Martin Soukenka

On a dynamics in rational polygonal billiards

In: Jan Chleboun and Petr Přikryl and Karel Segeth and Jakub Šístek (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 6-11, 2010. Institute of Mathematics AS CR, Prague, 2010. pp. 177–182.

Persistent URL: http://dml.cz/dmlcz/702757

ON A DYNAMICS IN RATIONAL POLYGONAL BILLIARDS*

Martin Soukenka

1 Introduction

A polygonal billiard table is a planar simply connected compact polygon P. The billiard flow $\{T_t\}_{t \in R}$, in P is generated by the free motion of a mass-point subject to the elastic reflection in the boundary. This means that the point moves along a straight line in P with a constant speed until it hits the boundary. At a smooth boundary point the billiard ball reflects according to the well known law of geometrical optics: the angle of incidence equals to the angle of reflection. If the billiard ball hits a corner, (a non-smooth boundary point), its further motion is not defined. Additionally to a corner, the billiard trajectory is not defined for a direction tangent to a side. By D we denote the group generated by the reflections in the lines through the origin, parallel to the sides of the polygon P. It is known that the group D is finite, when all the angles of P are of the form $\pi m_i/n_i$ with distinct coprime integers m_i, n_i. In this case $D = D_N$ the dihedral group is generated by the reflections in lines through the origin that meet at angles π/N, where N is the least common multiple of n_i's and a trajectory changes its direction by $2N$ directions of the group D_N. In this case the polygon is called *rational*.

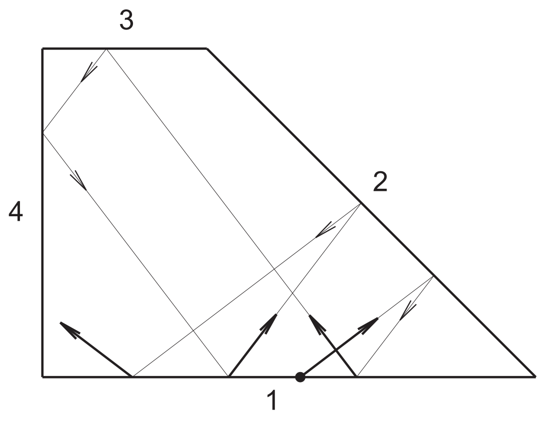

Fig. 1: *Elements of the group D_4 - directions of trajectory.*

*This work was supported by grant SGS 10-800020 of the Czech Technical University in Prague and research project of the Institute of Thermomechanics No AV0Z20760514.

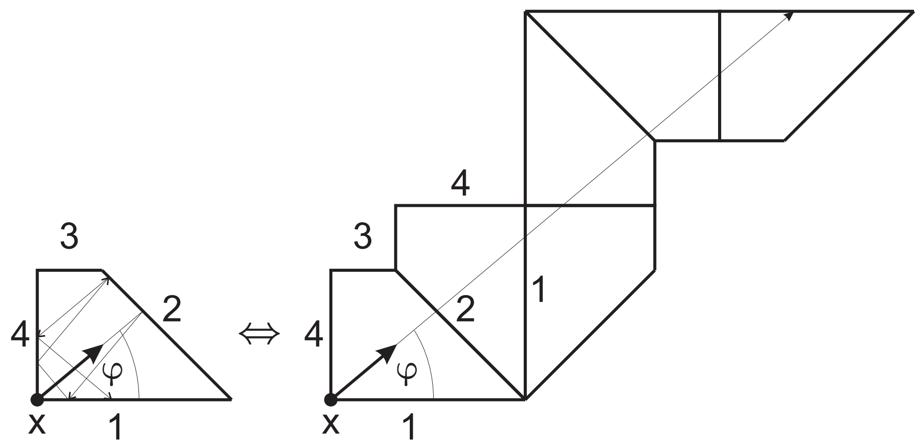

Fig. 2: *Unfolding of trajectory of* (x, φ) *with symbolic itinerary* $1, 2, 1, 4, 2, \ldots$

Example 1. Consider a polygon from Figure 1 with angles (counterclockwise) $\pi/2$, $\pi/4$, $3\pi/4$, $\pi/2$, hence $N = 4$, so the group contains eight directions (see seven of them in Figure 1).

