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DISCONTINUOUS GALERKIN METHOD FOR THE SIMULATION OF 3D VISCOUS COMPRESSIBLE FLOWS[*]

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1. Introduction

Our goal is to solve an unsteady viscous compressible flow which is described by the system of the *Navier-Stokes equations*. Our aim is to develop a sufficiently efficient, robust and accurate numerical method.

It is promising to use the *discontinuous Galerkin method* (DGM), which is based on a piecewise polynomial but discontinuous approximation, which is suitable for problems with discontinuities. We prefer the *discontinuous Galerkin finite element* (DGFE) method with *symmetric, nonsymmetric and/or incomplete* variant of stabilization and *interior* and *boundary penalty terms*. These schemes are usually denoted as SIPG, NIPG and IIPG, respectively.

The most usual approach for the time discretization is method of lines. Explicit methods such as the Runge-Kutta methods are very popular for their simplicity and a high order of accuracy. However, they suffer from strong time step restrictions. Fully implicit schemes lead to a system of highly nonlinear algebraic equations at each time step whose solution is complicated. To avoid this disadvantage we employ a semi-implicit method for the time discretization, which is based on a suitable linearization of the fluxes. The linear terms are treated implicitly and the nonlinear ones explicitly.

2. Compressible flow problem

For the description of motion of a viscous compressible flow we use the system of Navier-Stokes equations.

Let $\Omega \subset I\!\!R^3$ be a bounded domain and $T > 0$. We set $Q_T = \Omega \times (0, T)$ and by $\partial\Omega$ we denote the boundary of Ω which consists of several disjoint parts. We distinguish inlet Γ_I, outlet Γ_O and impermeable walls Γ_W on $\partial\Omega$. Using the Fourier law and some relations from physics, we can write these equations in the dimensionless form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \vec{f}(\boldsymbol{w}) = \sum_{s=1}^{3} \frac{\partial}{\partial x_s} \left(\sum_{k=1}^{3} \boldsymbol{K}_{sk}(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x_k} \right) \quad \text{in } Q_T, \tag{1}$$

where

$$\boldsymbol{w} = (w_1, \ldots, w_5)^{\mathrm{T}} = (\rho,\ \rho v_1,\ \rho v_2, \rho v_3,\ e)^{\mathrm{T}} \tag{2}$$

is the so-called *state vector*, $\vec{f} = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$,

$$
\begin{aligned}
\boldsymbol{f}_s(\boldsymbol{w}) &= (f_s^{(1)}(\boldsymbol{w}), \ldots, f_s^{(5)}(\boldsymbol{w}))^{\mathrm{T}} \\
&= (\rho v_s,\ \rho v_s v_1 + \delta_{s1} p,\ \rho v_s v_2 + \delta_{s2} p,\ \rho v_s v_3 + \delta_{s3} p,\ (e + p)\, v_s)^{\mathrm{T}},\ s = 1, 2, 3,
\end{aligned} \tag{3}
$$

are the so-called *inviscid (Euler) fluxes*, where ρ, p, and e stand for the density, the pressure, and the total energy, respectively, and δ is the Kronecker's delta. For description of the matrix $\boldsymbol{K}_{sk}(\boldsymbol{w}) : I\!\!R^5 \to I\!\!R^5 \times I\!\!R^5,\ s, k = 1, 2, 3$ see [6].

In order to close the system we use the following thermodynamical relations: *the state equation for perfect gas* and *the relation for the total energy*. The system is of *hyperbolic-parabolic* type. It is equipped with initial and boundary conditions. For more details see [1].

2.1. Properties of inviscid fluxes

From the expression of the Euler fluxes $\boldsymbol{f}_s,\ s = 1, 2, 3$ we find that $\boldsymbol{f}_s$ can be written (see [1]) in the form

$$\boldsymbol{f}_s(\boldsymbol{w}) = \boldsymbol{A}_s(\boldsymbol{w})\boldsymbol{w}, \quad s = 1, 2, 3, \tag{4}$$

where $\boldsymbol{A}_s(\boldsymbol{w})$ are the Jacobi matrices of the mappings $\boldsymbol{f}_s$.

