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THE DISCONTINUOUS GALERKIN METHOD FOR LOW-MACH FLOWS *

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1. Introduction

This work is concerned with the numerical solution of an inviscid compressible gas flow with the aid of the discontinuous Galerkin finite element method. Our goal is to develop a sufficiently accurate and robust method capable of solving flows with a wide range of Mach numbers – the Mach number is the ratio of the velocity of the flow to the local speed of sound. Since many numerical methods are capable of solving the high Mach number case and fail for low-Mach flows, this work is focused on overcoming the obstacles that arise when the Mach number tends to zero.

These obstacles include: *a)* severe limitations on the time step proportional to the Mach number (*CFL*-like condition) when using explicit time discretization, *b)* the fact that as the Mach number tends to zero, the compressible Euler equations tend to a *incompressible* limit, therefore the implications of the theory of incompressible flows must be taken into account (e.g. Babuška-Brezzi condition), *c)* a correct treatment of inlet/outlet boundary conditions, which must be transparent for acoustic effects (perturbations of density and pressure) *d)* the matrices involved in the numerical solution become very ill conditioned.

To solve these problem a semi-implicitly linearized method derived in [4] is used, allowing for much greater time steps than the explicit algorithms. However, this method gives very bad results – it is shown that the choice of boundary conditions is crucial in this matter. New boundary conditions are therefore proposed and tested and a simple block Jacobi preconditioner is applied to the linear solver. The semi-implicit time stepping combined with appropriate boundary conditions give a robust algorithm capable of solving both high and low Mach number flows (Mach number as low as 10^{-6}).

2. Continuous problem

We shall be concerned with inviscid compressible two-dimensional flows. Let $T > 0$, $\Omega \subset I\!R^2$ and $Q_T = \Omega \times (0, T)$. We define disjoint boundary components $\Gamma_I, \Gamma_O, \Gamma_W$, the *inlet*, *outlet* and *impermeable wall* respectively, such that $\partial\Omega = \Gamma_I \cup \Gamma_O \cup \Gamma_W$.

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We also define $\Gamma_{IO} = \Gamma_I \cup \Gamma_O$. The system of Euler equations describing 2D inviscid compressible flow can be written in the form of a conservation law for the *state vector* $\mathbf{w}(x,t)$:

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{f}_s(\mathbf{w})}{\partial x_s} = 0 \quad \text{in } Q_T, \tag{1}$$

where $\boldsymbol{f}_s$, $s = 1, 2$, are the *inviscid fluxes* and

$$\begin{gathered} \mathbf{w} = (\rho, \rho v_1, \rho v_2, e)^{\mathrm{T}} \in I\!R^4, \\ \boldsymbol{f}_i(w) = (f_{i1}(\mathbf{w}), \dots, f_{i4}(\mathbf{w}))^{\mathrm{T}} = (\rho v_i, \rho v_1 v_i + \delta_{1i} p, \rho v_2 v_i + \delta_{2i} p, (e+p) v_i)^{\mathrm{T}}. \end{gathered} \tag{2}$$

Here the following notation has been used: ρ - density, p - pressure, $\boldsymbol{v} = (v_1, v_2)$ - velocity, e - total energy. Furthermore we add the following relation derived from the equation of state:

$$p = (\gamma - 1)(e - \rho |\boldsymbol{v}|^2/2), \tag{3}$$

where $\gamma > 1$ is the Poisson adiabatic constant.

3. Discontinuous Galerkin discretization

Let $\mathcal{T}_h$ be a partition of the closure $\overline{\Omega}$ into a finite number of triangles, whose interiors are mutually disjoint. We define an index set $I \subset \mathbb{Z}^+ = \{0, 1, 2, \dots\}$ such that all elements of $\mathcal{T}_h$ are numbered by indices from I. If two elements $K_i, K_j \in \mathcal{T}_h$ share a common face, we call them *neighbours* and set $\Gamma_{ij} = \partial K_i \cap \partial K_j$. For $i \in I$ we define $s(i) = \{j \in I; K_j \text{ is a neighbour of } K_i\}$. By $\mathbf{n}_{ij}$ we denote the unit outer normal to ∂K_i on the face Γ_{ij}.

