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A COMPARISON OF SOME A POSTERIORI ERROR ESTIMATES FOR FOURTH ORDER PROBLEMS*

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Abstract

A lot of papers and books analyze analytical a posteriori error estimates from the point of view of robustness, guaranteed upper bounds, global efficiency, etc. At the same time, adaptive finite element methods have acquired the principal position among algorithms for solving differential problems in many physical and technical applications. In this survey contribution, we present and compare, from the viewpoint of adaptive computation, several recently published error estimation procedures for the numerical solution of biharmonic and some further fourth order problems including computational error estimates.

1 Introduction

In the *hp*-adaptive finite element method, there are two possibilities to assess the error of the computed solution a posteriori: to construct an *analytical error estimate* or to obtain, by the same procedure as the approximate solution, a *computational error estimate*. In the latter case, a *reference solution* is computed in a systematically refined mesh and, at the same time, with polynomial degree of all elements increased by 1 (see, e.g., [4], [9]).

In the paper, we are concerned with several formulations of the biharmonic problem and a general 4th order elliptic problem on a 2D domain. We present analytical a posteriori error estimates of different nature found in literature for these problems. We are primarily concerned with the computability of the right-hand parts of the estimates. In conclusion, we assess the advantages and drawbacks of the analytical as well as computational estimates in general.

We use common notation based primarily on the book [3]. For the lack of space, we sometimes only refer to the notation introduced in the papers quoted. The complete hypotheses of the theorems presented should be also looked for there. A more detailed version of the paper should appear elsewhere.

2 Dirichlet and second problems for biharmonic equation

2.1 Dirichlet problem

Let $\Omega \subset R^2$ have a polygonal boundary Γ. We consider the two dimensional biharmonic problem

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$$\Delta^2 u = f \quad \text{in} \quad \Omega, \tag{2.1}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma \tag{2.2}$$

with $f \in L_2(\Omega)$ that models, e.g., the vertical displacement of the mid-surface of a clamped plate subject to bending.

We use the standard formulation of the weak solution $u \in X = H_0^2(\Omega)$ and approximate solution $u_h \in X_h$ written in the form $\langle F(u), v\rangle = 0$ and $\langle F_h(u_h), v_h\rangle = 0$. Denote by k, $k \geq 1$, the maximum degree of polynomials in X_h. Further, put $f_h = \sum_{T\in\mathcal{T}_h} \pi_{l,T} f$, where T is a triangle of the triangulation $\mathcal{T}_h$, $\mathcal{E}_h$ is the set of all its edges, P_l, $l \geq 0$ fixed, is the space of polynomials of degree at most l and $\pi_{l,S}$, $S \in \mathcal{T}_h \cup \mathcal{E}_h$, is the L_2 projection of $L_1(S)$ onto $P_{l|S}$.

Put $\varepsilon_T = \|f - f_h\|_{0;T}$. Let h_T be the diameter of the triangle T and h_E the length of the edge E, $\mathcal{E}(T)$ the set of all edges of the triangle T, and $\mathcal{E}_{h,\Omega}$ the set of all inner edges of $\mathcal{T}_h$. Denote by n_E the normal to the edge E and by $[q]_E$ the jump of the function q over the edge E. Defining the *local residual a posteriori error estimator*

$$\eta_{\mathrm{V},T} = \left(h_T^4 \|\Delta^2 u_h - f_h\|_{0;T}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} (h_E \|[\Delta u_h]_E\|_{0;E}^2 + h_E^3 \|[n_E \cdot \nabla\Delta u_h]_E\|_{0;E}^2) \right)^{1/2}$$

for all $T \in \mathcal{T}_h$, we have the following theorem [11].