3. Discretization

For discretization we employ the *discontinuous Galerkin finite element method* (DGFEM), which takes advantages from finite element method as well as from finite volume method. DGFEM is based on piecewise polynomial approximation without any requirement on interelement continuity what is suitable for problems where shock waves and contact discontinuities appear.

Let $\mathcal{T}_h$ $(h > 0)$ be a partition of the domain Ω into a finite number of open three-dimensional mutually disjoint simplexes and/or parallelograms K i.e., $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$. We call $\mathcal{T}_h$ a *triangulation* of Ω and do not require the conforming properties from the finite element method. We define the set of faces $\mathcal{F}_h$, $\mathcal{F}_h^D$, $\mathcal{F}_h^{ID}$ and a unit normal vector $\boldsymbol{n}_\Gamma$, as can be seen in [7].

Over the triangulation $\mathcal{T}_h$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K) \ \forall\, K \in \mathcal{T}_h\}, \tag{5}$$

where $H^k(K) = W^{k,2}(K)$ denotes the (classical) Sobolev space on element K.

We introduce the following notation $v|_\Gamma$, $\langle v \rangle_\Gamma$ and $[v]_\Gamma$ for trace, mean value and jump, respectively, of function v over the edge Γ, see [2].

There are several variant of DGFEM. A particular role is played by the *symmetric and nonsymmetric interior penalty Galerkin* variant, denoted by SIPG and NIPG,

respectively. The main idea of SIPG and NIPG is to append artificial integral to each boundary integral

$$\int_{\Gamma} \langle \nabla u \cdot \vec{n} \rangle [\varphi] \, \mathrm{d}S, \qquad \nabla u \in [L^2(\Gamma)]^3, \ \varphi \in L^2(\Gamma), \tag{6}$$

arising from the use of Green's theorem in the case of linear diffusion simply by formal exchange of u and φ. We can see that this integrals vanish in case of regular solution. In our case we use the linearization formed by the terms $\boldsymbol{K}_{sk}$, see [6], or employ the so-called *incomplete interior penalty Galerkin* (IIPG) method, see [7].

Similarly, as in [3] for $\boldsymbol{w}, \boldsymbol{\varphi} \in [H^2(\Omega, \mathcal{T}_h)]^5$, we define the forms:

$$
\begin{aligned}
\tilde{\boldsymbol{a}}_h(\boldsymbol{w}, \boldsymbol{\varphi}) \ = \ & \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{3} \left(\sum_{k=1}^{3} \left(\boldsymbol{K}_{sk}(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x_k} \right) \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \right) \mathrm{d}x \\
& - \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_{\Gamma} \sum_{s=1}^{3} \left(\langle \sum_{k=1}^{3} \boldsymbol{K}_{sk}(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x_k} \rangle_{\Gamma} n_s \right) \cdot [\boldsymbol{\varphi}]_{\Gamma} \, \mathrm{d}S \\
& - \ \Theta \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_{\Gamma} \sum_{s=1}^{3} \left(\langle \sum_{k=1}^{3} \boldsymbol{K}_{sk}(\boldsymbol{w}) \frac{\partial \boldsymbol{\varphi}}{\partial x_k} \rangle_{\Gamma} n_s \right) \cdot [\boldsymbol{w}]_{\Gamma} \, \mathrm{d}S \qquad (7) \\
& + \ \Theta \sum_{\Gamma \in \mathcal{F}_h^{D}} \int_{\Gamma} \sum_{s=1}^{3} \left((\sum_{k=1}^{3} \boldsymbol{K}_{sk}(\boldsymbol{w}) \frac{\partial \boldsymbol{\varphi}}{\partial x_k}) n_s \right) \cdot \boldsymbol{w}_B(t) \, \mathrm{d}S, \\
\bar{\boldsymbol{b}}_h(\boldsymbol{w}, \boldsymbol{\varphi}) \ = \ & - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{3} \boldsymbol{f}_s(\boldsymbol{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, \mathrm{d}x \\
& + \sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} H \left(\boldsymbol{w}|_{\Gamma}^{(p)}, \boldsymbol{w}|_{\Gamma}^{(n)}, \vec{n}_{\Gamma} \right) [\boldsymbol{\varphi}]_{\Gamma} \, \mathrm{d}S, \qquad (8) \\
J_h^{\sigma}(\boldsymbol{w}, \boldsymbol{\varphi}) \ = \ & \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_{\Gamma} \sigma [\boldsymbol{w}]_{\Gamma} \cdot [\boldsymbol{\varphi}]_{\Gamma} \, \mathrm{d}S - \sum_{\Gamma \in \mathcal{F}_h^{D}} \int_{\Gamma} \sigma \, \boldsymbol{w}_B(t) \cdot \boldsymbol{\varphi} \, \mathrm{d}S, \qquad (9)
\end{aligned}
$$

