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ACCURACY INVESTIGATION OF A STABILIZED FEM FOR SOLVING FLOWS OF INCOMPRESSIBLE FLUID*

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Abstract

In computer fluid dynamics, employing stabilization to the finite element method is a commonly accepted way to improve the applicability of this method to high Reynolds numbers. Although the accompanying loss of accuracy is often referred, the question of quantifying this defect is still open. On the other hand, practitioners call for measuring the error and accuracy. In the paper, we present a novel approach for quantifying the difference caused by stabilization.

Dedicated to Professor Ivo Babuška on the occasion of his 80th birthday.

1. Introduction

The finite element method equipped with stabilization has proven to be a powerful tool for solving flows of incompressible fluids with high Reynolds numbers. But applying stabilization can lead to a change of the approximate solution in a serious way, as was discussed in [2].

The aim of our present research is to quantify the difference and find a way to predict it. Application of a posteriori error estimates seems to be a promising way to face these tasks.

Several numerical examples are presented to show the effect of stabilization and to investigate the accuracy.

2. Mathematical model

The considered mathematical model is the system of Navier-Stokes equations in two space dimensions (1) accompanied by the continuity equation (2). The aim is to search the vector of velocity $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t)) \in [\mathcal{C}^2(\overline{\Omega})]^2$ and pressure $p(\mathbf{x},t) \in \mathcal{C}^1(\overline{\Omega})/\mathbb{R}$ such that

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega\times[0,T]\,, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times[0,T]\,, \tag{2}$$

where ν denotes kinematic viscosity and $\mathbf{f}(\mathbf{x},t)$ stands for intensity of volume force.

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Boundary conditions (3)–(4) are allowed. For time dependent problems, initial condition (5) is considered.

$$
\begin{aligned}
\mathbf{u} &= \mathbf{g} \;\text{ on } \Gamma_g \times [0,T] && (3)\\
-\nu(\nabla \mathbf{u})\mathbf{n} + p\mathbf{n} &= \mathbf{0} \;\text{ on } \Gamma_h \times [0,T] && (4)\\
\mathbf{u} &= \mathbf{u}_0 \;\text{ in } \Omega,\; t = 0 && (5)
\end{aligned}
$$

For the solution by the finite element method, we consider the weak formulation of the problem (1)–(2). We introduce function spaces based on Sobolev spaces

$$
\begin{aligned}
V_g &= \Big\{ \mathbf{v} = (v_1, v_2) \mid \mathbf{v} \in [H^1(\Omega)]^2; \mathbf{Tr}\ v_i = g_i, i = 1,2, \text{ on } \Gamma_g \Big\},\\
V &= \Big\{ \mathbf{v} = (v_1, v_2) \mid \mathbf{v} \in [H^1(\Omega)]^2; \mathbf{Tr}\ v_i = 0, i = 1,2, \text{ on } \Gamma_g \Big\}.
\end{aligned}
$$

Now, we seek velocity $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t)) \in V_g$ such that $\mathbf{u} - \mathbf{u}_g \in V$ and pressure $p(\mathbf{x},t) \in L_2(\Omega)/\mathbb{R}$ for $t \in [0,T]$ satisfying

$$
\int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} \mathrm{d}\Omega + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \mathrm{d}\Omega + \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \mathrm{d}\Omega - \int_\Omega p \nabla \cdot \mathbf{v} \mathrm{d}\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v} \mathrm{d}\Omega \quad (6)
$$

$$
\int_\Omega \psi \nabla \cdot \mathbf{u} \mathrm{d}\Omega = 0 \quad (7)
$$

for any $\mathbf{v} \in V$ and $\psi \in L_2(\Omega)$. Operation ":" used in (6) is defined as

$$
\nabla \mathbf{u} : \nabla \mathbf{v} = \frac{\partial u_x}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial u_x}{\partial y}\frac{\partial v_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial v_y}{\partial x} + \frac{\partial u_y}{\partial y}\frac{\partial v_y}{\partial y}\,. \quad (8)
$$

