\big\{(r,\varphi,z)\ ;\ a \le r \le b\,, -\pi < \varphi \le \pi\,,\ -\mathsf{h}/2 \le z \le \mathsf{h}/2\big\}$ describes the plate in three dimensions *in cylindrical coordinates* (r,φ,z).

Let the elastic axisymmetric annular thin plate be represented by the domain $(a,b) \subset \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers. The body thickness h will be involved in the equilibrium equation as a constant (see [1] for details).

In general, the operator ψ is defined for the deflection function u and it is of the form

$$\psi(u) = \sum_{i=1}^{m} k_{Ni} u^+ \chi_{A_i} - \sum_{j=1}^{n} k_{Pj} u^- \chi_{B_j} \qquad m,n \in \mathbb{N}\,, \tag{1}$$

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where k_{Ni} and k_{Pj} are non-negative functions defined on (a,b) and χ_{A_i}, χ_{B_j} are characteristic function of the closed subintervals $A_i, B_j \subset (a,b)$. The functions u^+ and u^- are positive and negative part of the function u, respectively, i.e. $u^+ := \frac{1}{2}(|u|+u)$ and $u^- := \frac{1}{2}(|u|-u)$.

Classical formulation. According to the definition, the *classical solution* $u = u(r)$ satisfies the *equilibrium equation*

$$\mathsf{h}^2 D\, [r[\tfrac{1}{r}[r\cdot u']']']' + r\,\psi(u) = r\,\hat{f}, \qquad r \in (a,b) \tag{2}$$

with classical *boundary conditions* (for $r \in \{a,b\}$) of the following types

$$\text{Dirichlet conditions} \qquad u(r) = \hat{u}_r^{(0)} \qquad \text{and} \qquad u'(r) = \hat{u}_r^{(1)} \tag{3a}$$

or

$$\text{Neumann conditions} \qquad \mathcal{M}u(r) = \hat{m}_r \qquad \text{and} \qquad \mathcal{T}u(r) = \hat{t}_r \tag{3b}$$

or any reasonable combination of them. The symbol $[\,\cdot\,]'$ means $\frac{\mathrm{d}}{\mathrm{d}r}(\,\cdot\,)$ and the constant D is the combination of the elastic material coefficients namely the Young's modulus E and the Poisson's ratio μ. The function $\hat{f} = \hat{f}(r)$ describes given volume forces. The operators $\mathcal{T}$ and $\mathcal{M}$ represent *shear forces* and *bending moments* on the boundary, respectively. They are defined through the identities $\mathcal{T}(u) := -\mathsf{h}^3 D\,(r\,u''' + u'' - \frac{1}{r}u')$ and $\mathcal{M}(u) := -\mathsf{h}^3 D\,(r\,u'' + \mu\,u')$. The values $\hat{u}_r^{(0)}$, $\hat{u}_r^{(1)}$, $\hat{m}_r$, and $\hat{t}_r$ are given.

This is the linear elasticity problem. Accordingly, the components of the *small strain tensor* $\boldsymbol{\varepsilon}$ for the homogenous izotropic plate are in the forms $\varepsilon_{rr} = -z\,u''(r)$, $\varepsilon_{\varphi\varphi} = -z\,\frac{1}{r}\,u'(r)$, and $\varepsilon_{\alpha\beta} = 0$ for α,β otherwise, where $\alpha,\beta \in \{r,\varphi,z\}$ and $z \in (-\frac{\mathsf{h}}{2}, \frac{\mathsf{h}}{2})$.

The unstable *Neumann boudary conditions* with $\hat{m}_a = 0$, $\hat{m}_b = 0$, and $\hat{t}_a = 0$, $\hat{t}_b = 0$ will be considered in the following paragraphs. These conditions correspond to the so-called "free" plate.

