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TWO-SIDED A POSTERIORI ESTIMATES OF GLOBAL AND LOCAL ERRORS FOR LINEAR ELLIPTIC TYPE BOUNDARY VALUE PROBLEMS*

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Abstract

The paper is devoted to the problem of reliable control of accuracy of approximate solutions obtained in computer simulations. This task is strongly related to the so-called a posteriori error estimates, giving computable bounds for computational errors and detecting zones in the solution domain, where such errors are too large and certain mesh refinements should be performed. Mathematical model described by a linear elliptic (reaction-diffusion) equation with mixed boundary conditions is considered. We derive in a simple way two-sided (upper and lower) easily computable estimates for global (in terms of the energy norm) and local (in terms of linear functionals with local supports) control of the computational error, which is understood as the difference between the exact solution of the model and the approximation. Such two-sided estimates are completely independent of the numerical technique used to obtain approximations and can be made as close to the true errors as resources of a concrete computer used for computations allow.

Keywords: a posteriori error estimation, error control in energy norm, error control in terms of linear functionals, reaction-diffusion equation, mixed boundary conditions.

MSC: 65N15, 65N30

1. Introduction

Many physical and mechanical phenomena can be described by means of mathematical models presenting boundary value problems of elliptic type [7, 15]. Various numerical techniques (the finite difference method, the finite element method (FEM), the finite volume method etc.) are well developed for finding approximate solutions for such problems, see, e.g., [6]. However, in order to be practically meaningful, computer simulations always require an accuracy verification of computed approximations. Such a verification is the main purpose of a posteriori error estimation methods.

In the present paper, we recall two different ways of measuring the computational error, which is understood as the difference $u - \bar{u}$ between the exact solution u and approximation $\bar{u}$, in the global (energy) norm and in terms of linear bounded functionals. These two ways of measurement (and also of control – via a posteriori error

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estimation procedures) of the error are very natural and commonly used nowadays in both mathematical and engineering communities. The global error estimation (see [1, 2, 3, 4, 12, 13, 16, 18, 19, 25, 26] and references therein) normally gives a general presentation on the quality of approximation and a stopping criterion to terminate the calculations. However, practitioners are often interested not only in the value of the overall error, but also in errors over certain critical (and usually local) parts of the solution domain (for example, in fracture mechanics – see [23, 24] and references therein). This reason initiated another trend in a posteriori error estimation which is based on the concept of control of the computational error locally. One common way to perform such a control is to introduce a suitable linear functional ℓ related to subdomain of interest and to construct a posteriori computable estimate for $\ell(u - \bar{u})$, see [4, 5, 8, 11, 14, 20].

It is worth to mention here that most of estimates proposed so far strongly rely on the fact that the computed solutions are true finite element (FE) approximations which, in fact, rarely happens in real computations, e.g., due to quadrature rules, forcibly stopped iterative processes, various round-off errors, or even possible bugs in FE codes.

In this work, on the base of a model elliptic problem with mixed (Dirichlet/Neumann) boundary conditions, we present two relatively simple technologies for obtaining *computable guaranteed two-sided (upper and lower) a posteriori error estimates* needed for reliable control in both global (in the energy norm) and local (in terms of linear functionals) ways. The estimates derived are valid for *any conforming approximations* independently of numerical methods used to obtain them, and can be made *arbitrarily close* to the true errors. In real-life calculations this closeness only depends on resources of a concrete computer used. Some variant of the present paper was published as a preprint [9] in February 2006 (see also [10]).

2. Formulation of problem

For standard definitions of functional spaces and finite element terminology used in the paper we refer to [6].

2.1. Model problem

We introduce the model elliptic problem which consists of the governing equation (1) and mixed (Dirichlet/Neumann) boundary conditions (2)–(3): Find a function u such that

$$-\mathrm{div}(A\nabla u) + cu = f \quad \text{in } \Omega, \tag{1}$$

$$u = u_0 \quad \text{on } \Gamma_D, \tag{2}$$

$$\nu^T \cdot A\nabla u = g \quad \text{on } \Gamma_N, \tag{3}$$

where Ω is a bounded domain in $\mathbf{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$, such that $\overline{\partial\Omega} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\mathrm{meas}_{d-1}\,\Gamma_D > 0$ and ν is the outward normal to the boundary.