3. Approximation of the problem by FEM

We use Hood-Taylor finite elements, which lead to the following function spaces

$$
\begin{aligned}
V_{gh} &= \Big\{ \mathbf{v}_h = (v_{h_1}, v_{h_2}) \in [\mathcal{C}(\overline{\Omega})]^2;\; v_{h_i}\,|_K \in R_2(\overline{K}),\; i = 1,2,\; \mathbf{v}_h = \mathbf{g} \text{ in nodes on } \Gamma_g \Big\}\\
Q_h &= \Big\{ \psi_h \in \mathcal{C}(\overline{\Omega});\; \psi_h\,|_K \in R_1(\overline{K}) \Big\}\\
V_h &= \Big\{ \mathbf{v}_h = (v_{h_1}, v_{h_2}) \in [\mathcal{C}(\overline{\Omega})]^2;\; v_{h_i}\,|_K \in R_2(\overline{K}),\; i = 1,2,\; \mathbf{v}_h = \mathbf{0} \text{ in nodes on } \Gamma_g \Big\}
\end{aligned}
$$

where V_{gh} is the space for approximation of velocities, Q_h for pressure and test functions for the continuity equation, and V_h for test functions for momentum equations. Here

$$
R_m(\overline{K}) = \begin{cases} P_m(\overline{K}), & \text{if } K \text{ is a triangle}\\ Q_m(\overline{K}), & \text{if } K \text{ is a quadrilateral} \end{cases}
$$

and P_m, Q_m have the usual meaning. Among all the advantages of these elements, we consider it to be rather important, that they lead to functions satisfying Babuška-Brezzi (*inf-sup*) stability condition (9).

$$
\exists C_B > 0, const.\ \forall \psi_h \in Q_h\ \exists \mathbf{v}_h \in V_h \quad (\psi_h, \nabla \cdot \mathbf{v}_h)_0 \geq C_B \|\psi_h\|_0 \|\mathbf{v}_h\|_1 \quad (9)
$$

4. SemiGLS stabilization technique

In [4], semiGLS stabilization technique was derived as a modification of Galerkin Least Squares method, proposed by Hughes, Franca, and Hulbert [3]. We search the approximate velocity $\mathbf{u}_h \in V_{gh}$ and pressure $p_h \in Q_h$ satisfying in Ω

$$B_{sGLS}(\mathbf{u}_h, p_h; \mathbf{v}_h, \psi_h) = L_{sGLS}(\mathbf{v}_h, \psi_h), \quad \forall \mathbf{v}_h \in V_h, \ \ \forall \psi_h \in Q_h \,, \tag{10}$$

where

$$\begin{aligned} B_{sGLS}(\mathbf{u}_h, p_h; \mathbf{v}_h, \psi_h) \equiv & \int_\Omega \frac{\partial \mathbf{u}_h}{\partial t} \cdot \mathbf{v}_h \mathrm{d}\Omega + \int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \mathrm{d}\Omega \\ + \ \ \nu \int_\Omega \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \mathrm{d}\Omega & - \int_\Omega p_h \nabla \cdot \mathbf{v}_h \mathrm{d}\Omega + \int_\Omega \psi_h \nabla \cdot \mathbf{u}_h \mathrm{d}\Omega + \\ + \ \ \sum_{K=1}^{N} \int_K \left[\frac{\partial \mathbf{u}_h}{\partial t} + (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nu \Delta \mathbf{u}_h + \nabla p_h \right] & \cdot \tau \left[(\mathbf{u}_h \cdot \nabla)\mathbf{v}_h - \nu \Delta \mathbf{v}_h + \nabla \psi_h \right] \mathrm{d}\Omega, \end{aligned}$$

$$L_{sGLS}(\mathbf{v}_h, \psi_h) \ \ \equiv \ \ \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \mathrm{d}\Omega + \sum_{K=1}^{N} \int_K \mathbf{f} \cdot \tau \left[(\mathbf{u}_h \cdot \nabla)\mathbf{v}_h - \nu \Delta \mathbf{v}_h + \nabla \psi_h \right] \mathrm{d}\Omega \,.$$

Here τ denotes stabilization parameter. The way to determine it is mentioned in [2]. Index $sGLS$ is an abbreviation of semiGLS.

5. Evaluating of the accuracy

A straightforward way to evaluate the effect of stabilization is to compute the difference between solution with and without stabilization. This method was proposed in [2] accompanied by numerical examples and is applicable in the range of Reynolds numbers, where we can solve the problem both with and without stabilization. Such difference represents "pure distortion" caused by stabilization.

