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ON THE LONGEST-EDGE BISECTION ALGORITHM*

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There are many methods for refining finite element simplicial partitions in R^d, $d \in \{2, 3, \dots\}$. One of them is the longest-edge bisection algorithm. It is very popular for its simplicity, especially in the three-dimensional space. It chooses the longest edge in a given simplicial partition. Dividing this edge by its midpoint, we can define a locally refined partition by simplices that surround this midpoint. Repeating this process, we obtain a family of nested face-to-face partitions (see Figures 1, 2, and 4). This approach is much simpler (especially for $d > 2$) than the standard local refinement of simplicial partitions that uses red and green subdivisions (see, e.g., [3], [7]). Note that this family is never uniquely defined, since during the refinement process there appear many new edges having the same length due to the bisections. For instance, the last but one bisection in Figure 1 is not uniquely determined.

There is an extensive literature devoted to numerical analysis of the longest-edge bisection algorithm, see [1]–[20]. For instance, Rosenberg and Stenger [16] for $d = 2$ show that angles of triangles do not tend to zero for infinitely many steps of a bisection algorithm. A somewhat stronger result has been achieved by M. Stynes [20] who showed that the repeated bisection process yields only a finite number of similarity-distinct subtriangles. This number is bounded when the discretization parameter h tends to zero. However, Stynes admits the so-called hanging nodes which do not appear in face-to-face partitions considered in this paper.

Without loss of generality we can analyse the longest-edge bisection algorithm only for one simplex from a given initial simplicial partition. In Figures 1 and 2, we observe subsequent partitions of a triangle and a tetrahedron by the longest-edge bisection algorithm.

The worst case from the point of degeneracy happens when the regular simplex is bisected (see [5]). For instance, for the equilateral triangle, the minimal angle is halved. On the other hand, this situation does not occur while bisecting obtuse and right triangles. In the next theorem we prove that in this case the minimal angle does not change.

Theorem 1. *Let α be the smallest angle of a nonacute triangle. Bisecting the longest edge determines two triangles whose all angles are not less than α.*

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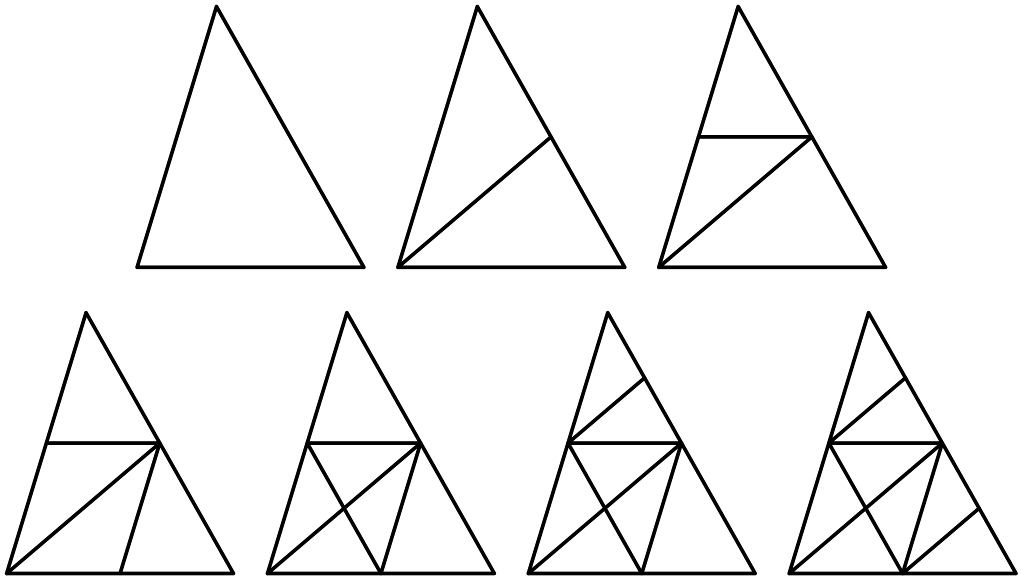