Weak formulation. In order to get the weak form of the problem, we introduce the *finite energy function space.* This is the weighted Sobolev space which is defined as

$$H^2\big((a,b);[r,\tfrac{1}{r},r]\big) := \{v = v(r) \mid v, v'' \in L^2_r(a,b) \text{ and } v' \in L^2_{\frac{1}{r}}(a,b)\}\,. \tag{4}$$

Weighted Lebesgue spaces $L^2_{\varrho(r)}(a,b)$ are Hilbert spaces with the norm $|\cdot|_{\varrho(r)}$ induced by the inner product $(u,v)_{\varrho(r)} := \int_a^b u(r)\,v(r)\,\varrho(r)\,\mathrm{d}r$. The norm $\|\cdot\|_{[r,\frac{1}{r},r]}$ of the Hilbert space $H^2\big((a,b);[r,\frac{1}{r},r]\big)$ is related to the inner product $(u,v)_{[r,\frac{1}{r},r]} := (u,v)_r + (u',v')_{\frac{1}{r}} + (u'',v'')_r$. See [3] for the details concerning weighted spaces.

The linear space $H^2\big((a,b);[r,\frac{1}{r},r]\big)$ is not only finite energy function space but also virtual displacement space for the problem. Indeed, the Neumann boundary conditions were choosen in the previous text.

We introduce the following forms for $w, v \in V = H^2\big((a,b);[r,\frac{1}{r},r]\big)$:

$$a_0(w,v) := D\mathsf{h}^2\Big((w',v')_{\frac{1}{r}} + (w'',v'')_r + \mu(w'',v')_1 + \mu(w',v'')_1 \Big) \,, \tag{5a}$$

$$a_\psi(w,v) := a_0(w,v) + (\psi(w),v)_r \,, \tag{5b}$$

$$\mathcal{F}(v) := (\hat{f},v)_r \,. \tag{5c}$$

We say that a function u in V is a *weak solution* of the problem (2), (3) whenever

$$a_\psi(u,v) = \mathcal{F}(v) \qquad \forall v \in V. \tag{6}$$

For the sake of brevity, we consider the operator ψ from (1) in the special case: $m = 1$, $k_{N_1} \equiv k_N$, and $k_{P_j} \equiv 0$ for $\forall j$, i.e.

$$\psi(u) = k_N u^+ \,, \tag{7}$$

where $k_N \in L^\infty\big((a,b)\big)$, $k_N(r) \geq \widehat{k}_N$, $\widehat{k}_N \in \mathbb{R}^+$. This form of ψ describes *the nonlinear unilateral upper subsoil* of the Winkler's type. A sufficient solvability condition of this problem is the inequality $\mathcal{F}(1) > 0$. See [2].

3. Discretization and numerical realization

We assume the discretization of the closed domain $\langle a,b\rangle$ as follows

$$a = r_1 < r_2 < \cdots < r_N < r_{N+1} = b$$

for $N \in \mathbb{N}$. The discretization parameter h is defined as $h := \max\limits_{i=1,\dots N}(r_{i+1} - r_i)$.

Finite element method has been used for the discrete formulation of the problem. For any h, we introduce the finite dimensional space V_h which consists of piecewise cubic smooth functions

$$V_h := \{ v_h \in C^1\big((a,b)\big) \;:\; v_h|_{\langle r_i,r_{i+1}\rangle} \in P_3 \;\; \forall i = 1,\dots N \} \,. \tag{8}$$

The standard basis in V_h will be denoted by $\{\varphi_k\}_{k=1}^{2N+2}$, for details see [4]. Therefore, every function $v_h \in V_h$ is represented by

$$v_h(r) = \sum_{k=1}^{N+1} \Big(v_{2k-1}\varphi_{2k-1}(r) + v_{2k}\varphi_{2k}(r) \Big) \,, \tag{9}$$

where $v_{2k-1} = v_h(r_k)$ and $v_{2k} = v_h'(r_k)$.

We look for the *discrete solution* which is defined as the function $u_h \in V_h$ satisfying the identity

$$a_\psi^h(u_h,v_h) = \mathcal{F}^h(v_h) \qquad \forall v_h \in V_h \,, \tag{10}$$

where forms a_ψ and $\mathcal{F}$ are defined in (5). The superscript ^{h} means the usage of the numerical quadrature procedure on every subinterval $\langle r_i, r_{i+1}\rangle$ instead of the exact integration. The two points Gaussian quadrature rule was used.