To get the idea about achieved accuracy of our solution, it is also suitable to apply a posteriori error estimates. We use following estimate derived for Hood-Taylor elements

$$\mathcal{U}^2(u_1 - u_{1h}, u_2 - u_{2h}, p - p_h) \le \mathcal{E}^2(u_{1h}, u_{2h}, p_h) \,, \tag{11}$$

where the terms represent

$$\mathcal{U}^2(u_1 - u_{1h}, u_2 - u_{2h}, p - p_h) = \|(e_{u_1}, e_{u_2})\|^2_{1,K} + \|e_p\|^2_{0,K} \,,$$

$$\begin{aligned} \mathcal{E}^2(u_{1h}, u_{2h}, p_h) \ \ = \ \ & C\Big[h_K^2 \int_K \big(r_1^2(u_{1h}, u_{2h}, p_h) + r_2^2(u_{1h}, u_{2h}, p_h) \big) \, \mathrm{d}\Omega \\ + \ \ & \int_K r_3^2(u_{1h}, u_{2h}, p_h) \mathrm{d}\Omega \Big] \,, \end{aligned}$$

and $r_1(u_{1h}, u_{2h}, p_h)$, $r_2(u_{1h}, u_{2h}, p_h)$, and $r_3(u_{1h}, u_{2h}, p_h)$ stand for residuals of the system (1)–(2); (u_1, u_2, p) denotes an exact solution, (u_{1h}, u_{2h}, p_h) an approximate solution computed by FEM, and $(e_{u_1}, e_{u_2}, e_p) = (u_1 - u_{1h}, u_2 - u_{2h}, p - p_h)$ an error of approximate solution. Constant C is determined from a numerical experiment described in [1], as well as details on the a posteriori estimates.

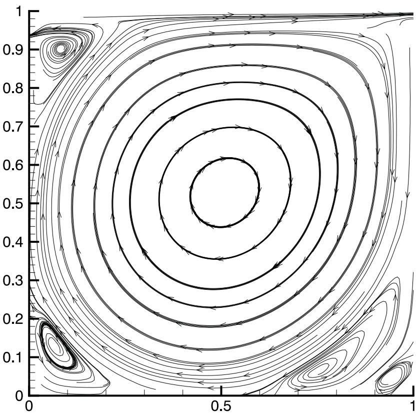
Such approach is applicable for any Reynolds number, for which we can find solution by the stabilized method and estimates the whole difference between a stabilized solution and an exact one.

6. Results of numerical experiments

To demonstrate the approach using a posteriori error estimates, we present results for a problem of a lid driven cavity and a channel with a sudden extension of diameter. Both problems are steady and the results for measuring distortion caused by the stabilization can be found in [2].

In Figures 1 – 2, we can observe the effect of stabilization on streamlines inside cavity for three levels of mesh fineness. A posteriori error estimates in the cavity are presented in Figures 3 – 5. They represent the relative error in percents. We can observe, that while the regions of higher error are decreasing for the Newton method without stabilization when refining the mesh, they remain almost independent of refinement for the stabilized method.

Geometry of the channel is described in Figure 6. Streamlines in the channel for Reynolds number 1,000 are presented in Figure 7, and in Figure 8, there are a posteriori error estimates to compare the differences.



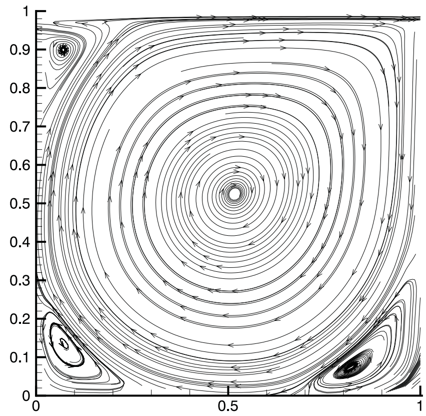


Fig. 1: *Streamlines, Re = 10,000, mesh 32×32 without stabilization (left) and by semiGLS (right).*

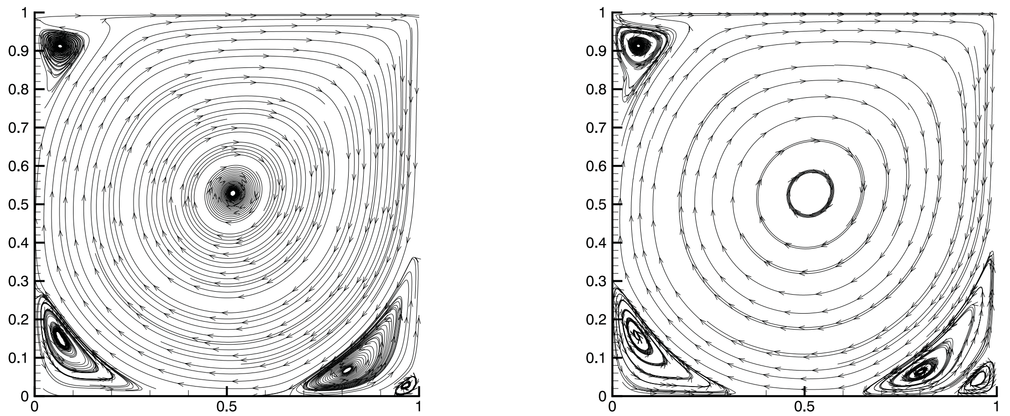


Fig. 2: *Streamlines by semiGLS, Re = 10,000, mesh 64×64 (left) and 128×128 (right).*