It is common practice to pose problem (1)–(3) in the so-called weak form: Find $u \in u_0 + H^1_{\Gamma_D}(\Omega)$ such that

$$\int_\Omega A\nabla u \cdot \nabla w \, dx + \int_\Omega cuw \, dx = \int_\Omega fw \, dx + \int_{\Gamma_N} gw \, ds \qquad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{4}$$

where

$$H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}.$$

For the purposes of the weak formulation, we assume, that $f \in L_2(\Omega)$, $u_0 \in H^1(\Omega)$, $g \in L_2(\Gamma_N)$, $c \in L_\infty(\Omega)$, the coefficient matrix A is symmetric, with entries $a_{ij} \in L_\infty(\Omega)$, $i, j = 1, \ldots, d$, and is such that

$$C_2|\xi|^2 \geq A(x)\xi \cdot \xi \geq C_1|\xi|^2 \qquad \forall \xi \in \mathbf{R}^d \quad \text{for a.e. } x \in \Omega. \tag{5}$$

In addition, the coefficient c is assumed to be either zero or bounded away from zero by a positive constant c_0, i.e. $c \equiv 0$ in $\Omega \setminus \overline{\Omega^c}$, where

$$\Omega^c := \text{supp } c = \{x \in \Omega \mid c(x) \geq c_0 > 0\}. \tag{6}$$

If we define bilinear form $a(\cdot,\cdot)$ and linear form $F(\cdot)$ as follows

$$\begin{aligned} a(v, w) &:= \int_\Omega A\nabla v \cdot \nabla w \, dx + \int_\Omega cvw \, dx, \quad v, w \in H^1(\Omega), \\ F(w) &:= \int_\Omega fw \, dx + \int_{\Gamma_N} gw \, ds, \quad w \in H^1(\Omega), \end{aligned}$$

then weak formulation (4) can be written in a short form: Find $u = u_0 + u^*$, where $u^* \in H^1_{\Gamma_D}(\Omega)$, such that $a(u, w) = F(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega)$.

Remark 2.1 *The weak solution defined by* (4) *exists and is unique in view of well-known Lax-Milgram lemma (see, e.g.,* [6]*).*

The so-called *energy functional* J of problem (4) is defined as follows

$$J(w) := \frac{1}{2}a(w, w) - \bar{F}(w), \qquad w \in H^1(\Omega), \tag{7}$$

where $\bar{F}(w) := F(w) - a(u_0, w)$, and the corresponding *energy norm* is defined as $\sqrt{a(\cdot,\cdot)}$.

Remark 2.2 *It is well-known that problem* (4) *(namely, finding the function u^*) is equivalent to the problem of finding the minimizer (which is equal to u^*) of the energy functional* (7) *over the space $H^1_{\Gamma_D}(\Omega)$.*

2.2. Types of error control

Let $\bar{u} = u_0 + \bar{u}^*$ be *any function* from $u_0 + H^1_{\Gamma_D}(\Omega)$ (e.g., computed by some numerical method) considered as an approximation of u. It is a natural practice to measure the overall accuracy of the approximation $\bar{u}$ in terms of the above-defined energy norm. Thus, our first goal is to construct reliable and easily computable two-sided estimates for controlling the following value

$$a(u-\bar{u}, u-\bar{u}) = \int_\Omega A\nabla(u-\bar{u}) \cdot \nabla(u-\bar{u})\, dx + \int_\Omega c(u-\bar{u})^2\, dx. \tag{8}$$

