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ANALYSIS OF THE DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD APPLIED TO A SCALAR NONLINEAR CONVECTION-DIFFUSION EQUATION*

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Abstract

We deal with a scalar nonstationary convection-diffusion equation with nonlinear convective as well as diffusive terms which represents a model problem for the solution of the system of the compressible Navier-Stokes equations describing a motion of viscous compressible fluids. We present a discretization of this model equation by the discontinuous Galerkin finite element method. Moreover, under some assumptions on the nonlinear terms, domain partitions and the regularity of the exact solution, we introduce a priori error estimates in the $L^\infty(0,T;L^2(\Omega))$-norm and in the $L^2(0,T;H^1(\Omega))$-seminorm. A sketch of the proof is presented.

1. Introduction

Our goal is to develop a sufficiently robust, accurate and efficient numerical method for the solution of the system of the compressible Navier-Stokes equations describing a motion of viscous compressible fluids. Due to the lack of the theory concerning an existence of the solution of the Navier-Stokes equations we consider the model problem represented by a nonstationary two-dimensional convection-diffusion equation with nonlinear convection as well as diffusion.

Among a wide class of numerical methods, the *discontinuous Galerkin finite element method* (DGFEM) seems to be a promising technique for the solution of convection-diffusion problems. DGFEM is based on a piecewise polynomial but discontinuous approximation, for a survey, see, e.g., [2], [3]. Within this paper we deal with the space semidiscretization of the model problem with the aid of the three variants of DGFEM. Namely nonsymmetric (NIPG), symmetric (SIPG) and incomplete interior penalty Galerkin (IIPG) techniques, see [1].

This article represents a generalization of research papers [5], [6], [7], and [8], where the linear diffusion term was considered. Moreover, let us cite works [4], [9], and [10], where simpler forms of nonlinear diffusion were analysed.

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2. Problem formulation

We consider the following unsteady nonlinear convection-diffusion problem. Let $\Omega \subset I\!R^2$ be a bounded polygonal domain and $T > 0$. We seek a function $u : Q_T \to I\!R$, $Q_T = \Omega \times (0,T)$, such that

$$\text{(a)} \qquad \frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} = \mathrm{div}(I\!K(u)\,\nabla u) + g \quad \text{in } Q_T, \tag{1}$$

$$\text{(b)} \qquad u|_{\partial\Omega\times(0,T)} = u_D, \tag{2}$$

$$\text{(c)} \qquad u(x,0) = u^0(x), \quad x \in \Omega, \tag{3}$$

where $g : Q_T \to I\!R$, $u_D : \partial\Omega \times (0,T) \to I\!R$, $u^0 : \Omega \to I\!R$ are given functions, $f_1, f_2 \in C^1(I\!R)$ represent convective terms and the matrix $I\!K(u) \in I\!R^{2,2}$ plays a role of nonlinear anisotropic diffusive coefficients. If $I\!K(u) = \varepsilon I\!I$, where ε is a positive constant and $I\!I \in I\!R^{2,2}$ the unit matrix, then problem (1) reduces to the equation considered in [5], [6], [7], [8]. For simplicity we prescribe the Dirichlet condition on the whole boundary but it is also possible to consider mixed Dirichlet-Neumann boundary conditions.

Formally, we introduce a weak solution u of the problem (1) by

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),v) + b(u(t),v) + a(u(t),v) = (g(t),v) \quad \forall\, v \in H_0^1(\Omega), \text{ a.e. } t \in (0,T), \tag{4}$$

where $u(t)$ denotes the function on Ω such that $u(t)(x) = u(x,t)$, $x \in \Omega$. Further, $(\cdot,\cdot)$ denotes the L^2-scalar product, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are nonlinear forms representing the diffusive and convective terms, respectively. We also consider appropriate representation of initial and boundary conditions. For details see [4], [7].