Fig. 1:

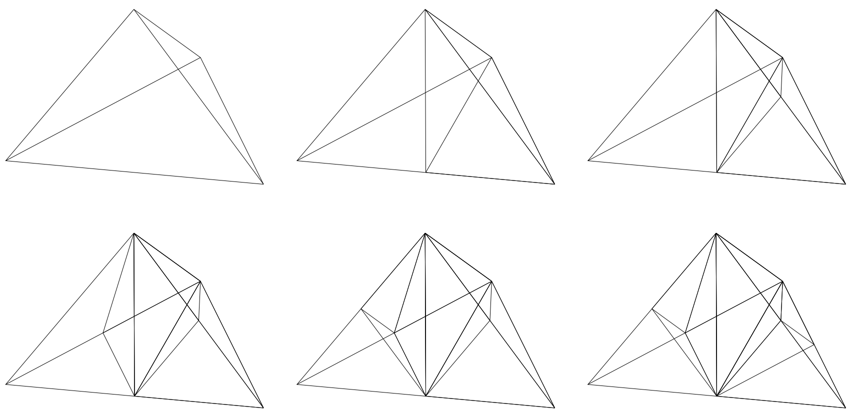


Fig. 2:

P r o o f . Let a nonacute triangle be given. Denote its angles so that

$$\alpha \leq \beta \leq \frac{\pi}{2} \leq \gamma \tag{1}$$

and let

$$a \leq b \leq c \tag{2}$$

be the associated edges.

Now bisect the triangle by the median t to the longest edge c. Denote the new angles by $\alpha_1, \beta_1, \gamma_1$, and γ_2 as illustrated in Figure 3. We show that all these angles are not less than α.

By the Cosine theorem we see that

$$\begin{aligned} a^2 &= t^2 + \left(\frac{c}{2}\right)^2 - tc\cos\alpha_1, \\ b^2 &= t^2 + \left(\frac{c}{2}\right)^2 - tc\cos\beta_1. \end{aligned}$$

From this and (2) we find that $\cos\alpha_1 \geq \cos\beta_1$. Since $\alpha_1 + \beta_1 = \pi$ and the function cos is decreasing on the whole interval $[0, \pi]$, we have

$$\alpha_1 \leq \frac{\pi}{2} \leq \beta_1. \tag{3}$$

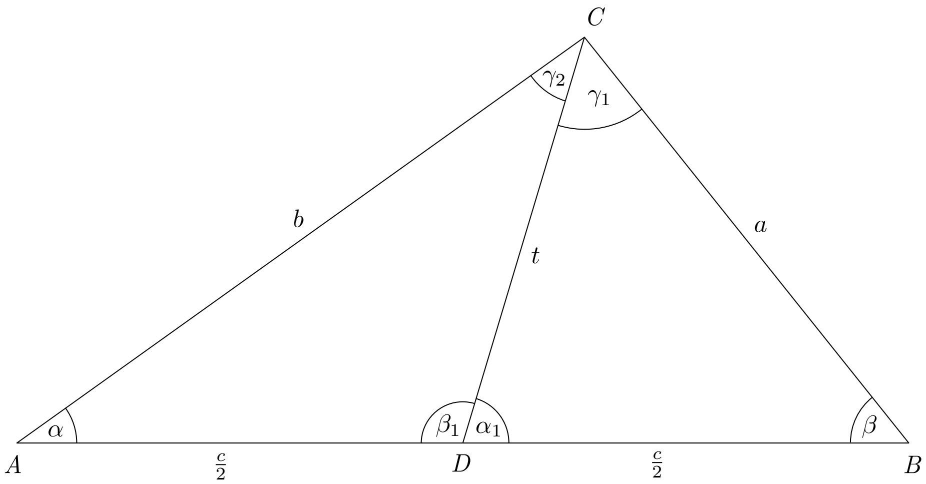


Fig. 3:

Denote vertices of the original triangle ABC as marked in Figure 3. Let D be the midpoint of the segment AB and let C' be such a point that D is the midpoint of the segment CC', i.e., $ACBC'$ is a parallelogram. Using the triangle inequality for the triangle ACC' and relation (2), we get $2t < a + b \leq 2b$, i.e.,

$$t < b.$$

From this and the Sine theorem we obtain

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} < \frac{\sin\beta}{t} = \frac{\sin\alpha_1}{a},$$

which implies that

$$\alpha \leq \alpha_1. \tag{4}$$

Finally, by (1) we know that $\gamma \geq \frac{\pi}{2}$, and therefore, $t \leq \frac{c}{2}$. Using again the Sine theorem, we come to

$$\alpha \leq \gamma_2, \quad \beta \leq \gamma_1. \tag{5}$$

From this, (1), (3), and (4) the lemma follows. $\square$

Remark 1. It is $\gamma_2 \leq \gamma_1$, since by (5) and (1) we have

$$\frac{2\sin\gamma_2}{c} = \frac{\sin\alpha}{t} \leq \frac{\sin\beta}{t} = \frac{2\sin\gamma_1}{c}.$$

Remark 2. From the inequality

$$b \geq \frac{a+b}{2} > \frac{c}{2}$$

we observe that the edge b will be bisected in the next step.

Theorem 2. *Let α_0 be the minimum angle in a given triangulation. Then the longest-edge bisection algorithm yields the following lower bound for any angle α of refined triangles:*

$$\alpha \geq \frac{\alpha_0}{2}.$$

The proof is quite complicated and technical. It is based on some ideas from [16]. We see that for the equilateral triangle the above lower bound $\alpha_0/2$ is attainable. Let us point out that a similar theorem, which guarantees a nondegeneracy in $d = 3$, is still an open problem, even though all triangles on surfaces of all tetrahedra in the partition will be bisected in the same way as for $d = 2$.

Numerical tests. In Figure 4, we observe the initial triangulation and the result of the longest-edge bisection algorithm after 10 and 1000 refining steps.

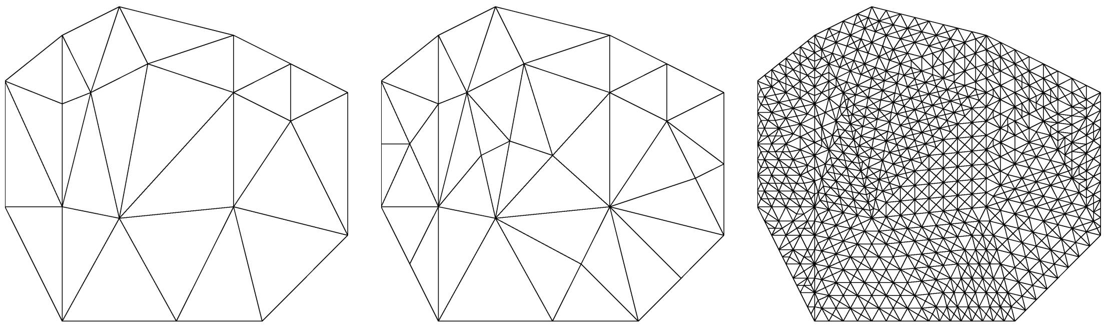
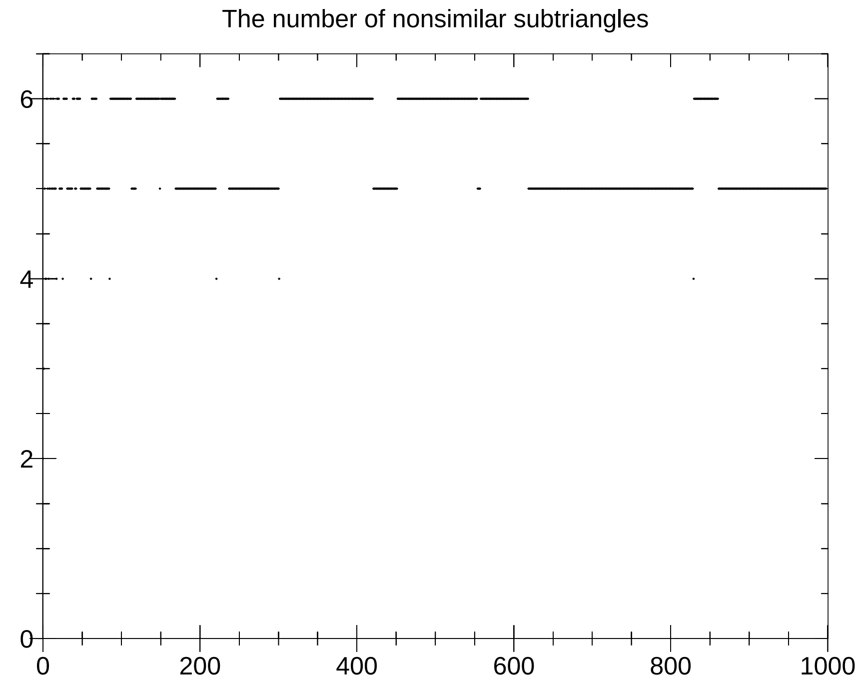