The second type of error control considered in the paper is two-sided estimation of the value of the difference $u - \bar{u}$ in terms of a linear bounded functional ℓ

$$\ell(u-\bar{u}). \tag{9}$$

Remark 2.3 *It is clear that existence of an estimate for* (9) *also allows to estimate the value $\ell(u)$ (often called quantity of interest or goal-oriented quantity* [1]*). Really, $\ell(u) = \ell(u-\bar{u}) + \ell(\bar{u})$ where $\ell(\bar{u})$ is computable and $\ell(u-\bar{u})$ is estimated. The value of $\ell(u)$ can be sometimes more important to know than the solution u itself (see* [23, 24]*).*

Remark 2.4 *If the functional ℓ in* (9) *is defined as some integral over small subdomain (or line) in $\overline{\Omega}$, then reliable two-sided estimation of $\ell(u-\bar{u})$ helps to control the behaviour of the error $u-\bar{u}$ locally in that subdomain (or over the line). For example, one can be interested in estimation of $\ell(u-\bar{u}) = \int_S \varphi(u-\bar{u})\, dx$ with S be a subdomain in Ω or a line in Γ_N (where the solution is also unknown), see* [11] *for more details and numerical results in this respect.*

2.3. Inequalities and constants

In what follows we shall need the Friedrichs inequality

$$\|w\|_{0,\Omega} \le C_{\Omega,\Gamma_D} \|\nabla w\|_{0,\Omega} \quad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{10}$$

and the inequality in the trace theorem

$$\|w\|_{0,\partial\Omega} \le C_{\partial\Omega} \|w\|_{1,\Omega} \quad \forall w \in H^1(\Omega), \tag{11}$$

where C_{Ω,Γ_D} and $C_{\partial\Omega}$ are positive constants, depending only on Ω, Γ_D, and $\partial\Omega$. The above used denotation $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{1,\Omega}$ stand for the standard norms in $L_2(\Omega)$ and $H^1(\Omega)$, respectively. The symbol $\|\cdot\|_{0,\partial\Omega}$ means the norm in $L_2(\partial\Omega)$. Proofs of inequalities (10) and (11) can be found, e.g., in [17].

3. Two-sided estimates of error in energy norm

In this section we shall employ the denotation χ_S for a characteristic function of set S, i.e., $\chi_S(x) = 1$ if $x \in S$, and $\chi_S(x) = 0$ if $x \notin S$. We also define $|\!|\!|\mathbf{y}|\!|\!|_\Omega := \sqrt{\int_\Omega A\mathbf{y} \cdot \mathbf{y}\, dx}$ for $\mathbf{y} \in L_2(\Omega, \mathbf{R}^d)$.

3.1. Upper estimate

Proposition 3.1 *For the error in the energy norm* (8) *we have the following upper estimate*

$$a(u-\bar u,u-\bar u) \le \Big\| \frac{1}{\sqrt{c}}(f+\operatorname{div}\,\mathbf{y}^*-c\bar u)\Big\|^2_{0,\Omega^c} + \\ + (1+\alpha)|\!|\!|A^{-1}\mathbf{y}^*-\nabla\bar u|\!|\!|^2_\Omega + \Big(1+\frac{1}{\alpha}\Big)(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f+\operatorname{div}\mathbf{y}^*\|^2_{0,\Omega\setminus\overline{\Omega}^c} \\ + \Big(1+\frac{1}{\alpha}\Big)\Big(1+\frac{1}{\beta}\Big)C^2_{\Omega,\partial\Omega}\|g-\nu^T\cdot\mathbf{y}^*\|^2_{0,\Gamma_N}, \quad (12)$$

where α and β are arbitrary positive real numbers, $\mathbf{y}^*$ is any function from

$$H_N(\Omega,\operatorname{div}) := \big\{\mathbf{y}\in L_2(\Omega,\mathbf{R}^d)\mid \operatorname{div}\mathbf{y}\in L_2(\Omega),\ \nu^T\cdot\mathbf{y}\in L_2(\Gamma_N)\big\},$$

and $C_{\Omega,\partial\Omega} := C_{\partial\Omega}\sqrt{1+C^2_{\Omega,\Gamma_D}}/\sqrt{C_1}$.