Theorem 2.1 *Let $u \in X$ be the unique weak solution of the problem* (2.1), (2.2) *and let $u_h \in X_h$ be an approximate solution of the corresponding discrete problem. Then we have the a posteriori estimates*

$$\|u - u_h\|_2 \leq c_1 \left(\sum_{T\in\mathcal{T}_h} \eta_{\mathrm{V},T}^2\right)^{1/2} + c_2 \left(\sum_{T\in\mathcal{T}_h} h_T^4 \varepsilon_T^2\right)^{1/2} + c_3 \|F(u_h) - F_h(u_h)\| + c_4 \|F_h(u_h)\|$$

and

$$\eta_{\mathrm{V},T} \leq c_5 \|u - u_h\|_{2;\omega_T} + c_6 \left(\sum_{T'\subset\omega_T} h_{T'}^4 \varepsilon_{T'}^2\right)^{1/2}$$

for all $T \in \mathcal{T}_h$. The quantities $c_1, \ldots, c_6$ depend only on h_T/ρ_T, and the integers k and l. Here ω_T is the set of all neighbors of the triangle T and ρ_T the diameter of the circle inscribed to T.

The proof is given in [11].

2.2 Dirichlet problem in mixed formulation

Let $\Omega \subset R^2$ be a convex polygon with boundary Γ. Again, we consider the biharmonic problem (2.1), (2.2) with $f \in H^{-1}(\Omega)$. The problem is concerned in practice with both linear plate analysis and incompressible flow simulation.

We employ the Ciarlet-Raviart weak formulation of the problem (2.1) and (2.2) for the solution $\{w = \Delta u, u\}$ and the corresponding conforming second order approximate solution $\{w_h, u_h\}$. Let us put $f_h = \Pi_h f$ where Π_h denotes the L_2 orthogonal projection on the set of piecewise constant functions on triangles.

The local residuals $\mathcal{P}_T$, $\mathcal{R}_T$, $\mathcal{P}_E$, and $\mathcal{R}_E$ are defined in [2]. Denoting the area of the triangle T by $|T|$, we introduce the *local residual a posteriori error estimators*

$$\eta^2_{\mathrm{C},T} = |T| \|\mathcal{P}_T(u_h)\|^2_{0;T} + \tfrac{1}{2} \sum_{E \in \mathcal{E}(T)} h_E \|\mathcal{P}_E(u_h)\|^2_{0;E}$$

and $\widetilde{\eta}_{\mathrm{C},T}$ computed from $\mathcal{R}_T$ and $\mathcal{R}_E$. We put $e_h(u) = u - u_h$ and $e_h(w) = w - w_h$. Then the following theorem holds [2].

Theorem 2.2 *Let $\{w, u\}$ be the unique mixed weak solution of the problem* (2.1) *and* (2.2)*, and let $\{w_h, u_h\}$ be an approximate solution of the corresponding discrete problem. For $T \in \mathcal{T}_h$ we then have the a posteriori estimates*

$$\|e_h(u)\|_1 + h\|e_h(w)\|_0 \le C_1 \left(\left(\sum_{T \in \mathcal{T}_h} \eta^2_{\mathrm{C},T} \right)^{1/2} + h^2 \left(\sum_{T \in \mathcal{T}_h} \widetilde{\eta}^2_{\mathrm{C},T} \right)^{1/2} \right),$$

$$\eta_{\mathrm{C},T} + h^2 \widetilde{\eta}_{\mathrm{C},T} \le C_2 \left(|e_h(u)|_{1;\omega_T} + h_T \|e_h(w)\|_{0;\omega_T} + h_T^3 \sum_{T' \subset \omega_T} \varepsilon_{T'} \right)$$

with some positive constants C_1 and C_2 independent of h.

The proof is given in [2].

2.3 Second problem in mixed formulation

Let $\Omega \subset R^2$ be a convex polygon with boundary Γ. We consider the two dimensional second biharmonic problem for the equation (2.1) with the boundary condition

$$u = \Delta u = 0 \quad \text{on} \quad \Gamma \tag{2.3}$$

with $f \in L_2(\Omega)$ that models the deformation of a simply supported thin elastic plate.

