Jiří Fürst

Finite volume WLSQR scheme and its applications to transonic flows

In: Jan Chleboun and Karel Segeth and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Prague, May 28-31, 2006. Institute of Mathematics AS CR, Prague, 2006. pp. 86–91.

Persistent URL: http://dml.cz/dmlcz/702823

FINITE VOLUME WLSQR SCHEME AND ITS APPLICATIONS TO TRANSONIC FLOWS*

Jiří Fürst

Abstract

This article describes the development of a high order numerical method for the solution of compressible transonic flows. The discretisation in space is based on the standard finite volume method of Godunov's type. A higher order of accuracy is achieved by a piecewise polynomial interpolation similar to the ENO or weighted ENO methods (see e.g. [8]).

1. Introduction

The weighted least square reconstruction (WLSQR) of pointwise data at the cell faces from given cell averages is developed with the aim to simplify the implementation of the standard ENO procedure especially for the case of unstructured meshes. The reconstruction procedure uses single stencil and computes an interpolation polynomial by minimizing the weighted interpolation error over the cells in this stencil.

The complete finite volume scheme equipped with the piecewise linear reconstruction was successfully used for the solution many transonic flow problems (see e.g. [4, 5]). This article presents the basic analytical results as well as some new numerical experiments with the WLSQR scheme especially for the case of inviscid 3D flows and turbulent flows in 2D. The WLSQR reconstruction has been used for the conservative variables as well as for the model of turbulence.

The flow is described by the set of the Euler or the Navier-Stokes equations in conservative form

$$W_t + F(W)_x + G(W)_y = F^v(W)_x + G^v(W)_y + S(W), \tag{1}$$

where $W = [\rho, \rho u, \rho v, e]^T$ is the vector of conservative variables, $F(W)$ and $G(W)$ are the inviscid fluxes, $F^v(W)$ and $G^v(W)$ are the viscous fluxes ($F^v = G^v = 0$ for the case of the Euler equations) and $S(W)$ is a source term, for more details see [2].

The equations equipped with proper boundary conditions are solved numerically using an unstructured mesh and a finite volume scheme with all unknowns located at cell centers. The fluxes through the cell interfaces are approximated by the Gauss quadrature with the physical fluxes replaced by the numerical ones

$$\int_{C_i \cap C_j} (F(W), G(W)) \cdot d\vec{S} \approx \sum_{q=1}^{J} \omega_q F^{AUSMPW+}(W^L_{ijq}, W^R_{ijq}, \vec{S_{ijq}}). \tag{2}$$

*This work was supported by grant No. 201/05/0005 of the Grant Agency of the Czech Republic and by the Research Plan MSM No. 6840770010.

Here W_{ijq}^L and W_{ijq}^R denotes the values of the vector of unknowns interpolated to the Gauss point q of the interface $C_i \cap C_j$ from the left cell or from the right cell, respectively. $F^{AUSMPW+}$ denotes the numerical flux described in [9] and ω_q are the weights of the Gauss quadrature. The resulting finite volume scheme for inviscid case can be then written in semi-discrete form

$$|C_i| \frac{dW_i}{dt} = - \sum_{j \in \mathcal{N}_i} \sum_{q=1}^{J} \omega_q F^{AUSMPW+}(W_{ijq}^L, W_{ijq}^R, \vec{S_{ijq}}). \tag{3}$$

Here C_i is the i-th cell, $W_i = \int_{C_i} W(\vec{x}, t) d\vec{x}$, and $\mathcal{N}_i = \{j : \dim(C_i \cap C_j) = 1\}$.

The basic first order scheme can be obtained by setting $J = 1$, $W_{ijq}^L = W_i$, and $W_{ijq}^R = W_j$.

2. The WLSQR interpolation

However the basic first order scheme posses very good mathematical properties, it is well known, that it is very diffusive. Therefore a use of higher order schemes is preferred, especially for the viscous flow calculations. The higher order scheme can be constructed within this framework simply by improving the interpolation of W^L and W^R. There exist several methods for the construction of a stable interpolation, the most known are the limited least squares of Barth [1], the ENO/WENO schemes [8], or the TVD schemes [7].