The aim of this text is to illustrate to non-experts some open problems in rational billiards and to point out that even in theoretical problems usage of numerical experiments can be at the beginning of the exact theoretical solution. We note that there are many open problems in rational billiards, most of which are studied by both nuclear and theoretical physicists and mathematicians [2], [1]. Let us first give some notations and basic facts.

For a simply connected polygon P with k sides consider counterclockwise orientation of its boundary ∂P and denote by $[p_i, p_{i+1}]$ a closed arc with outgoing endpoint p_i and incoming endpoint p_{i+1} where p_i is i-th corner of P. Let us denote by $1 := [p_1, p_2], 2 := [p_2, p_3], \ldots, k := [p_k, p_1]$ the sides of P (Figure 1). Consider a pair (x, φ), where $x \in \partial P$ (so called a *foot point*) and φ is a direction. By a *symbolic itinerary* $a_0, a_1, a_2, \ldots$ of (x, φ) we mean the sequence of visited numbers $a_i \in \{1, 2, \ldots, k\}$ of sides of ∂P by trajectory $T_t x$. There is a simple way how to visualize a trajectory of a given (x, φ), see Figure 2 - draw the straight line starting at $x \in \partial P$ under the angle φ from the side $\partial P \ni x$ and reflect in successive steps the polygon by the sides crossing the line. This is called *unfolding* of a trajectory in P.

2 Dynamics in hammer polygons

In the rest of this text we consider very special shaped polygon P from Figure 1 and Example 1 - a *hammer polygon* (shortly h-polygon). Let us now present two open problems in h-polygons. In what follows by f^n of a map f we mean $f(f^{n-1})$.

2.1 Is dynamics in h-polygons Li-York chaotic?

Let us start with the famous problem in polygonal billiards - an existence of so called *Li-York chaos*.

Definition 1. Let X be a compact metric space with metric ϱ and $f : X \to X$ continuous map. A pair (x, y) of points $x, y \in X, x \neq y$ is called *Li-York pair*, if

$$\liminf_{n\to\infty} \varrho(f^n x, f^n y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} \varrho(f^n x, f^n y) > 0.$$

If there is a Li-York pair, then the dynamical system (X, f) is called *Li-York chaotic.*

It can be shown that an existence of Li-York pair in billiard dynamics can be formulated as follows: let P be a fixed h-polygon and $u := (x, \varphi)$, $x \in \partial P$, resp. $v := (y, \psi)$, $y \in \partial P$, $u \neq v$ both generate infinite symbolic itineraries $a_0, a_1, a_2, \ldots$, resp. $b_0, b_1, b_2, \ldots$ in P. Then (u, v) is Li-York pair, if lengths of blocks of the same numbers a_i, b_i are (non-monotonically) increasing to infinity as $i \to \infty$, see Figure 3.