Proof: First of all, we notice that it actually holds, cf. (6),

$$a(u-\bar u,u-\bar u) = |\!|\!|\nabla(u-\bar u)|\!|\!|^2_\Omega + \|\sqrt{c}(u-\bar u)\|^2_{0,\Omega^c}. \quad (13)$$

Further, using the fact that $u-\bar u\in H^1_{\Gamma_D}(\Omega)$ and identity (4) we observe that

$$a(u-\bar u,u-\bar u) = \int_\Omega f(u-\bar u)dx + \int_{\Gamma_N} g(u-\bar u)\,ds - \int_\Omega A\nabla\bar u\cdot\nabla\,(u-\bar u)\,dx \\ - \int_\Omega c\bar u(u-\bar u)\,dx = \int_\Omega (f-c\bar u)(u-\bar u)\ dx + \int_{\Gamma_N} g(u-\bar u)\,ds \\ - \int_\Omega (A\nabla\bar u-\mathbf{y}^*)\cdot\nabla(u-\bar u)\ dx - \int_\Omega \mathbf{y}^*\cdot\nabla(u-\bar u)\ dx, \quad (14)$$

where $\mathbf{y}^*$ is any function from the space $H_N(\Omega,\operatorname{div})$ defined in the formulation of the theorem. Applying the Green's formula to the last term in above gives

$$\int_\Omega \mathbf{y}^*\cdot\nabla(u-\bar u)\ dx = \int_{\Gamma_N}(\nu^T\cdot\mathbf{y}^*)(u-\bar u)\,ds - \int_\Omega \operatorname{div}\mathbf{y}^*(u-\bar u)\ dx.$$

Using this identity and equation (14) we obtain

$$a(u-\bar u,u-\bar u) = \int_\Omega A(A^{-1}\mathbf{y}^*-\nabla\bar u)\cdot\nabla(u-\bar u)\ dx + \int_\Omega (f+\operatorname{div}\mathbf{y}^*-c\bar u)(u-\bar u)\ dx \\ + \int_{\Gamma_N}(g-\nu^T\cdot\mathbf{y}^*)(u-\bar u)\,ds. \quad (15)$$

Now, we proceed by estimating the three terms in the right-hand side (RHS) of equality (15). The first term can be estimated by the Cauchy-Schwarz inequality as

$$\int_\Omega A(A^{-1}\mathbf{y}^* - \nabla\bar{u}) \cdot \nabla(u-\bar{u})\, dx \le \| A^{-1}\mathbf{y}^* - \nabla\bar{u} \|_\Omega \, \| \nabla(u-\bar{u}) \|_\Omega. \tag{16}$$

The second term in the RHS of equality (15) can be estimated using Friedrichs inequality (10), ellipticity condition (5), denotation (6), and a simple inequality $a\,b \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ as follows

$$\begin{aligned}
&\int_\Omega (f + \operatorname{div}\mathbf{y}^* - c\bar{u})(u-\bar{u})\, dx \\
&= \int_{\Omega^c} \frac{1}{\sqrt{c}}(f + \operatorname{div}\mathbf{y}^* - c\bar{u})\, \sqrt{c}(u-\bar{u})\, dx + \int_\Omega \chi_{\Omega\setminus\overline{\Omega}^c}(f + \operatorname{div}\mathbf{y}^* - c\bar{u})\,(u-\bar{u})\, dx \\
&\le \|\sqrt{c}(u-\bar{u})\|_{0,\Omega^c} \left\| \frac{1}{\sqrt{c}}(f + \operatorname{div}\mathbf{y}^* - c\bar{u}) \right\|_{0,\Omega^c} \\
&\qquad\qquad + \|\chi_{\Omega\setminus\overline{\Omega}^c}(f + \operatorname{div}\mathbf{y}^* - c\bar{u})\|_{0,\Omega}\, \|u-\bar{u}\|_{0,\Omega} \\
&\le \frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c} + \frac{1}{2}\left\| \frac{1}{\sqrt{c}}(f + \operatorname{div}\mathbf{y}^* - c\bar{u}) \right\|^2_{0,\Omega^c} \\
&\qquad\qquad + \frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}} \|f + \operatorname{div}\mathbf{y}^* - c\bar{u}\|_{0,\Omega\setminus\overline{\Omega}^c} \| \nabla(u-\bar{u}) \|_\Omega.
\end{aligned} \tag{17}$$