The use of limiters as in the TVD or the Barth's schemes usually cut the order of accuracy near extrema and may also hamper the convergence to a steady state. On the other hand, the implementation of ENO/WENO schemes is relatively complicated for unstructured meshes. Therefore a novel reconstruction procedure was introduced in [5]. Denote by ϕ a component of W. Then the interpolation polynomial $P_i(\vec{x}; \phi)$ for the cell C_i is constructed by minimizing the weighted interpolation error [1]

$$\text{err} := \sum_{j \in \mathcal{M}_i} \left[w_{ij} \left(\int_{C_j} \tilde{P}(\vec{x}; \phi) \, d\vec{x} - |C_j| \phi_j \right) \right]^2 \tag{4}$$

with respect to the conservativity constraint

$$\int_{C_j} P_i(\vec{x}; \phi) \, d\vec{x} = |C_j| \phi_j. \tag{5}$$

The weights w_{ij} are chosen in such a way, that the magnitude of w is big whenever the solution is smooth and w is close to zero when the solution is discontinuous, see formula (6). The single stencil $\mathcal{M}_i$ is selected according to the order of the polynomial P.

[1] Herefrom comes the name of the method - the Weighted Least Square Reconstruction.

2.1. The second order scheme

The formally second order scheme can be obtained by using linear polynomials P_i. For this case, the choice of $\mathcal{M}_i := \mathcal{M}_i^1 = \{j : C_i \cap C_j \neq \emptyset\}$ (i.e. cells touching C_i at least by a vertex) has been tested together with the weights

$$w_{ij} = \sqrt{\frac{h^{-r}}{\left|\frac{\phi_i - \phi_j}{h}\right|^p + h^q}},\ j \in \mathcal{M}_i, \tag{6}$$

with h being the distance between cell centers of C_i and C_j and $p = 4$, $q = -3$, and $r = 3$. The analysis of simplified cases has been carried out in [4] showing a stability of WLSQR interpolation for special discontinuous data.

2.2. The third order scheme

This approach can be extended to a scheme which has formally third order of accuracy by using quadratic polynomials P_i. It is also necessary to enlarge the stencil to $\mathcal{M}_i := \mathcal{M}_i^2 = \mathcal{M}_i^1 \cup \{j : C_j \cap \mathcal{M}_i^1 \neq \emptyset\}$ (i.e. the stencil is extended by the cells touching $\mathcal{M}_i^1$). Although there are no analytical results for quadratic reconstruction, the same definition of w_{ij}, $j \in \mathcal{M}_i^2$ has been used successfully.

2.3. Analysis of weights in WLSQR interpolation

The complete analysis of this three-parametric family of weights is very difficult task, therefore we investigate here only effects of p and q. The value of r was kept constant $r = 3$ in this work.

In [3] the theoretical analysis of 1D piecewise linear reconstruction using regular mesh has been developed with the following results:

Lemma 2.1 *Assume a sufficiently smooth function $u(x)$ having cell averages u_i and weights $w \neq 0$. Then the piecewise linear WLSQR interpolation polynomial approximates $u(x)$ with second order of accuracy, i.e.*

$$P(x; u) = u(x) + \mathcal{O}(h^2). \tag{7}$$

In the case of discontinuous data the total variation of the interpolant for $u(x)$ defined as $u(x) = 1$ for $x < x_{shock}$ and $u(x) = 0$ for $x \geq x_{shock}$ has been analyzed and the following TV-estimate has been proven

$$TV(P(x; u)) \leq TV(u) + 6h^{1+q/p}. \tag{8}$$

Several numerical experiments for piecewise linear WLSQR method in [3] have shown, that the choices $p, q, r = 4, -2, 3$ or $4, -3, 3$ are appropriate at least for inviscid transonic flows in test channel. Therefore we chose here $p, q, r = 4, -2, 3$ also for the piecewise quadratic WLSQR method.

2.4. Numerical experiments with the WLSQR scheme

The numerical analysis of the order of accuracy of an upwind scheme with WLSQR interpolation has been done in [3] for the case of linear advection in 2D and for the non-linear Burgers equation in 2D. The numerical experiments proved, that the order of accuracy corresponds well to the order of the reconstruction for the case of smooth data i.e. the scheme without reconstruction has order of accuracy almost 1, the scheme with piecewise linear reconstruction almost 2, and finally the scheme with quadratic reconstruction almost 3. On the other hand, the order of accuracy drops to one as soon as there are moving discontinuities.