The computional problem. The detailed analysis of the form a_ψ^h enables us to see, where the main problem is. The second term $([u_h]^+, v_h)_r$ has the following form

$$\Big(k_N(r)\Big[\sum_{k=1}^{2N+2} u_k\varphi_k(r)\Big]^+ , v_h(r)\Big)_r^h =$$
$$= \Big(k_N(r)\,\frac{1}{2}\sum_{k=1}^{2N+2} u_k\varphi_k(r) + k_N(r)\,\frac{1}{2}\Big|\sum_{k=1}^{2N+2} u_k\varphi_k(r)\Big| , v_h(r)\Big)_r^h .$$

The necessity of the "C^0–function" u^+ expression in terms of the base $\{\varphi_k\}_k$ is the origin of numerical difficulties in the computational algorithm. We propose a possible way how to overcome this difficulty. It is based on the following equivalent formulation of (6):

$$a_\psi^h(u_h, v_h) = \mathcal{F}^h(v_h) \qquad \forall v_h \in V_h\,,$$
$$a_0^h(u_h, v_h) + (k_N\, u_h^+, v_h)_r^h = \mathcal{F}^h(v_h)\,,$$
$$a_0^h(u_h, v_h) + (k_N\, u_h, v_h)_r^h = \mathcal{F}^h(v_h) - (k_N\, u_h^-, v_h)_r^h\,,$$
$$a_0^h(u_h, \varphi_k) + (k_N\, u_h, \varphi_k)_r^h = (\widehat{f}, \varphi_k)_r^h - (k_N\, u_h^-, \varphi_k)_r^h, \qquad k = 1, \dots 2N+2\,,$$
$$\sum_{j=1}^{2N+2} \big(a_0^h(\varphi_j, \varphi_k) + (k_N\, \varphi_j, \varphi_k)_r^h\big) u_j = (\widehat{f}, \varphi_k)_r^h - (k_N\, u_h^-, \varphi_k)_r^h,$$
$$(K + k_N M)\vec{u} = \vec{f} - (k_N\, u_h^-, \varphi_k)_r^h\,.$$

The relation $u = u^+ - u^-$ has been used. Note that the matrix $K + k_N M$ is positive definite. More general types of the form ψ can be used in the previous procedure. Other boundary conditions do not affect the last identity. Hereafter, we could formulate the computational algorithm.

The suggested computational algorithm

1. Setting of the system stiffness matrix $K_N = K + k_N M$; standard modifying of K_N with respect to the given boundary conditions (see [4]),
2. setting of the vector $\vec{f}$ without any boundary conditions adjustments,
3. the initial choice of $\vec{u}_0$,
4. procedure in the n^{th} iteration:
   (a) setting of the vector $(k_N(u_{n-1})^-, \vec{\varphi})_r^h$, which represents the reaction of the subsoil active parts,
   (b) setting of the right side vector $\vec{f}_n = \vec{f} - (k_N(u_{n-1})^-, \vec{\varphi})_r$; standard modifying of the vector with the respect to the given boundary conditions (see [4]),

(c) solving of the system $K_N\vec{u}_n = \vec{f}_n$,

(d) the termination criterion $\dfrac{\|\vec{u}_n - \vec{u}_{n-1}\|}{\|\vec{u}_n\|} \leq tol.$

3.1. Numerical examples

We illustrate the efficiency of the suggested computational algorithm by two numerical examples. We suppose the axisymmetric annular elastic thin plate with the following characteristics. The length of inner radius is $a = 1\text{m}$ and of outer one is $b = 5\text{m}$. The plate thickness is $\mathsf{h} = 0.01\text{m}$ and the elastic constants are $E = 10^7\text{N/m}^2$ and $\mu = 0.5$. The magnitudes E and μ are choosen to get a small deformation representation of very ellastic material in order to get a better visual verification.

Example 1: The zero Dirichlet boundary conditions (3a) for all $r \in \{a, b\}$ have been prescribed. The data in equation (6) are the following: $\mathcal{F}(v) = (-10^3\frac{1}{r}, v(r))_r$, the operator $\psi(u) = -10^7 u^- \chi_B$, where $B = \langle 2.25\text{m}, 3.75\text{m}\rangle$. Note that the choosen boundary conditions are stable, so we do not require anything in order to get the solution existence. The table of iterations and the solution diagram follow.