Over $\mathcal{T}_h$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K) \; \forall K \in \mathcal{T}_h\} \tag{4}$$

and for $v \in H^1(\Omega, \mathcal{T}_h)$ we set $v|_{\Gamma_{ij}} =$ *trace* of $v|_{K_i}$ on Γ_{ij}. The *discontinuous Galerkin finite element method* uses a weak form of equation (1) in the sense of the space $[H^k(\Omega, \mathcal{T}_h)]^4$ and we approximate this space by the space of discontinuous vector-valued piecewise polynomial functions

$$S_h = S^{p,-1}(\Omega, \mathcal{T}_h) = \{v; v|_K \in P_p(K) \; \forall K \in \mathcal{T}_h\}, \tag{5}$$

where $P_p(K)$ is the space of all polynomials on K of degree $\leq p$.

We multiply (1) by a test function $\boldsymbol{\varphi} \in [H^1(\Omega, \mathcal{T}_h)]^4$ and integrate over $K_i \in \mathcal{T}_h$. With the aid of Green's theorem and summing over all $i \in I$, we obtain

$$\begin{aligned} \frac{d}{dt} \sum_{K_i \in \mathcal{T}_h} \int_{K_i} \mathbf{w} \cdot \boldsymbol{\varphi} \, dx &= \sum_{K_i \in \mathcal{T}_h} \int_{K_i} \sum_{s=1}^{2} \boldsymbol{f}_s(\mathbf{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, dx \\ &\quad - \sum_{K_i \in \mathcal{T}_h} \sum_{j \in S(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \boldsymbol{f}_s(\mathbf{w}) n_{ij}^{(s)} \cdot \boldsymbol{\varphi} \, dS. \end{aligned} \tag{6}$$

In the second right-hand side term, we use the approximation

$$\int_{\Gamma_{ij}} \sum_{s=1}^{2} \boldsymbol{f}_s(\mathbf{w}) n_s \cdot \boldsymbol{\varphi}\, dS \approx \int_{\Gamma_{ij}} \mathbf{H}(\mathbf{w}|_{\Gamma_{ij}}, \mathbf{w}|_{\Gamma_{ji}}, \mathbf{n}_{ij}) \cdot \boldsymbol{\varphi}\, dS, \tag{7}$$

incorporating a numerical flux $\mathbf{H}$, as known from the finite volume method. Now we introduce the form

$$\begin{aligned} b_h(\mathbf{w},\boldsymbol{\varphi}) = &-\sum_{i\in I} \int_{K_i} \sum_{s=1}^{2} \boldsymbol{f}_s(\mathbf{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s}\, dx \\ &+ \sum_{i\in I} \sum_{j\in S(i)} \int_{\Gamma_{ij}} \mathbf{H}(\mathbf{w}|_{\Gamma_{ij}}, \mathbf{w}|_{\Gamma_{ji}}, \mathbf{n}_{ij}) \cdot \boldsymbol{\varphi}\, dS, \end{aligned} \tag{8}$$

where $\mathbf{w}|_{\Gamma_{ji}}$ for $\Gamma_{ij} \subset \partial\Omega$ is defined using appropriate boundary conditions. We can define the DGFEM scheme as:

$$\frac{d}{dt}(\mathbf{w}_h(t), \boldsymbol{\varphi}_h) + b_h(\mathbf{w}_h(t), \boldsymbol{\varphi}_h) = 0, \quad \forall \boldsymbol{\varphi}_h \in \mathbf{S}_h,\ \forall t \in (0,T) \tag{9}$$

with appropriate initial conditions.

4. Time discretization

Let $0 < t_0 < t_1 < \ldots$ be a partition of the time interval $(0,T)$ and $\tau_k = t_{k+1} - t_k$. We seek $\mathbf{w}_h^k \approx \mathbf{w}_h(t_k)$ such that

$$\left(\frac{\mathbf{w}_h^{k+1} - \mathbf{w}_h^k}{\tau_k}, \boldsymbol{\varphi}_h \right) + b_h(\mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) = 0, \quad \forall \boldsymbol{\varphi}_h \in \mathbf{S}_h,\ k = 0,1,\ldots, \tag{10}$$

This backward Euler scheme however leads to a large system of highly nonlinear equations. Therefore in [4] a simplified linearization of the form b_h from problem (10) is presented in order to obtain a large (sparse) system of linear equations instead.