Again, we employ the Ciarlet-Raviart weak formulation of the problem (2.1) and (2.3). We introduce the quantities ε_1, ε_2, the gradient recovery operator Gv_h, and the *gradient recovery a posteriori error estimator* η_{L} like in [6]. Then the following theorem holds.

Theorem 2.3 *Let $\{w, u\}$ be the unique weak solution of the problem* (2.1) *and* (2.3), *and let $\{w_h, u_h\}$ be an approximate solution of the corresponding discrete problem. Then we have the a posteriori estimates*

$$c\eta_{\mathrm{L}}^2 - C_2\varepsilon_2^2 \le |w - w_h|_1^2 + |u - u_h|_1^2 \le C\eta_{\mathrm{L}}^2 + C_1\varepsilon_1^2$$

with some positive constants c, C, C_1, and C_2 independent of h.

The proof is given in [6].

2.4 Kirchhoff plate bending problem

We consider the bending problem of an isotropic linearly elastic plate. We employ the Kirchhoff plate bending model for the deflection $u \in H_0^2$ of the plate in the weak formulation [1]. The nonconforming finite element approximation of the problem is done in the *discrete Morley space* W_h of second degree piecewise polynomial functions on $\mathcal{T}_h$ [1].

Let us introduce the norm $|||w|||_h$ in $W_h \cup H^2$ and define the *local a posteriori error estimator* $\eta_{\mathrm{M},T}$ like in [1]. Then the following theorem holds.

Theorem 2.4 *Let $u \in H_0^2$ be the unique weak solution of the Kirchhoff plate bending problem and let $u_h \in W_h$ be an approximate solution of the corresponding discrete problem. Then we have the a posteriori estimates*

$$|||u - u_h|||_h \le C\left(\sum_{T\in\mathcal{T}_h} \eta_{\mathrm{M},T}^2 + \sum_{T\in\mathcal{T}_h} h_T^4\varepsilon_T^2\right)^{1/2} \quad \textit{and} \quad \eta_{\mathrm{M},T} \le |||u - u_h|||_{h;T} + h_T^2\varepsilon_T$$

with some positive constant C independent of h and for all $T \in \mathcal{T}_h$.

The proof is given in [1].

3 Dirichlet problem for fourth order elliptic equation

3.1. Let D^2u denote the Hessian matrix of a function $u : \Omega \to R$, $u \in H^2(\Omega)$. Let the matrix-valued function $\Lambda = [\lambda_{ik}]$, $\Lambda : \Omega \times R^{n\times n} \to R^{n\times n}$ be measurable and bounded with respect to the variable $x \in \Omega$ and of class C_2 with respect to the matrix variable $\Theta \in R^{n\times n}$.

Let $\Omega \subset R^n$ have a piecewise C_1 boundary. We consider the fourth order problem

$$\mathrm{div}^2\Lambda(x, \mathrm{D}^2u) = f \quad \text{in} \quad \Omega \tag{3.1}$$

with the boundary condition (2.2) and $f \in L_2(\Omega)$.

We assume that Λ' is positive definite with constants $0 < m \le M$. We introduce the weak solution $u \in H_0^2(\Omega)$ in the usual way.

Let $\bar{u}$ be an arbitrary function from $H_0^2(\Omega)$ considered as an approximation of the solution u. We measure the error of $\bar{u}$ by the functional $E(\bar{u})$ depending on Λ, D^2, and f [5].

For an arbitrary matrix-valued function $\Psi \in H(\mathrm{div}^2, \Omega) \cap L_\infty(\Omega, R^{n\times n})$ and an arbitrary scalar-valued function $w \in H_0^2(\Omega)$, define the *global a posteriori error estimator* $\eta_{\mathrm{K}}(\Psi, w, \bar{u})$ depending on m, M, the constant from the Friedrichs inequality for D^2 on $H_0^2(\Omega)$, and the Lipschitz continuity constant of Λ' [5]. Then the following theorem holds.