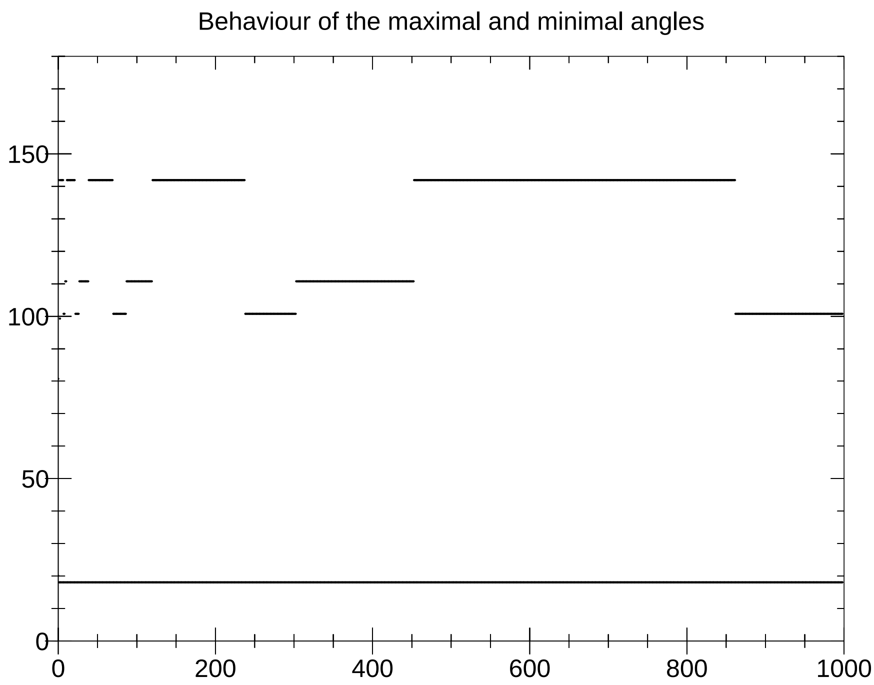
Fig. 4:

To illustrate that repeated bisection process yields only a finite number of similarity-distinct subtriangles, we have chosen the initial triangle with vertices (0,0), (10,0), and (9,3.2). Numerical results in Figure 5 indicate that this number is bounded when $h \to 0$ (cf. [20] for a different approach which produces hanging nodes, in general). In this test we performed 1000 bisections. In Figure 6 we observe values of the maximal and minimal angles from the interval $(0°, 180°)$ during the 1000 bisections. The minimal angle $\approx 18°$ does not change.



The number of nonsimilar subtriangles
6
4
2
0
0
200
400
600
800
1000

Fig. 5:



Behaviour of the maximal and minimal angles
150
100
50
0
0
200
400
600
800
1000

Fig. 6:

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