Finally, the third term can be estimated using inequalities (10) and (11) and the ellipticity condition (5) as

$$\begin{aligned}
&\int_{\Gamma_N} (g - \nu^T \cdot \mathbf{y}^*)(u-\bar{u})\, ds \le \|g - \nu^T \cdot \mathbf{y}^*\|_{0,\Gamma_N} \|u-\bar{u}\|_{0,\Gamma_N} \\
&\quad \le C_{\partial\Omega} \|g - \nu^T \cdot \mathbf{y}^*\|_{0,\Gamma_N} \|u-\bar{u}\|_{1,\Omega} \le C_{\Omega,\partial\Omega} \|g - \nu^T \cdot \mathbf{y}^*\|_{0,\Gamma_N} \| \nabla(u-\bar{u}) \|_\Omega.
\end{aligned} \tag{18}$$

Using (16), (17), and (18) to estimate the terms on the RHS of (15), we obtain

$$\begin{aligned}
&a(u-\bar{u}, u-\bar{u}) \\
&\le \frac{1}{2}\Big(\| A^{-1}\mathbf{y}^* - \nabla\bar{u}) \|_\Omega + C_{\Omega,\partial\Omega}\|g - \nu^T \cdot \mathbf{y}^*\|_{0,\Gamma_N} + \frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}} \|f + \operatorname{div}\mathbf{y}^* - c\bar{u}\|_{0,\Omega\setminus\overline{\Omega}^c} \Big)^2 \\
&\qquad + \frac{1}{2}\| \nabla(u-\bar{u}) \|^2_\Omega + \frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c} + \frac{1}{2}\left\| \frac{1}{\sqrt{c}}(f + \operatorname{div}\mathbf{y}^* - c\bar{u}) \right\|^2_{0,\Omega^c}.
\end{aligned} \tag{19}$$

Using now (13) and the final inequality (19), multiplying by two and regrouping, we immediately get for the error in the energy norm that

$$\begin{aligned}
&a(u-\bar{u}, u-\bar{u}) = \| \nabla(u-\bar{u}) \|^2_\Omega + \|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c} \le \left\| \frac{1}{\sqrt{c}}(f + \operatorname{div}\mathbf{y}^* - c\bar{u}) \right\|^2_{0,\Omega^c} \\
&+\Big(\| A^{-1}\mathbf{y}^* - \nabla\bar{u} \|_\Omega + \frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}} \|f + \operatorname{div}\mathbf{y}^*\|_{0,\Omega\setminus\overline{\Omega}^c} + C_{\Omega,\partial\Omega}\|g - \nu^T \cdot \mathbf{y}^*\|_{0,\Gamma_N} \Big)^2.
\end{aligned} \tag{20}$$

Finally, using two times the inequality $(a+b)^2 \le (1+\lambda)a^2 + (1+\frac{1}{\lambda})b^2$, valid for any $\lambda > 0$, for the terms in the round brackets in (20), we get estimate (12).

□

3.2. Lower estimate

Proposition 3.2 *For the error in the energy norm* (8) *we have the following lower bound*

$$a(u-\bar{u}, u-\bar{u}) \geq 2(J(\bar{u}^*) - J(w)), \tag{21}$$

where w is any function from $H^1_{\Gamma_D}(\Omega)$ and the functional J is defined in (7).