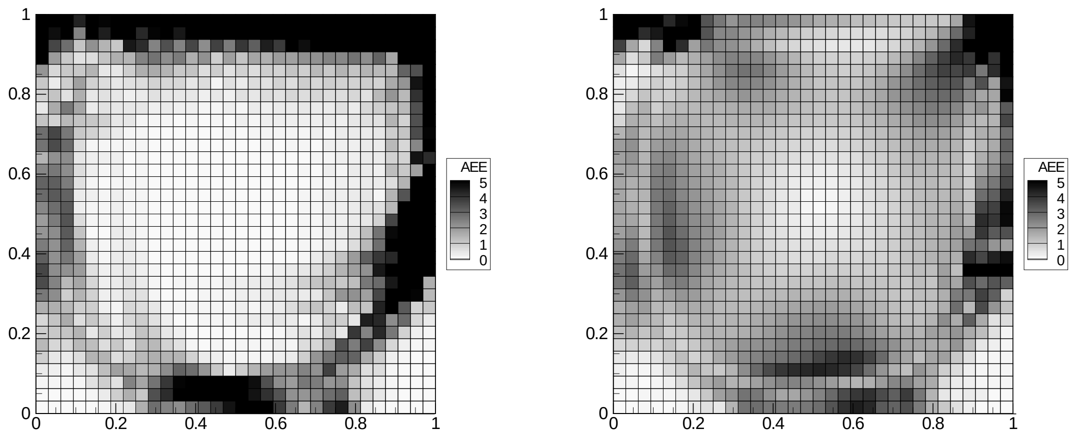


Fig. 3: *A posteriori errors on elements, Re = 10,000, mesh 32×32 without stabilization (left) and by semiGLS (right).*

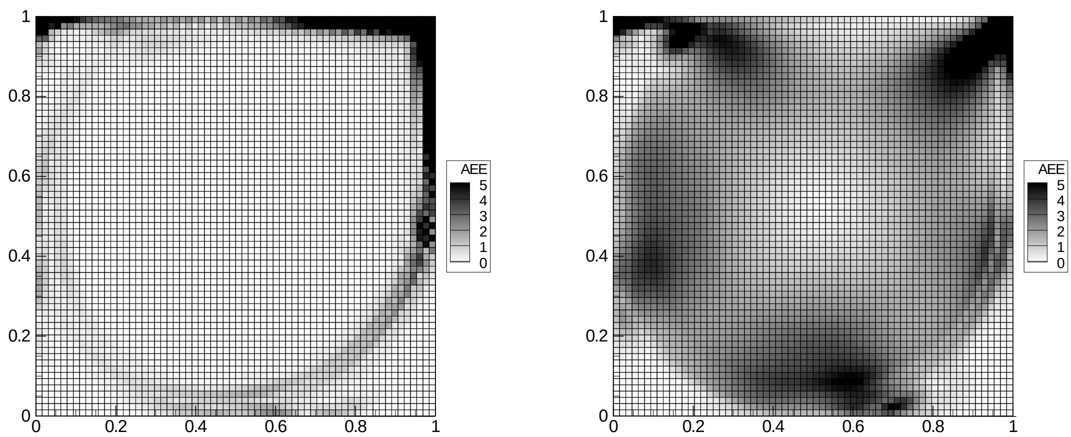


Fig. 4: *A posteriori errors on elements, Re = 10,000, mesh 64×64 without stabilization (left) and by semiGLS (right).*