3. Applications in turbomachinery

The above mentioned numerical method has been applied to the solution of transonic flows in 2D turbine cascades. The compressible viscous flow is described by the set of the Euler equations or the Favre averaged Navier-Stokes equations (RANS) coupled with the TNT $k-\omega$ model of turbulence (see [10]). The turbulent transonic flow through a 2D turbine cascade was solved using a hybrid mesh with quadrilaterals around the profile, in the mixing region behind the outlet edge and at the outlet part of boundary. The remaining part of the domain was filled up with triangles. The total number of elements was 24087 with $y_1^+ < 1$ (here y_1^+ is the size of the first cell near the wall in normal direction in wall coordinates, see [11]).

Figure 1 shows the isolines of the Mach number the detail of isolines of entropy

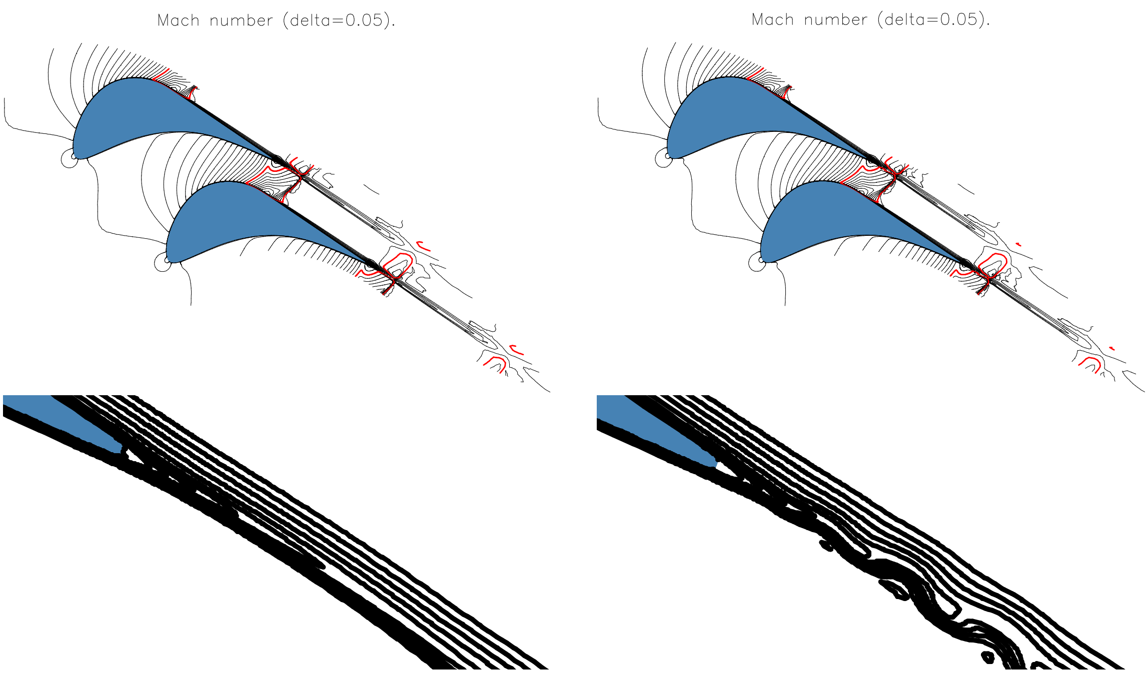

Fig. 1: *Isolines of Mach number (above) and entropy (below) in 2D turbine cascade, second (left) and third (right) order solution.*