Proof: First, we prove that

$$a(u-\bar{u}, u-\bar{u}) = 2(J(\bar{u}^*) - J(u^*)). \tag{22}$$

Really, we have

$$\begin{aligned} 2(J(\bar{u}^*) - J(u^*)) &= a(\bar{u}^*, \bar{u}^*) - 2\bar{F}(\bar{u}^*) - a(u^*, u^*) + 2\bar{F}(u^*) \\ = a(\bar{u}^*, \bar{u}^*) - a(u^*, u^*) &+ 2\bar{F}(u^* - \bar{u}^*) = a(\bar{u}^*, \bar{u}^*) - a(u^*, u^*) + 2a(u^*, u^* - \bar{u}^*) \\ &= a(\bar{u}^*, \bar{u}^*) + a(u^*, u^*) - 2a(u^*, \bar{u}^*) = a(u-\bar{u}, u-\bar{u}). \end{aligned}$$

Since u^* minimizes the energy functional, we have $J(u^*) \leq J(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega)$, which proves (21). □

Remark 3.1 *The estimate* (21) *has a practical meaning only if w satisfies $J(w) \leq J(\bar{u}^*)$. For example, if $\bar{u}^*$ comes from a FE-solution obtained using mesh S_h, suitable w can be constructed, e.g., by solving the weak problem* (4) *on a hierarcially refined mesh S_τ.*

3.3. Comments on two-sided estimates (12) and (21)

- In order to derive the upper (12) and the lower (21) estimates, we did not specify the function $\bar{u}$ to be a finite element approximation (or computed by some another numerical method). In fact, it is simply any function from the set $u_0 + H^1_{\Gamma_D}(\Omega)$.

- The upper estimate (12) cannot be improved. Really, if one takes $\mathbf{y}^* = A\nabla u$, which obviously belongs to $H_N(\Omega, \mathrm{div})$, then the last two terms in the right-hand side of (12) vanish. Further, taking $\alpha = 0$, we finally observe that the inequality (12) holds as equality. To prove that the lower estimate (21) cannot be improved either, we should, obviously, take $w = u^* \in H^1_{\Gamma_D}(\Omega)$ and use (22).

- The upper estimate (12) contains only two global constants, C_{Ω,Γ_D} and $C_{\partial\Omega}$, which do not depend on the computational process. They have to be computed (or accurately estimated from above) only once when the problem is posed.

- In many works, devoted to a posteriori error estimation, one usually takes $c \equiv 0$. In this case $a(u-\bar{u}, u-\bar{u}) = \|\nabla(u-\bar{u})\|^2_\Omega$, the set $\Omega^c = \emptyset$, and the estimate (12) takes a simpler form

$$a(u-\bar{u},u-\bar{u}) \leq (1+\alpha)\|\!|A^{-1}\mathbf{y}^*-\nabla\bar{u}\|\!|^2_\Omega+\Big(1+\frac{1}{\alpha}\Big)(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f+\operatorname{div}\mathbf{y}^*\|^2_{0,\Omega}$$
$$+\Big(1+\frac{1}{\alpha}\Big)\Big(1+\frac{1}{\beta}\Big)C^2_{\Omega,\partial\Omega}\|g-\nu^T\cdot\mathbf{y}^*\|^2_{0,\Gamma_N}. \quad (23)$$

- For the pure Dirichlet boundary condition, the third term in RHS of (23) vanishes, and, since the estimate is valid for any positive β, we can take it to be zero. Then, we get the estimate

$$a(u-\bar{u},u-\bar{u}) \leq (1+\alpha)\|\!|A^{-1}\mathbf{y}^*-\nabla\bar{u}\|\!|^2_\Omega+\Big(1+\frac{1}{\alpha}\Big)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f+\operatorname{div}\mathbf{y}^*\|^2_{0,\Omega}. \quad (24)$$

- The upper estimate (24) was first obtained in [19] using complicated tools of the duality theory, and later it was also obtained in [21] for the Poisson equation, using the Helmholz decomposition of $L_2(\Omega,\mathbf{R}^d)$. The estimate (23) is derived in [22] using the duality theory again. Our approach of derivation of the estimates is different from those used in the above mentioned works and is simplier.