We treat the interior and boundary terms in (10) separately:

$$\begin{aligned} b_h(\mathbf{w}_h^{k+1},\boldsymbol{\varphi}_h) = &-\underbrace{\sum_{i\in I} \int_{K_i} \sum_{s=1}^{2} \boldsymbol{f}_s(\mathbf{w}_h^{k+1}) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}\, dx}_{:=\tilde{\sigma}_1} \\ &\underbrace{+ \sum_{i\in I} \sum_{j\in S(i)} \int_{\Gamma_{ij}} \mathbf{H}(\mathbf{w}_h^{k+1}|_{\Gamma_{ij}}, \mathbf{w}_h^{k+1}|_{\Gamma_{ji}}, \mathbf{n}_{ij}) \cdot \boldsymbol{\varphi}_h\, dS}_{:=\tilde{\sigma}_2}\,. \end{aligned} \tag{11}$$

For $\tilde{\sigma}_1$ we use the *homogeneity* of the Euler fluxes, which implies $\boldsymbol{f}_s(\mathbf{w}) = \mathbb{A}_s(\mathbf{w})\mathbf{w}$, where $\mathbb{A}_s(\mathbf{w}) = D\boldsymbol{f}_s(\mathbf{w})/D\mathbf{w},\ s = 1,2$, and set

$$\sigma_1 = \sum_{i\in I} \int_{K_i} \sum_{s=1}^{2} \mathbb{A}_s(\mathbf{w}_h^k)\, \mathbf{w}_h^{k+1} \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}\, dx. \tag{12}$$

In order to treat the term $\tilde{\sigma}_2$, the Vijayasundaram numerical flux is chosen. This numerical flux is written in the form

$$\mathbf{H}_{VS}(\mathbf{w}_L, \mathbf{w}_R, \mathbf{n}) = \mathbb{P}^+\left(\frac{\mathbf{w}_L + \mathbf{w}_R}{2}, \mathbf{n}\right)\mathbf{w}_L + \mathbb{P}^-\left(\frac{\mathbf{w}_L + \mathbf{w}_R}{2}, \mathbf{n}\right)\mathbf{w}_R, \tag{13}$$

which is suitable for the linearization of the terms in $\tilde{\sigma}_2$. For interior edges this reads:

$$\sum_{i\in I}\sum_{j\in s(i)}\int_{\Gamma_{ij}}\left[\mathbb{P}^+\big(\langle\mathbf{w}_h^k\rangle_{ij}, \mathbf{n}_{ij}\big)\,\mathbf{w}_h^{k+1}|_{\Gamma_{ij}} + \mathbb{P}^-\big(\langle\mathbf{w}_h^k\rangle_{ij}, \mathbf{n}_{ij}\big)\,\mathbf{w}_h^{k+1}|_{\Gamma_{ji}}\right]\cdot\boldsymbol{\varphi}_h\,dS, \tag{14}$$

where $\langle\mathbf{w}\rangle_{ij} = (\mathbf{w}_{ij} + \mathbf{w}_{ji})/2$. For $\Gamma_{ij} \subset \Gamma_{IO}$ we have no other choice than to treat $\mathbf{w}_h^{k+1}|_{\Gamma_{ji}}$ explicitly, i.e. $\mathbf{w}_h^{k+1}|_{\Gamma_{ji}} \approx \mathbf{w}_h^{k}|_{\Gamma_{ji}}$. For $\Gamma_{ij} \subset \Gamma_W$ the *no-stick* condition is prescribed. This implies the form of the numerical flux on Γ_W for extrapolated pressure p:

$$H_W(\mathbf{w}_L, \mathbf{w}_R, \mathbf{n}) = p(0, n_1, n_2, 0)^{\mathrm{T}}. \tag{15}$$

Written in conservative variables, (15) can be linearized, using again the homogeneity of inviscid fluxes.

5. Boundary conditions

The choice of appropriate boundary conditions is a delicate problem which plays a key role in the presented algorithm. Boundary conditions are incorporated into the DGFEM, as in the finite volume method, via the choice of $H(\mathbf{w}_L, \mathbf{w}_R, \mathbf{n})$ or $\mathbf{w}_R = \mathbf{w}|_{\Gamma_{ji}}$ for boundary edges. In the case of Γ_W, we prescribe the boundary flux (15). The situation is much more problematic on the inlet and outlet. One choice often used in practice is to prescribe ρ and $\mathbf{v}$ and extrapolate p on the inlet and to prescribe p and extrapolate ρ and $\mathbf{v}$ on the outlet. However these standard boundary conditions reflect acoustic effects coming from the inside of Ω. This behavior is nonphysical and the reflected interfering density and pressure waves corrupt the solution in the low-Mach number case. To cure this disease new *characteristic based* boundary conditions are derived, which reflect the hyperbolic character of the Euler equations and are transparent to acoustic effects.