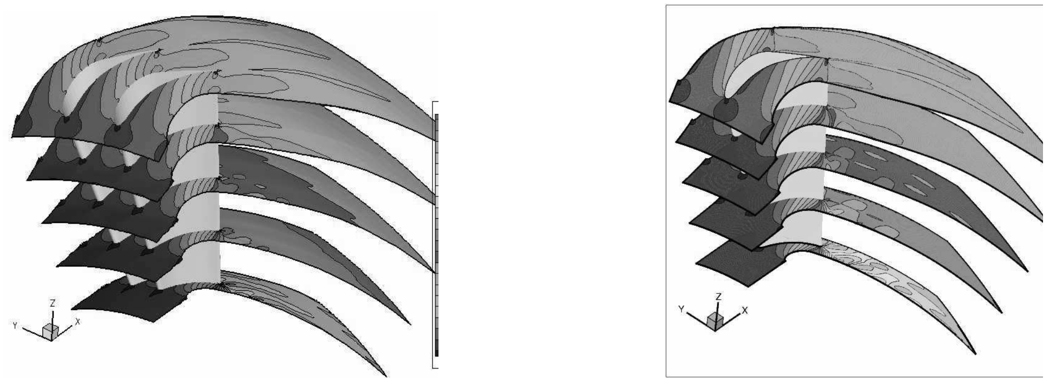

Fig. 2: *Isolines of Mach number for inviscid flow through a 3D turbine stator, WLSQR method on the left (coarser mesh), TVD MC scheme on the right (finer mesh).*

near the outlet edge obtained with the help of the second and the third order method for the flow characterized by the outlet Mach number $M_{2i} = 0.906$ and Reynolds number $Re = 848000$. The isolines of entropy document clearly the difference between those two results - the second order scheme gives stationary solution whereas the wake is unsteady for the third order solution.

Last example is the inviscid transonic flow through 3D turbine cascade. We assume that the flow is periodic from blade to blade and therefore it is possible to solve the flow field just in one period. The domain is discretized using a structured mesh with hexahedral cells. The inflow and outflow conditions depend on the radius. Figure 2 compares the distribution of Mach number obtained with the piecewise linear WLSQR method with AUSM flux using a structured mesh with $100 \times 20 \times 20$ cell. It can be seen, that the solution is comparable to the reference solution obtained with TVD MacCormack scheme with finer mesh having $200 \times 40 \times 40$ cells. Similar results were also obtained by J. Halama [6] using cell vertex Ni's scheme with Jameson's artificial viscosity.

4. Conclusion

The article describes briefly the weighted least-square reconstruction procedure. The proposed WLSQR reconstruction posses good stability even for the case of transonic turbulent flows and is easily extensible to 3D case as well as to third order of accuracy. The difference between second and third order scheme was demonstrated for the case of 2D flows through a turbine. The third order scheme uses less numerical dissipation and produces an unsteady solution in this case.

References

[1] T.J. Barth, D.C. Jesperson: *The design and application of upwind schemes on unstructured meshes.* AIAA Paper 89–0366, AIAA, Jan 1989.

[2] M. Feistauer, J. Felcman, I. Straškraba: *Mathematical and computational methods for compressible flow.* Numerical Mathematics and Scientific Computation. Oxford University Press, 2003.

[3] J. Fürst: *Numerical solution of inviscid and viscous flows using modern schemes and quadrilateral or triangular mesh.* In: M. Beneš, (ed.), Proceedings of the Czech-Japanese Seminar in Applied Mathematics, Czech Technical University in Prague, August 2004.

[4] J. Fürst: *A finite volume scheme with weighted least square reconstruction.* In: S. Raghay F. Benkhaldoun, D. Ouazar, (eds), Finite Volumes for Complex Applications IV, Hermes Science, 2005, 345–354.

[5] J. Fürst, K. Kozel: *Second and third order weighted ENO scheme on unstructured meshes.* In: F. Benkhaldoun and R. Vilsmeier, (eds), Finite Volumes for Complex Applications. Problems and Perspectives, Hermes, July 2002.

[6] J. Halama, T. Arts, J. Fořt: *Numerical solution of steady and unsteady transonic flow in turbine cascades and stages.* Computers and Fluids **33**, 2004, 729–740.

[7] A. Harten: *High resolution schemes for hyperbolic conservation laws.* Journal of Computational Physics **49**, 1983, 357–393.

[8] C. Hu, C.-W. Shu: *Weighted essentially non-oscillatory schemes on triangular meshes.* Journal of Computational Physics **150**, 1999, 97–127.

[9] K.H. Kim, C. Kim, O.-H. Rho: *Methods for the accurate computations of hypersonic flows i. AUSMPW+ scheme.* Journal of Computational Physics **174**, 2001, 38–80.

[10] J.C. Kok: *Resolving the dependence on free stream values for the k-omega turbulence model.* Technical Report NLR-TP-99295, NLR, 1999.

[11] D.C. Wilcox: *Turbulence modeling for CFD.* DCW Industries, Inc., second edition edition, 1998.