- In the case of pure Dirichlet conditions, only the constant C_{Ω,Γ_D} has to be computed or estimated from above.

- In the case of pure Dirichlet condition and if $c \geq c_0 > 0$ in Ω, we need not estimate any constants at all.

In what follows we shall use the following denotations for the upper and lower bounds of the error in the energy norm (8)

$$M^\oplus(\bar{u},\mathbf{y}^*,\alpha,\beta) = \Big\|\frac{1}{\sqrt{c}}(f+\operatorname{div}\mathbf{y}^*-c\bar{u})\Big\|^2_{0,\Omega^c}+(1+\alpha)\|\!|A^{-1}\mathbf{y}^*-\nabla\bar{u}\|\!|^2_\Omega$$
$$+\Big(1+\frac{1}{\alpha}\Big)(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f+\operatorname{div}\mathbf{y}^*\|^2_{0,\Omega\setminus\overline{\Omega}^c}+\Big(1+\frac{1}{\alpha}\Big)\Big(1+\frac{1}{\beta}\Big)C^2_{\Omega,\partial\Omega}\|g-\nu^T\cdot\mathbf{y}^*\|^2_{0,\Gamma_N},$$

and

$$M^\ominus(\bar{u},w) = 2(J(\bar{u})-J(w)).$$

Sometimes we shall use only a short denotation $M^\oplus$ or $M^\ominus$ for the corresponding bounds if it does not lead to misunderstanding.

4. Two-sided estimates for local errors

Two-sided estimates for controlling the error $u-\bar{u}$ in terms of linear functional (9) are essentially based on the usage of an auxiliary (often called *adjoint*) problem formulated below.

Adjoint problem: Find $v \in H^1_{\Gamma_D}(\Omega)$ such that

$$\int_\Omega A\nabla v \cdot \nabla w \, dx + \int_\Omega cvw \, dx = \ell(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega).$$

The adjoint problem can be rewritten in a shorter form similarly to the main problem (4): Find $v \in H^1_{\Gamma_D}(\Omega)$ such that $a(v,w) = \ell(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega)$. In particular this means, that the bilinear forms of the main and adjoint problems coincide.

The adjoint problem is uniquely solvable due to the assumption that ℓ is a linear bounded functional. However, the exact solution v of it is usually very hard (or even impossible) to find in analytical form and, thus, we only have some approximation for v, which we denote by the symbol $\bar{v}$ in what follows, assuming again only that $\bar{v} \in H^1_{\Gamma_D}(\Omega)$.

Proposition 4.1 *(cf.* [8]*) The following error decomposition holds*

$$\ell(u-\bar{u}) = E_0(\bar{u},\bar{v}) + E_1(u-\bar{u}, v-\bar{v}),$$

where

$$E_0(\bar{u},\bar{v}) = \int_\Omega f\bar{v}\, dx + \int_{\Gamma_N} g\bar{v}\, ds - \int_\Omega A\nabla\bar{v}\cdot\nabla\bar{u}\, dx - \int_\Omega c\bar{v}\bar{u}\, dx, \tag{25}$$

$$E_1(u-\bar{u}, v-\bar{v}) = \int_\Omega A\nabla(u-\bar{u})\cdot\nabla(v-\bar{v})\, dx + \int_\Omega c(u-\bar{u})(v-\bar{v})\, dx.$$

The first term E_0 is, obviously, directly computable once we have $\bar{u}$ and $\bar{v}$ computed, but the term E_1 contains unknown gradients ∇u and ∇v. In order to estimate it, we notice first that $E_1(u-\bar{u}, v-\bar{v}) \equiv a(u-\bar{u}, v-\bar{v})$. Further, the following relation obviously holds for any positive α:

$$\begin{aligned} 2E_1(u-\bar{u}, v-\bar{v}) = a\Big(\alpha(u-\bar{u}) + \frac{1}{\alpha}(v-\bar{v}), \alpha(u-\bar{u}) + \frac{1}{\alpha}(v-\bar{v})\Big) \\ - \alpha^2 a(u-\bar{u}, u-\bar{u}) - \frac{1}{\alpha^2} a(v-\bar{v}, v-\bar{v}). \end{aligned} \tag{26}$$