3. Discretization

Let $\mathcal{T}_h$ $(h > 0)$ be a family of the partitions of the domain $\Omega \subset I\!R^2$ into triangular elements. We do not require the conformity of the mesh, i.e., the so-called hanging nodes are allowed. However, more general elements (even non-convex) can be considered within the frame of DGFEM, see [7]. By $\mathcal{F}_h$ we denote the smallest possible set of all edges of all elements $K \in \mathcal{T}_h$. Furthermore, let $\mathcal{F}_h^I$ and $\mathcal{F}_h^D$ represent the *interior* and the *boundary* edges of $\mathcal{T}_h$, respectively. Obviously $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D$. Finally, for each $\Gamma \in \mathcal{F}_h$ we define a unit normal vector $\vec{n}_\Gamma$. We assume that $\vec{n}_\Gamma$, $\Gamma \subset \partial\Omega$ has the same orientation as the outer normal of $\partial\Omega$. For $\vec{n}_\Gamma$, $\Gamma \in \mathcal{F}_I$ the orientation is arbitrary but fixed for each edge.

The approximate solution is sought in a space of piecewise polynomial but discontinuous functions

$$S_{hp} \equiv S_{hp}\,(\Omega, \mathcal{T}_h) = \{v; v|_K \in P_p(K)\ \forall\, K \in \mathcal{T}_h\}, \tag{5}$$

where $P_p(K)$ denotes the space of all polynomials on K of degree $\le p$, $K \in \mathcal{T}_h$.

For each $\Gamma \in \mathcal{F}_h^I$ there exist two elements K_L, $K_R \in \mathcal{T}_h$ such that $\Gamma \subset K_L \cap K_R$. We use a convention that K_R lies in the direction of $\vec{n}_\Gamma$ and K_L in the opposite direction of $\vec{n}_\Gamma$. For $v \in S_{hp}$, by

$$v|_\Gamma^{(L)} = \text{ trace of } v|_{K_L} \text{ on } \Gamma, \quad v|_\Gamma^{(R)} = \text{ trace of } v|_{K_R} \text{ on } \Gamma \tag{6}$$

we denote the *traces* of v on edge Γ, which are different in general. Additionally,

$$[v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}, \qquad \langle v \rangle_\Gamma = \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)} \right), \tag{7}$$

denotes the *jump* and the *mean value* of function v over the edge Γ, respectively. For $\Gamma \subset \partial\Omega$ there exists an element $K_L \in \mathcal{T}_h$ such that $\Gamma \subset K_L \cap \partial\Omega$. Then for $v \in S_{hp}$, we put: $v|_\Gamma^{(L)} =$ trace of $v|_{K_L}$ on Γ, $\quad \langle v \rangle_\Gamma = [v]_\Gamma = v|_\Gamma^{(L)}$. In case that $[\cdot]_\Gamma$ and $\langle \cdot \rangle_\Gamma$ are arguments of $\int_\Gamma \ldots \,\mathrm{d}S$, $\Gamma \in \mathcal{F}_h$ we omit the subscript Γ and write simply $[\cdot]$ and $\langle \cdot \rangle$, respectively.

Similarly as in [5], it is possible to derive the space semi-discretization of (1). A particular attention should be paid to the nonlinear diffusive term. In order to replace the interelement continuity, we add some stabilization and penalty terms into formulation of the discrete problem. The convective term is approximated with the aid of a numerical flux $H(\cdot,\cdot,\cdot)$, known from the finite volume method.

Therefore, we say that $u_h \in C^1(0,T;S_{hp})$ is the *semi-discrete solution* of (1) if $(u_h(0), v_h) = (u^0, v_h)\ \forall v_h \in S_{hp}$ and

$$\left(\frac{\partial u_h(t)}{\partial t}, v_h \right) + b_h(u_h(t), v_h) + a_h^\Theta(u_h(t), v_h) + \alpha J_h^\sigma(u_h(t), v_h) = \ell_h^\Theta(u_h(t), v_h)\,(t) \tag{8}$$

$$\forall\, v_h \in S_{hp},\ \forall\, t \in (0,T),$$

where

$$\begin{aligned} a_h^\Theta(u,v) &= \sum_{K \in \mathcal{T}_h} \int_K I\!K(u)\,\nabla u \cdot \nabla v \,\mathrm{d}x - \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \langle I\!K(u)\,\nabla u \cdot \vec{n} \rangle [v] \,\mathrm{d}S \\ &+ \Theta \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \langle I\!K(u)\,\nabla v \cdot \vec{n} \rangle [u] \,\mathrm{d}S, \end{aligned} \tag{9}$$