Using the rotational invariance and homogeneity we write the Euler equations in the nonconservative form

$$\frac{\partial\mathbf{q}}{\partial t} + \mathbb{A}_1(\mathbf{q})\frac{\partial\mathbf{q}}{\partial\tilde{x}_1} = 0, \tag{16}$$

where $\mathbf{q} = \mathbb{Q}(\mathbf{n})\mathbf{w}$ and $\mathbb{Q}(\mathbf{n})$ is a standard 4×4 rotational matrix (see [1]). We linearize this system around the state $\mathbf{q}_i = \mathbb{Q}(\mathbf{n})\mathbf{w}_i$ and obtain the linear system

$$\frac{\partial\mathbf{q}}{\partial t} + \mathbb{A}_1(\mathbf{q}_i)\frac{\partial\mathbf{q}}{\partial\tilde{x}_1} = 0, \tag{17}$$

which will be considered in the set $(-\infty, 0)\times(0, \infty)$ and equipped with the initial and boundary conditions

$$\mathbf{q}(\tilde{x}_1, 0) = \mathbf{q}_i,\ \tilde{x}_1 \in (-\infty, 0) \quad \text{and} \quad \mathbf{q}(0, t) = \mathbf{q}_j,\ t > 0. \tag{18}$$

The goal is to choose $\mathbf{q}_j$ in such a way that this initial-boundary problem is well posed, i.e. has a unique solution. This linearized system has a solution which can be obtained using the method of characteristics. We get the following result:

We shall take some state $\mathbf{q}_j^0 = \mathbb{Q}(\mathbf{n})\mathbf{w}_j^0$. The state $\mathbf{w}_j^0$ is the state vector of the far-field flow, or the incoming fluid at the inlet, or the initial condition, depending on the situation and interpretation. We calculate the eigenvectors $\mathbf{r}_s, s = 1, \dots, 4$ of the matrix $\mathbb{A}_1(\mathbf{q}_i)$, arrange them as columns in the matrix $\mathbb{T}$ and calculate $\mathbb{T}^{-1}$ (explicit formulae can be found in [1] or [2]). We calculate

$$\boldsymbol{\beta} = \mathbb{T}^{-1}\mathbf{q}_i, \quad \boldsymbol{\alpha} = \mathbb{T}^{-1}\mathbf{q}_j^0. \tag{19}$$

Now we calculate the state $\mathbf{q}_j$ according to the presented process:

$$\mathbf{q}_j := \sum_{s=1}^{4} \gamma_s \mathbf{r}_s = \mathbb{T}\boldsymbol{\gamma}, \quad \gamma_s = \begin{cases} \alpha_s, & \lambda_s \geq 0, \\ \beta_s, & \lambda_s < 0 \end{cases} \tag{20}$$

and $\lambda_s,\ s = 1, \dots, 4$ are eigenvalues of $\mathbb{A}_1(\mathbf{q}_i)$. Finally the sought boundary state is $\mathbf{w}_j = \mathbb{Q}^{-1}(\mathbf{n})\mathbf{q}_j$.

In the framework of the presented theory, these boundary conditions seem to give the natural choice for $\mathbf{w}_j$. Experiments show that this method applied to the inlet and outlet give natural results. Undesired density and pressure waves pass through the boundaries without *any* reflection, even when applied to low-Mach flows.

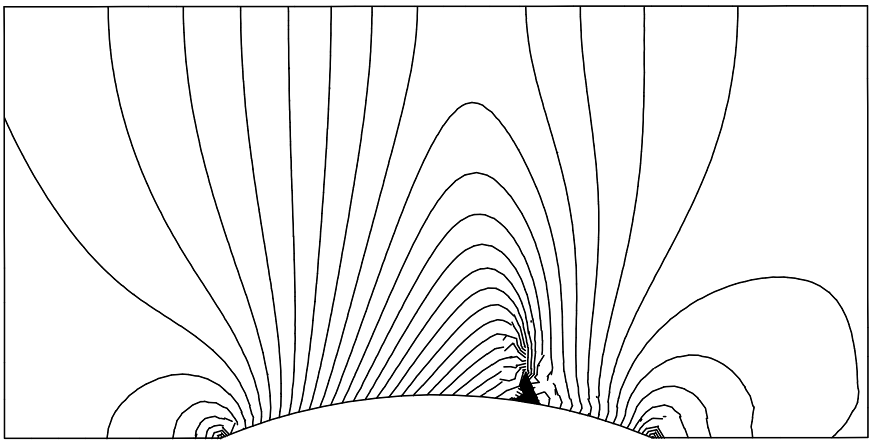

Fig. 1: *GAMM channel density isolines for inlet Mach number 0.67.*

6. Numerical experiments

The presented algorithm is tested on the GAMM channel (10% circular bump) for inlet Mach number ranging from 0.67 (Figure 1) to 10^{-6} (Figure 2). In the first case the characteristic Zierep singularity is present. In the latter case we see the algorithm gives good results for very low Mach numbers. The method is also