The last two terms in the above identity present the errors in the energy norm for main and adjoint problems. Thus, we can immediately use the two-sided estimates from Section 3, written in somewhat simplified form:

$$M^\ominus \le a(u-\bar{u}, u-\bar{u}) \le M^\oplus, \quad M^\ominus_{ad} \le a(v-\bar{v}, v-\bar{v}) \le M^\oplus_{ad},$$

where subindex "*ad*" means that the corresponding estimate is obtained for the adjoint problem.

As far it concerns the first term in the right-hand side of (26), we observe that

$$a\Big(\alpha(u-\bar u)+\frac{1}{\alpha}(v-\bar v),\alpha(u-\bar u)+\frac{1}{\alpha}(v-\bar v)\Big)=$$
$$=a\Big(\big(\alpha u+\frac{1}{\alpha}v\big)-\big(\alpha\bar u+\frac{1}{\alpha}\bar v\big),\big(\alpha u+\frac{1}{\alpha}v\big)-\big(\alpha\bar u+\frac{1}{\alpha}\bar v\big)\Big).$$

The function $\alpha u+\frac{1}{\alpha}v$ can be perceived as the solution of the following problem (called as the *mixed problem* in what follows): Find $u_\alpha\in u_0+H^1_{\Gamma_D}(\Omega)$ such that

$$\int_\Omega A\nabla u_\alpha\cdot\nabla w\,dx+\int_\Omega cu_\alpha w\,dx=\alpha F(w)+\frac{1}{\alpha}\ell(w)\quad\forall w\in H^1_{\Gamma_D}(\Omega),$$

which is uniquely solvable due to the fact that $\alpha F(w)+\frac{1}{\alpha}\ell(w)$ is, obviously, also linear bounded functional.

The function $\alpha\bar u+\frac{1}{\alpha}\bar v\in H^1_{\Gamma_D}(\Omega)$ can be considered as an approximation of u_α, and we can again apply the techniques of Section 3 in order to obtain the following two-sided estimates (writen again in simplified form)

$$M^{\ominus}_{mix}\le a\Big(\alpha(u-\bar u)+\frac{1}{\alpha}(v-\bar v),\alpha(u-\bar u)+\frac{1}{\alpha}(v-\bar v)\Big)\le M^{\oplus}_{mix},$$

where subindex "*mix*" means that the estimates are obtained for the mixed problem.

Further, we immediately observe that

$$\frac{1}{2}\Big(M^{\ominus}_{mix}-\alpha^2M^{\oplus}-\frac{1}{\alpha^2}M^{\oplus}_{ad}\Big)\le E_1(u-\bar u,v-\bar v),$$

and

$$E_1(u-\bar u,v-\bar v)\le\frac{1}{2}\Big(M^{\oplus}_{mix}-\alpha^2M^{\ominus}-\frac{1}{\alpha^2}M^{\ominus}_{ad}\Big).$$

The above considerations can be summarized as follows.

Proposition 4.2 *For the error in terms of linear functional $\ell(u-\bar u)$ we have the following upper estimate*

$$\ell(u-\bar u)\le E_0(\bar u,\bar v)+\frac{1}{2}\Big(M^{\oplus}_{mix}-\alpha^2M^{\ominus}-\frac{1}{\alpha^2}M^{\ominus}_{ad}\Big),$$

and the following lower estimate

$$\ell(u-\bar u)\ge E_0(\bar u,\bar v)+\frac{1}{2}\Big(M^{\ominus}_{mix}-\alpha^2M^{\oplus}-\frac{1}{\alpha^2}M^{\oplus}_{ad}\Big),$$

where the directly computable term $E_0(\bar u,\bar v)$ is defined in (25).

Remark 4.1 *For practical realisations of the above technologies, see e.g.* [8, 9, 21, 22].

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