where Θ is +1 in SIPG, -1 in NIPG and 0 in IIPG case and σ is a suitable coercive parameter and $\boldsymbol{w}_B(t)$ is the solution on the boundary, where Dirichlet condition is prescribed. $H \left(\boldsymbol{w}|_{\Gamma}^{(p)}, \boldsymbol{w}|_{\Gamma}^{(n)}, \vec{n}_{\Gamma} \right)$ is the so-called *numerical flux*, well-known in the finite volume method (see, e.g., [1, Section 3.2])

Now we can introduce the *semidiscrete* problem. The approximate solution of problem (1) with initial and boundary condition is sought at each instant time t in the space of discontinuous piecewise polynomial functions $\boldsymbol{S}_h$ defined by $\boldsymbol{S}_h \equiv [S_h]^5$, $S_h \equiv \{v; v|_K \in P^p(K) \ \forall K \in \mathcal{T}_h\}$, where p is a positive integer and $P^p(K)$ denotes the space of all polynomials on K of degree at most p.

In order to avoid the time step restriction and nonlinearity of the discretized problem, we carry out a linearization of the nonlinear forms $\tilde{\boldsymbol{a}}_h$ and $\bar{\boldsymbol{b}}_h$.

For $\bar{\boldsymbol{w}}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$ we define a new form $\boldsymbol{b}_h(\bar{\boldsymbol{w}}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h)$ using (4) and (8), which is linear with respect to the second and the third variable and consistent

with $\bar{\boldsymbol{b}}_h(\cdot, \cdot)$ by $\bar{\boldsymbol{b}}_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = \boldsymbol{b}_h(\boldsymbol{w}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h) \quad \forall \, \boldsymbol{w}_h, \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$. For more details see, e.g. [3].

In a similar way, as in the case of the form $\bar{\boldsymbol{b}}_h$, we define a new form $\boldsymbol{a}_h(\bar{\boldsymbol{w}}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h)$ for $\bar{\boldsymbol{w}}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$ using properties of the forms $\boldsymbol{K}_{sk}$ and (7), which is also linear with respect to its second and third variable. Moreover, it is consistent with $\tilde{\boldsymbol{a}}_h(\cdot, \cdot)$ by $\tilde{\boldsymbol{a}}_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = \boldsymbol{a}_h(\boldsymbol{w}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h) \quad \forall \, \boldsymbol{w}_h, \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$. The definition of $\boldsymbol{a}_h(\cdot, \cdot, \cdot)$ can be found in [3].