Theorem 3.1 *Let* $u \in H_0^2(\Omega)$ *be the unique weak solution of the problem* (3.1), (2.2) *and* $\bar{u} \in W^{2,\infty}(\Omega)$ *an arbitrary function. Then*

$$E(\bar{u}) \leq \eta_{\mathrm{K}}(\Psi, w, \bar{u}) \tag{3.2}$$

for any $\Psi \in H(\mathrm{div}^2, \Omega) \cap L_\infty(\Omega, R^{n\times n})$ *and* $w \in H_0^2(\Omega)$.

The proof of the theorem is based on a more general statement proven in [5]. An analogous result is obtained there for a similar error estimator easier to compute. There is an interesting question of optimizing the inequality (3.2) with respect to Ψ and w. Moreover, it is proven in [5] that the estimator η_{K} is sharp.

3.2. Let $\Omega \in R^n$ be a bounded connected domain and Γ its Lipschitz continuous boundary. We consider the 4th order elliptic problem for a scalar-valued function u,

$$\mathrm{div}\,\mathrm{Div}(\gamma\nabla\nabla u) = f \quad \text{in} \quad \Omega, \tag{3.3}$$

with the boundary condition (2.2) and $f \in L_2(\Omega)$, $\gamma = [\gamma_{ijkl}]_{i,j,k,l=1}^n$ and $\gamma_{ijkl} = \gamma_{jikl} = \gamma_{klij} \in L_\infty(\Omega)$.

We define the energy norm $|\!|\!|\Phi|\!|\!|$ in $L_2(\Omega, R^{n\times n})$ and the *global a posteriori error estimator* $\eta_{\mathrm{R}}(\beta, \Phi, \bar{u})$ like in [8], where β is an arbitrary positive real number and Φ an arbitrary smooth matrix-valued function. The estimator depends on the constant from the Friedrichs inequality for $\nabla\nabla$ on $H_0^2(\Omega)$. We then have the following theorem [8].

Theorem 3.2 *Let* $u \in H_0^2(\Omega)$ *be the weak solution of the problem* (3.3), (2.2) *and* $\bar{u} \in H_0^2(\Omega)$ *an arbitrary function. Then*

$$|\!|\!|\nabla\nabla(\bar{u} - u)|\!|\!|^2 \leq \eta_{\mathrm{R}}(\beta, \Phi, \bar{u}) \tag{3.4}$$

for any positive number β *and any matrix-valued function* $\Phi \in H(\mathrm{div}\,\mathrm{Div}, \Omega)$.

The proof of the theorem is based on a more general statement proven in [8]. There is an interesting question of optimizing the inequality (3.4) with respect to β and Φ. To avoid possible smoothness difficulties we can introduce a further global error estimator and prove the same statement as in Theorem 3.2 [8].

4 Conclusion

In the paper, we have presented several analytical a posteriori error estimators that appear in inequalities, usually with some unknown constants on the right-hand part. The quantitative properties of the estimators cannot be easily assessed and compared analytically. Only numerical experiment can be the means for this purpose. There are, however, global analytical error estimates for some classes of problems (see, e.g., [5], [7], [8]) that require as few unknown constants as possible. Some papers provide for the estimation of these constants. Analytical estimates are usually efficient in practice if they are asymptotically exact. The a posteriori estimates with unknown constants, however, are not optimal for the practical computation.

Exceptionally, there are analytical estimates containing really no unknown constants (see, e.g., [10] for a 2D linear 2nd order elliptic problem).

The paper is closely connected with the automatic hp-adaptivity that gives many h as well as p possibilities for the next step of the solution process. A single number provided by the local analytical a posteriori error estimator for each mesh element need not be enough information for the decision. This is the reason for using the computational error estimate (reference solution). The computation of the reference solution is rather time-consuming but it is obtained by the same software that is used to compute the approximate solution. We use reference solutions as robust error estimators with no unknown constants to control the adaptive strategies in the most complex finite element computations.

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