$$b_h(u,v) = -\sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{2} f_s(u) \frac{\partial v}{\partial x_s} \,\mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \vec{n}_\Gamma)\,[v] \mathrm{d}S, \tag{10}$$

$$J_h^\sigma(u,v) = \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \sigma [u]\,[v] \,\mathrm{d}S, \tag{11}$$

$$\ell_h^\Theta(u,v)(t) = \int_\Omega g(t)\, v \,\mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \Big(\Theta\, I\!K(u)\,\nabla v \cdot \vec{n}\, u_D(t) + \sigma\, u_D(t)\, v \Big) \,\mathrm{d}S. \tag{12}$$

Nonlinear forms $a_h^\Theta(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are the discrete variants of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, respectively. According to value of parameter Θ, we speak of SIPG ($\Theta = -1$), IIPG ($\Theta = 0$) or NIPG ($\Theta = 1$) variants of DGFEM. Penalty terms are represented by J_h^σ and the penalty parameter function σ in (11) is defined as $\sigma|_\Gamma = C_W \cdot |\Gamma|^{-1}$, $\Gamma \in \mathcal{F}_h$, where $C_W \geq 0$ is a suitable constant depending on the used variant of the scheme and on the degree of polynomial approximation. The value of multiplicative constant α before the penalty form J_h^σ will be specified in Section 4, assumption (14).

The problem (8) exhibits a system of ordinary differential equations for $u_h(t)$ which has to be discretized by a suitable ODE method.

If the numerical flux H is consistent with the convective fluxes f_1, f_2 (i.e., $H(u, u, \vec{n}) = f_1(u)n_1 + f_2(u)n_2 \; \forall u \in I\!R \; \forall \vec{n} = (n_1, n_2)$) then we find that the sufficiently regular exact solution u satisfies

$$\left(\frac{\partial u(t)}{\partial t}, v_h\right) + b_h(u(t), v_h) + a_h^\Theta(u(t), v_h) + \alpha J_h^\sigma(u(t), v_h) = \ell_h^\Theta(u(t), v_h)\,(t) \qquad (13)$$
$$\forall\, v_h \in S_{hp} \; \forall\, t \in (0, T),$$

4. Error analysis

To carry out the error analysis we need to specify the additional assumptions on mesh, nonlinear diffusion term and regularity of the solution u of the continuous problem. Therefore, we assume that:

(A1) The matrix $I\!K(v) = \{k_{ij}(v)\}_{i,j=1}^2$, $k_{ij}(v) : I\!R \to I\!R$, appearing in the diffusion terms satisfies

$$\begin{aligned}
&\text{(a)} \quad \|I\!K(v)\|_\infty \leq C_U < \infty \; \forall\, v \in I\!R, \\
&\text{(b)} \quad \|I\!K(v_1) - I\!K(v_2)\|_\infty \leq C_L|v_1 - v_2| \; \forall\, v_1, v_2 \in I\!R, \\
&\text{(c)} \quad \xi^T I\!K(v)\xi \geq \alpha\|\xi\|^2, \; \alpha > 0, \; \forall\, v \in I\!R, \; \forall\, \xi \in I\!R^2,
\end{aligned} \qquad (14)$$

where $\|\cdot\|_\infty$ represents the l^∞-matrix norm, i.e., $\|I\!K\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |k_{ij}|$.

(A2) The weak solution u is sufficiently regular, namely

$$\begin{aligned}
&\text{(a)} \quad u \in L^2(0, T; H^{p+1}(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0, T; H^p(\Omega)), \; p \geq 1 \\
&\text{(b)} \quad \|\nabla u(t)\|_{L^\infty(\Omega)} \leq C_D \quad \text{for a.a. } t \in (0, T),
\end{aligned} \qquad (15)$$

where $p \geq 1$ denotes the given degree of the polynomial approximation.

(A3) The triangulations $\mathcal{T}_h$, $h \in (0, h_0)$ are *locally quasi-uniform* and *shape-regular* (for detailed definitions see [4]).

Now, we are ready to formulate the main result of this paper.