Now we introduce the full space-time discrete problem. The main idea of the semi-implicit discretization is to treat the linear parts of forms $\boldsymbol{a}_h$ and $\boldsymbol{b}_h$ implicitly and their nonlinear parts explicitly. In order to obtain a sufficiently accurate approximation with respect to the time coordinate we use the so-called *backward difference formula* (BDF) for the solution of the ODE semidiscrete problem. Moreover, a suitable explicit higher order extrapolation is used in the nonlinear parts of $\boldsymbol{a}_h$ and $\boldsymbol{b}_h$.

Let $0 = t_0 < t_1 < \ldots < t_r = T$ be a partition of the interval $(0, T)$ and let $\tau_k \equiv t_{k+1} - t_k, \ k = 0, 1, \ldots, r-1$, be the time steps.

Definition 1 *Functions* $\boldsymbol{w}_h^{k+1}, \ k = 0, \ldots, r-1$ *are an approximate solution of problem (1) with some suitable initial and boundary conditions satisfying*

(a) $\quad \boldsymbol{w}_h^{k+1} \in \boldsymbol{S}_h$,

(b) $\quad \dfrac{1}{\tau_k} \left(\displaystyle\sum_{l=0}^{n} \alpha_l \boldsymbol{w}_h^{k+1-l}, \boldsymbol{\varphi}_h \right) + \boldsymbol{a}_h \left(\displaystyle\sum_{l=1}^{n} \beta_l \boldsymbol{w}_h^{k+1-l}, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h \right)$

$\qquad + \boldsymbol{b}_h \left(\displaystyle\sum_{l=1}^{n} \beta_l \boldsymbol{w}_h^{k+1-l}, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h \right) + \boldsymbol{J}_h \left(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h \right) = 0 \qquad (10)$

$\qquad \forall \, \boldsymbol{\varphi}_h \in \boldsymbol{S}_h, \ k = n-1, \ldots, r-1$,

(c) $\quad \boldsymbol{w}_h^0$ *is an* $\boldsymbol{S}_h$ *approximation of initial condition* $\boldsymbol{w}^0$,

(d) $\quad \boldsymbol{w}_h^l \in \boldsymbol{S}_h, \ l = 1, \ldots, n-1$ *are given by a suitable one-step method,*

where $n \geq 1$ *is the degree of the BDF scheme, the coefficients* $\alpha_l, \ l = 0, \ldots, n$, *and* $\beta_l, \ l = 1, \ldots, n$, *depend on time steps* $\tau_{k-l}, \ l = 0, \ldots, n$.

The problem (10), (a)–(d) represents a system of linear algebraic equations for each $k = n-1, \ldots, r-1$, which is solved by a suitable iterative solver (e.g. GMRES).

4. Stabilization

Application of this numerical scheme to transsonic flow leads to spurious overshoots and undershoots in computed quantities near shock waves. We use the stabilization of the scheme similar to [5]. For each $K_i \in \mathcal{T}_h$ we define quantity $g_{K_i}(\boldsymbol{w}_h)$, what measures the interelement jump of the function ρ_h, what is piecewise polynomial approximation of ρ. Moreover, we define the forms d_h and J_h which represent artificial viscosity and interior penalty, respectively. Both terms vanish in region where $\boldsymbol{w}_h$ is smooth. In [5], the stabilization for 2D problems is derived. The choice of exponents and another changes in forms d_h and J_h for 3D is in development.

5. Adaptive time step

In order to achieve a steady-state solution in an efficient way it is necessary to adapt the time step during the computational process. In [4], an adaptive choice of the time step based on a comparison of two BDF formulae was presented. However, this approach does not seem to be very efficient for viscous compressible flow simulations. Therefore, we employ an heuristic choice of the time step which is based on a idea to increase the time step when the "steady-state residuum" is decreasing. Hence we put: $\tau_k = \sqrt{\frac{const}{||\boldsymbol{w}_k - \boldsymbol{w}_{k-1}||}}$, where $const$ is a suitable constant (e.g. 10^{-6}).