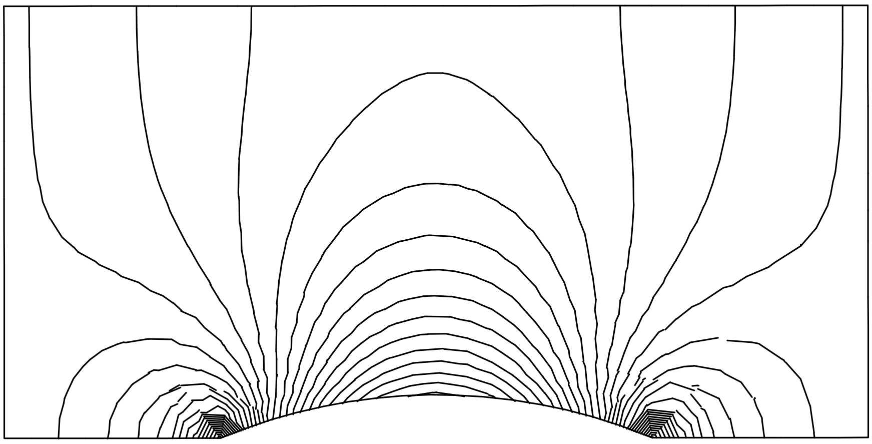

Fig. 2: *GAMM channel Mach number isolines for inlet Mach number* 10^{-6}.

tested for the production of numerical viscosity – flow around a cylinder should be symmetric. Figure 3 shows the presence of a small wake for inlet Mach number 10^{-2}. However, this effect is most likely caused by a insufficient representation of the curved boundary (we currently use bilinear mappings on the reference triangle). Figure 4 shows that this effect diminishes as the Mach number tends to zero. In general, the behavior of the presented method improves for very small Mach numbers.

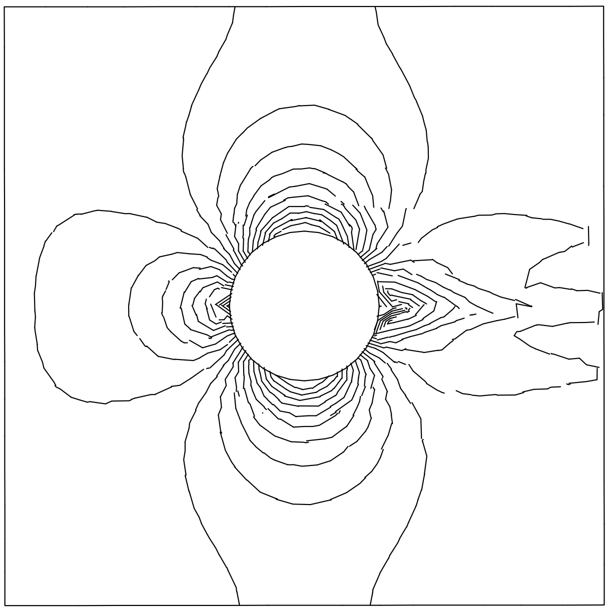

Fig. 3: *Flow around cylinder Mach number isolines for inlet Mach number* 10^{-2}.

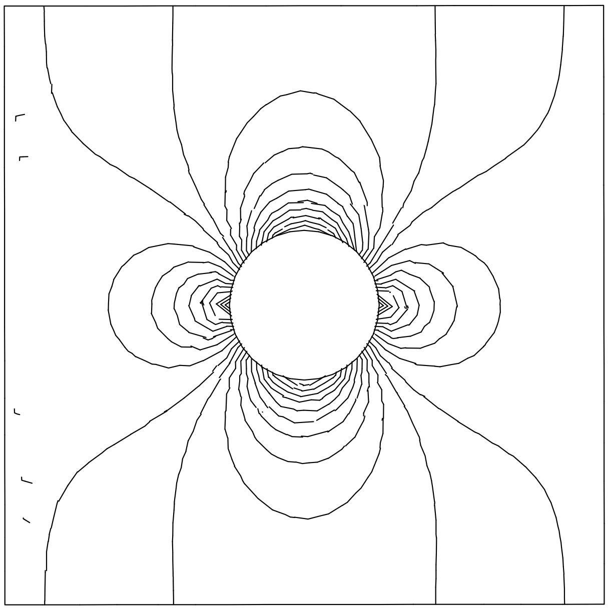

Fig. 4: *Flow around cylinder Mach number isolines for inlet Mach number* 10^{-6}.

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