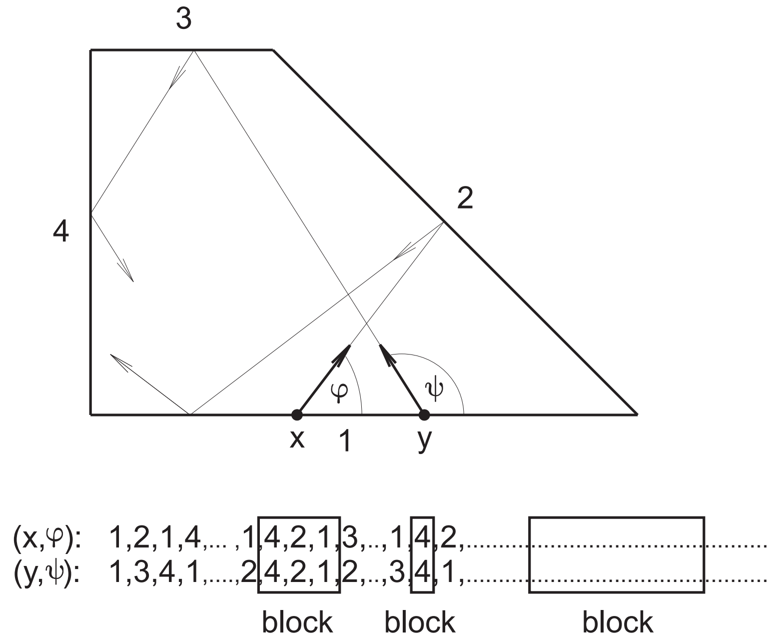

Fig. 3: *Blocks of the same numbers a_i, b_i of lengths* $3, 1, \ldots$

Problem 1. *Is billiard dynamical system in h-polygon Li-York chaotic?*

For h-polygon P of a fixed size we have been trying to find a Li-York pair in P. Numerical results of a number of the blocks of various lengths (denoting by L_{block}) bigger than 5 for some (u, v) in 10^9 iterations (thus in sequences $\{a_i\}, \{b_i\}$ for $i = 0, 1, \ldots, 10^9$) are listed below. Notice the extremely nonuniform frequencies

$L_{\text{block}} > 5$	frequency in 10^9 iter	$L_{\text{block}} > 5$	frequency in 10^9 iter
186	1	376	14180
360	1152	377	27073
361	480	378	20446
363	2040	379	5636
364	1008	380	2416
365	1056	382	3580
366	1824	383	12364
367	3704	385	5798
368	12828	386	136
369	1966	388	1152
370	4580	389	652
371	23954	393	176
373	43441	395	68
375	10066		

Tab. 1: *A number of the blocks of lengths bigger than 5 of the same numbers a_i, b_i in sequences $\{a_i\}, \{b_i\}$ for $i = 0, 1, \ldots, 10^9$.*

of lengths: there is no block of length bigger than 5 and less than 360 except one exception - value 186 occurs due to "initial conditions" - it represents first 186 iterates at all. What are the next values after 395 as increasing a number of iterates nobody knows.

Theoretical explanation of these mysterious behavior is a subject of our further (theoretical) research. However, detailed analysis, which goes out of the purpose of this text, allow us to postulate the following conjecture.

Conjecture 1. *Billiard dynamical system in h-polygon is Li-York chaotic.*

2.2 What is the maximal length of the same itinerary in quasisimilar h-polygons?

As it is a frequent situation in physics and mathematics, one can ask how the outcome (for instance, a solution of ODE's or PDE's) changes if the income (the right hand side of a given ODE's or PDE's) changes by a some (small) perturbation. According to this we shall consider so called *quasisimilar* $h-$polygons and ask how the dynamics can change in these polygons. Let us first give some notations.

Let P be h-polygon. For a point $x \in k \subset \partial P$ and direction φ we say that a pair (x, φ) is *minimal* for P if the first 7 reflections of the trajectory of (x, φ) generate 4 different directions of the group on side k (for instance, the pair "$(\bullet, \nearrow)$" in Figure 1 is minimal while that of (x, φ) in Figure 2 with $\varphi = \pi/2 - \varepsilon$ is not).