6. Implementation

The subject of this research is a part of the project ADIGMA supported by the European Commission. This project holds in the period 2006–2009 and is devoted to development, application and verification of higher order schemes for the simulation of viscous compressible flow.

The presented numerical method is now being implemented within the object oriented platform COOLFluid, developed at the Von Karman Institute in Brussel, see [8]. The main advantage of object oriented programing is in dynamic creation
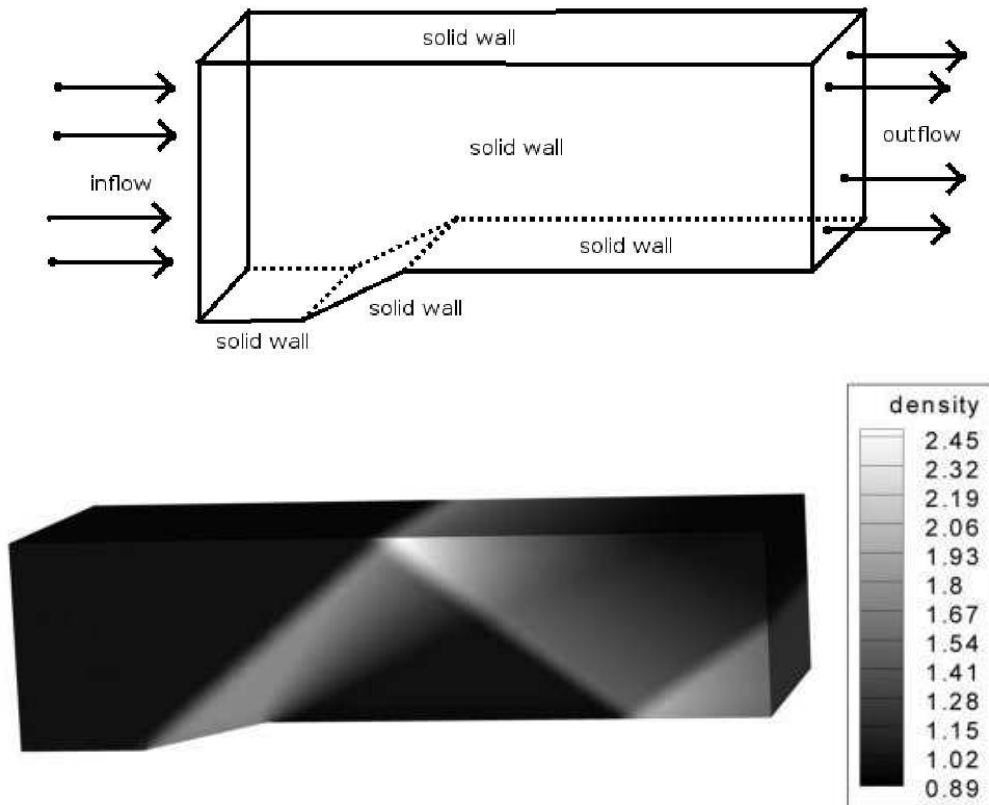
Fig. 1: *Distribution of density.*

of the object. There are templates of methods for discretization in time and space, for solving the system of equations, for any type of elements in 2D and 3D. But in the process of computation there are created only objects which are really needed for computation. It takes some time at the beginning for creation and initialization of objects, but the computer code is then shorter and simpler to write.

7. Example – Wedge 3D

Wedge 3D is one of COOLFluiD test cases. It is supersonic flow (MACH = 2.0) in channel forward facing oblique step. The initial condition is constant with values $\rho = 1.0, v = (2.366431913, 0.0, 0.0), e = 5.3$.

8. Conclusion

We described a numerical solution of the compressible Navier-Stokes equations by a combination of DGFEM and BDF. We presented SIPG, NIPG and IIPG variants of DGFEM. These schemes are theoretically unconditionally stable, have a high order of approximation with respect to space and time and lead to a linear algebraic systems at each time step.

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