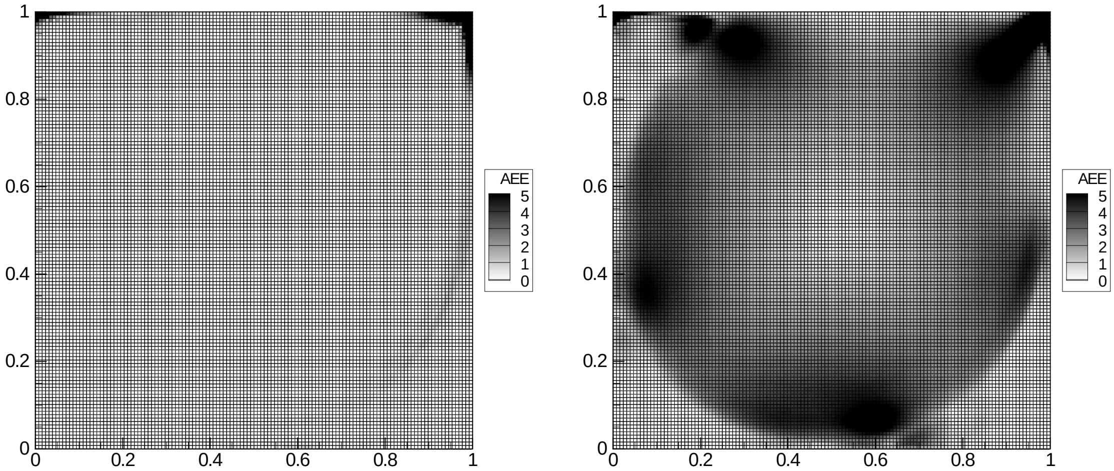


Fig. 5: *A posteriori errors on elements, Re = 10,000, mesh 128×128 without stabilization (left) and by semiGLS (right).*

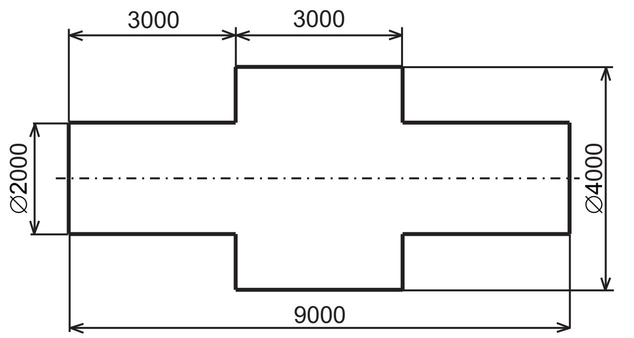


Fig. 6: *Geometry of the channel (dimensions in milimeters).*

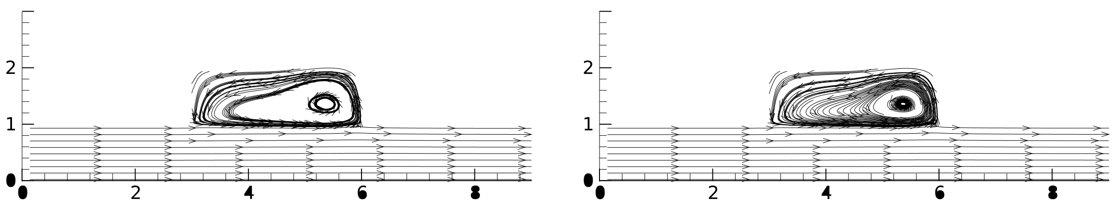


Fig. 7: *Streamlines in the channel by the Newton method without stabilization (left) and by the semiGLS algorithm (right), Re = 1,000.*

7. Conclusion

We have developed a stabilized method and tested it on various problems, where it provided promising results. This means, that we were able to reach markably higher Reynolds numbers using this method than using method without stabilization.

The cost for using stabilization is a loss of accuracy. This loss is hard to predict, but we are able to quantify it and estimate it a posteriori. We have presented two

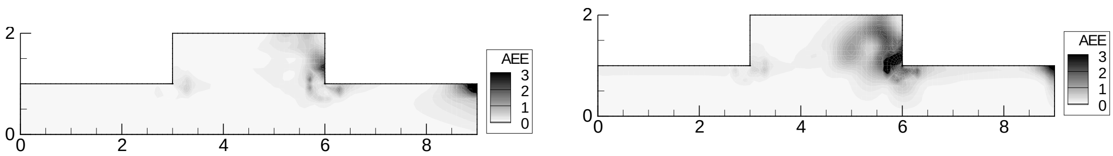


Fig. 8: *A posteriori error estimates in the channel by the Newton method without stabilization (left) and by the semiGLS algorithm (right), Re = 1,000.*

approaches for such evaluation, based on comparing approximate solutions with and wihout stabilization and on a posteriori error estimation.

As the main ideas resulting from the research we could mention, that for reaching higher Reynolds numbers, stabilization should be efficiently combined with mesh refinement, because both of these factors improve the stability of the method. We have shown, that residual stabilization is not as innocent in practice as available proofs of convergence claim, and people, who use stabilized methods, should be aware of this fact and always take care of the final accuracy of their computations.

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