Theorem *Let assumptions (A1) be satisfied, let u be the exact solution of the continuous problem satisfying (A2). Let $\mathcal{T}_h$, $h \in (0, h_0)$ be a family of triangulations satisfying (A3) and let the numerical flux H from (10) be consistent, conservative* *and Lipschitz continuous. Let $u_h \in S_{hp}$ be the solution of the discrete problem given by (8). Then the discretization error $e_h = u_h - u$ satisfies*

$$\max_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^T |||e_h(\vartheta)|||^2 \mathrm{d}\vartheta \le Ch^{2p}, \tag{16}$$

where $|||v|||^2 := \sum_{K\in\mathcal{T}_h} |v|^2_{H^1(K)} + J_h^\sigma(v,v)$ and $C > 0$ is a constant independent of h.

***Sketch of the proof*:** Let $u \in H^{p+1}(\Omega,\mathcal{T}_h)$ be the solution of the continuous problem. For $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the L^2-projection of v on S_{hp}. We set $\xi(t) = u_h(t) - \Pi_h u(t) \in S_{hp}$, $\eta(t) = \Pi_h u(t) - u(t)$, $e_h(t) = u_h(t) - u(t) = \xi(t) + \eta(t)$ for a.a. $t \in (0,T)$. We subtract (13) from (8), set $v_h := \xi$ and add terms $-a_h^\Theta(\Pi_h u,\xi) + \ell_h^\Theta(\Pi_h u,\xi)$ on both sides of this identity. Then we obtain

$$\begin{aligned}
&\left(\frac{\partial\xi}{\partial t},\xi\right) + \underbrace{a_h^\Theta(u_h(t),\xi) - a_h^\Theta(\Pi_h u,\xi) + \ell_h^\Theta(\Pi_h u,\xi) - \ell_h^\Theta(u_h,\xi) + \alpha J_h^\sigma(\xi,\xi)}_{=:\chi_1}\\
=\ & \underbrace{-\left(\frac{\partial\eta}{\partial t},\xi\right) + b_h(u,\xi) - b_h(u_h,\xi) - \alpha J_h^\sigma(\eta,\xi)}_{=:\chi_2} \qquad (17)\\
&+ \underbrace{a_h^\Theta(u,\xi) - a_h^\Theta(\Pi_h u,\xi) + \ell_h^\Theta(\Pi_h u,\xi) - \ell_h^\Theta(u,\xi)}_{=:\chi_3}.
\end{aligned}$$

With the aid of the *multiplicative trace inequality*, *inverse inequality* and *approximation properties of the space S_{hp}*, (see [7, Lemmas 4.2–4.4]), we estimate terms χ_1, χ_2 and χ_3:

$$|\chi_2| \le C\Big\{h^p|\partial u/\partial t|_{H^p(\Omega)}\|\xi\|_{L^2(\Omega)} + |||\xi|||(h^{p+1}|u|_{H^{p+1}(\Omega)} + \alpha h^p|u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)})\Big\}. \tag{18}$$

Analogously as in [9] we derive

$$|\chi_3| \le C\Big(C_U h^p|u|_{H^{p+1}(\Omega)} + C_D C_L h^{p+1}|u|_{H^{p+1}(\Omega)}\Big)|||\xi|||. \tag{19}$$

Finally, by a particular choice of the constant C_W we obtain

$$\chi_1 \ge \frac{\alpha}{2}|||\xi|||^2 - C\left(C_U h^p|u|_{H^{p+1}(\Omega)} + C_D C_L\|\xi\|_{L^2(\Omega)}\right)|||\xi|||. \tag{20}$$

In consequence, we use inequalities (18)–(20) in identity (17), apply Young's inequality, and integrate from 0 to t, $t \in [0,T]$. Finally application of the Gronwall's lemma leads to desirable error estimate (16). □

5. Conclusion

We presented a space semi-discretization of a scalar nonstationary convection-diffusion equation (with nonlinear convection as well as diffusion) with the aid of SIPG, IIPG and NIPG variants of DGFEM. We presented a priori error estimates which are optimal in the $L^2(0,T,H^1(\Omega))$-seminorm but suboptimal in the $L^\infty(0,T,L^2(\Omega))$-norm.

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