Example 2: The zero Neumann boundary conditions (3b) for all $r \in \{a, b\}$ have been prescribed. The given data in equation (6) are the following:

$$\mathcal{F}(v) = 5 \cdot 10^3\big(-r^{(1)}v(r^{(1)}) - r^{(2)}v(r^{(2)}) + r^{(3)}v(r^{(3)})\big)$$

iteration	rel. error	residual
1	5.98826×10^{-4}	1.49466×10^2
2	5.93289×10^{-4}	1.48116×10^2
$\vdots$	$\vdots$	$\vdots$
449	1.00296×10^{-5}	2.56704
450	9.93897×10^{-6}	2.54386

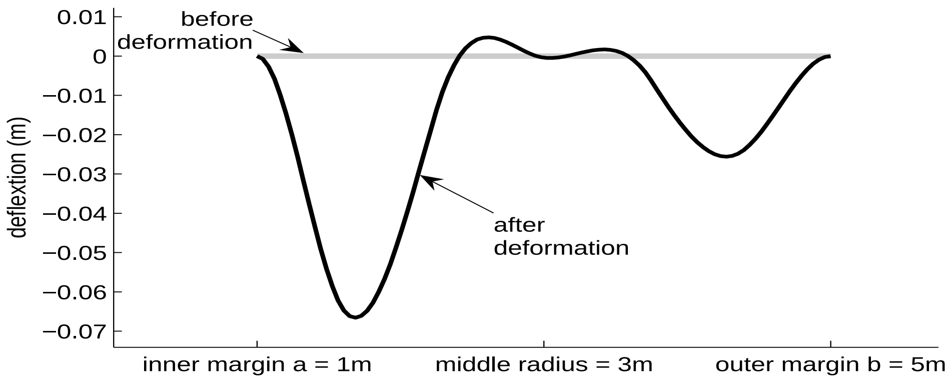

Fig. 1: *The character of resulting bending for the first problem.*

for $[r^{(i)}]_{i=1}^{3} = [1.5, 2.7, 3.7]$. This $\mathcal{F}$ describes the forces concerned on rings with radii $[r^{(i)}]_i$ and with the origin on the axis of the plate symmetry. Finally, the form ψ is the following:

$$\psi(u) = 10^7 u^+ \chi_{A_1} + 10^5 u^+ \chi_{A_2} - 10^5 u^- \chi_{B_1} - 10^7 u^- \chi_{B_2},$$

where $A_1 = \langle 2.0\mathrm{m}, 3.25\mathrm{m}\rangle$, $A_2 = \langle 4.0\mathrm{m}, 5.0\mathrm{m}\rangle$, $B_1 = \langle 1.0\mathrm{m}, 2.25\mathrm{m}\rangle$, and the last $B_2 = \langle 3.0\mathrm{m}, 4.0\mathrm{m}\rangle$. It is prescribed upper and lower subsoil. Hence, the weak solution of the problem exists. The table of iterations and the solution diagram follow.

iteration	rel. error	residual
1	5.11890×10^{-2}	1.60807×10^{8}
2	4.47150×10^{-2}	1.52994×10^{8}
⋮	⋮	⋮
1929	9.99668×10^{-6}	3.36741×10^{5}

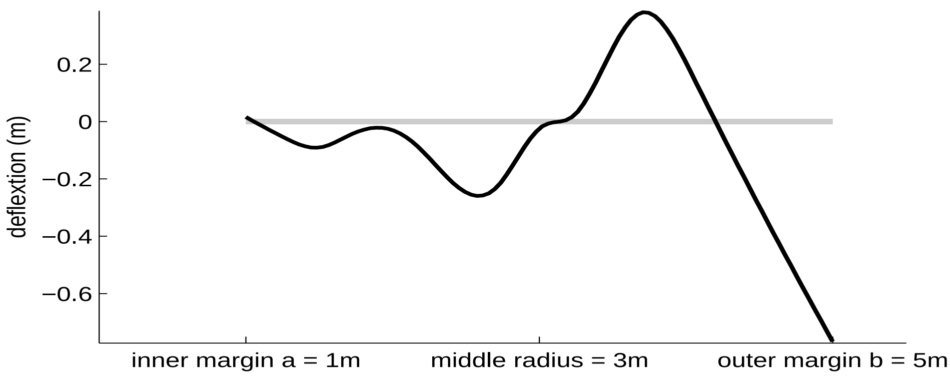

Fig. 2: *The character of resulting bending for the second problem.*

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