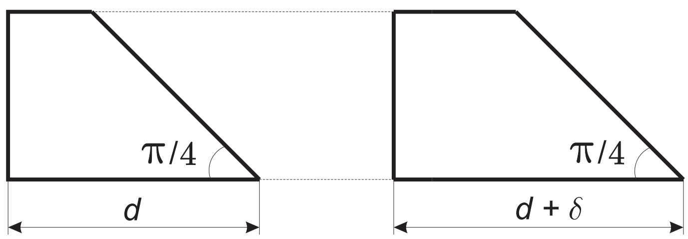

Fig. 4: *Quasisimilar h-polygons* P*,* P_δ *for perturbation* $\delta > 0$*.*

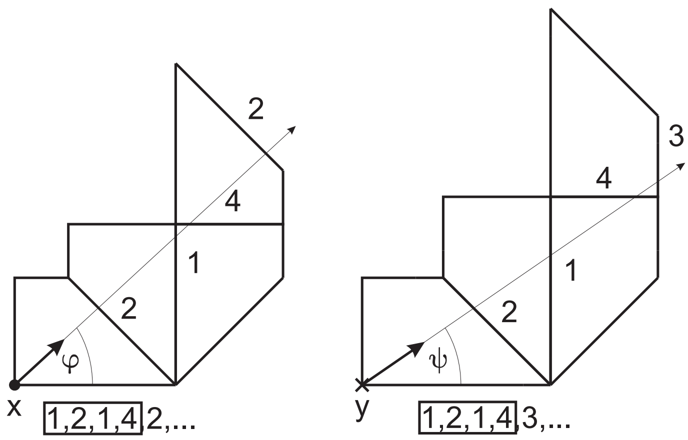

Fig. 5: *Unfoldings of trajectories of* $u = (x, \varphi)$*,* $v = (y, \psi)$ *with* $L_u(\delta, v) = 4$*.*

We consider *quasisimilar* to P polygon as in Figure 4 for perturbation $\delta > 0$, that is a polygon P_δ with the same angles as P including ordering but with different sizes of the sides.

For a symbolic itinerary $a_0, a_1, a_2, \dots$ of $u := (x, \varphi)$ in h-polygon P and a symbolic itinerary $b_0, b_1, b_2, \dots$ of $v := (y, \psi)$ in P_δ for some $\delta > 0$ we denote by

$$L_u(\delta, v) := \max_i \{i + 1\,;\, a_j - b_j = 0, j = 0, 1, \dots i\}$$

Example 2. Consider pair $u := (x, \varphi)$ in P, resp. $v := (y, \psi)$ in P_δ as in Figure 5. The itineraries are

$$\{a_n\}_{n=0}^{\infty} : \quad 1, 2, 1, 4, 2, \dots$$
$$\{b_n\}_{n=0}^{\infty} : \quad 1, 2, 1, 4, 3, \dots$$

Then $L_u(\delta, v) = \max\{1, 2, 3, 4\} = 4$.

Problem 2. *Consider a point* $x \in \partial P$ *as a corner of* P *and take* φ *such that* $u = (x, \varphi)$ *is fixed minimal pair in* P *generating an infinite symbolic itinerary*

SI : $a_0, a_1, a_2, \dots$ in P. Let $\delta > 0$. Consider $y \in \partial P_\delta$ as a corresponding fixed corner to that of x (that is, corners x, y have the same counterclockwise number, see Figure 5) and a symbolic itinerary of (y, ψ) in P_δ. What is the value of $L_{\max}(\delta) := \max_\psi L_u(\delta, \psi)$?

The answer to Problem 2 is far from trivial. However, detailed theoretical (geometrical) analysis allows us to count values $L_{\max}(\delta) = \max_\psi L_u(\delta, \psi)$ numerically. Numerical experiments show, that there are big jumps of $L_{\max}(\delta)$ for a fixed u in P when varying a value of a perturbation δ. A typical illustration of this jump for some fixed size of P and u, depending on δ, is listed bellow (compare $\delta = 0.50$ and $\delta = 0.51$). Theoretical explanation of these empirical results remains open.

Perturbation δ	$L_{\max}(\delta)$	Perturbation δ	$L_{\max}(\delta)$
0.47	1260	0.52	130
0.48	1254	0.53	85
0.49	1248	0.54	65
0.50	1242	0.55	55
0.51	255	0.56	45

Tab. 2: *Jumps of values of $L_{\max}(\delta)$ from Problem 2 when varying $\delta > 0$.*

3 Remark on numerical computation

As far as computing, all the codes we have used are written in Fortran 90 using the library for multiple precision computing FMLIB which allows one to set a number of significant digits. We have set 60 resp. 70 for controlling computation.

References

[1] Bobok, J. and Troubetzkoy, S.: Does a billiard orbit determine its (polygonal) table? Accepted to Fundamenta Mathematicae, 2010, 17 pp.

[2] Masur, H. and Tabachnikov, S.: Rational billiards and flat surfaces. In: B. Hasselblatt and A. Katok (Eds.), *Handbook of Dynamical Systems*, vol. 1A, Elsevier Science B.V., 2002.