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A QUADRATIC SPLINE-WAVELET BASIS ON THE INTERVAL

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Abstract

In signal and image processing as well as in numerical solution of differential equations, wavelets with short support and with vanishing moments are important because they have good approximation properties and enable fast algorithms. A B-spline of order m is a spline function that has minimal support among all compactly supported refinable functions with respect to a given smoothness. And recently, B. Han and Z. Shen constructed Riesz wavelet bases of $L_2(\mathbb{R})$ with m vanishing moments based on B-spline of order m. In our contribution, we present an adaptation of their quadratic spline-wavelets to the interval $[0, 1]$ which preserves vanishing moments.

1. Introduction

Wavelets are a widely accepted tool in signal and image processing as well as in numerical solution of operator equations. In this area, methods based on wavelets are successfully used for preconditioning of large systems of linear equations arising from discretization of elliptic partial differential equations, sparse representations of some types of operators and adaptive solving of operator equations. The performance of these methods strongly depends on the choice of a wavelet basis, in particular on its condition number.

Wavelet bases on a general domain are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by tensor product technique to the n-dimensional cube. Finally by splitting the domain into subdomains which are images of $(0, 1)^n$ under appropriate parametric mappings one can obtain wavelet bases on a fairly general domain. Thus, the properties of wavelet basis on the interval are important for the properties of resulting bases on general domains.

Here, we focus on quadratic spline-wavelets and we construct well-conditioned interval spline-wavelet bases. From the viewpoint of numerical stability, ideal wavelet bases are orthogonal ones. However, they are usually avoided mainly due to the lack of smoothness and their large support. Natural generalization of orthogonal wavelets are biorthogonal wavelets, but their construction and implementation is relatively complicated and wavelets usually have larger support than scaling functions. For more details see for instance [1]. In recent years, there appeared some interesting constructions of biorthogonal wavelets with globally supported dual wavelets [5, 7]. This seems not to cause any problem in numerical solution of linear PDEs because dual functions are not directly used. And recently, B. Han and Z. Shen [6] constructed a Riesz wavelet bases of $L_2(\mathbb{R})$ with m vanishing moments based on B-spline of order m. In our contribution, we present an adaptation of quadratic spline-wavelets proposed in [6] to the interval $[0,1]$ which preserves vanishing moments and compare their properties with quadratic spline wavelets constructed in [1].

2. B-splines

We use a scaling basis based on quadratic B-splines employed for example in [1, 2], because they are well-conditioned and can be easily adapted to the bounded interval by employing multiple knots at the endpoints. Let N be the desired order of polynomial exactness of scaling basis, $j \in \mathbb{N}_0$ and let $\mathbf{t}^j = \left(t_k^j\right)_{k=-N+1}^{2^j+N-1}$ be a Schoenberg sequence of knots defined by

$$\begin{aligned} t_k^j &:= 0, && k = -N+1, \ldots, 0, \\ t_k^j &:= \frac{k}{2^j}, && k = 1, \ldots, 2^j - 1, \\ t_k^j &:= 1, && k = 2^j, \ldots, 2^j + N - 1. \end{aligned}$$

The corresponding B-splines of order N are then defined by

$$B_{k,N}^j(x) := \left(t_{k+N}^j - t_k^j\right)\left[t_k^j, \ldots, t_{k+N}^j\right](t-x)_+^{N-1}, \quad x \in [0,1],$$

where $(x)_+ := \max\{0, x\}$. The symbol $[t_k, \ldots t_{k+N}] f(t)$ is the N-th divided difference of f which is recursively defined as

$$\begin{aligned} [t_k, \ldots, t_{k+N}] f(t) &= \frac{[t_{k+1}, \ldots, t_{k+N}] f(t) - [t_k, \ldots, t_{k+N-1}] f(t)}{t_{k+N} - t_k} && \text{if} \quad t_k \neq t_{k+N}, \\ &= \frac{f^{(N)}(t_k)}{N!} && \text{if} \quad t_k = t_{k+N}, \end{aligned}$$

with $[t_k] f(t) = f(t_k)$. Then, we define the set $\Phi_j = \{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\}$ of scaling functions where

$$\phi_{j,k} = 2^{j/2} B_{k,N}^j, \quad k = -N+1, \ldots, 2^j - 1, \quad j \geq 0.$$

Thus, there are $2^j - N + 1$ inner scaling functions and $N - 1$ functions at each boundary. The functions $\phi_{j,-N+1}$ and $\phi_{j,2^j-1}$ are the only functions which do not vanish at the boundaries. Therefore, scaling bases satisfying homogeneous Dirichlet boundary conditions are given by

$$\Phi_j^B = \left\{\phi_{j,k}, k = -N+2, \ldots, 2^j - 2\right\}.$$

Inner scaling functions are translations and dilations of one function ϕ corresponding to the primal scaling function constructed by Cohen, Daubechies and Feauveau in [4]. In the case of a quadratic spline-wavelet basis, there is only one boundary scaling function at each boundary. Specifically, the quadratic spline function $\phi(x)$ is defined by

$$\phi(x) = \begin{cases} \frac{x^2}{2} & x \in [0,1], \\ -x^2 + 3x - \frac{3}{2} & x \in [1,2], \\ \frac{x^2}{2} - 3x + \frac{9}{2} & x \in [2,3], \\ 0 & \text{otherwise.} \end{cases}$$

The left boundary function $\phi_B(x)$ is defined by

$$\phi_B(x) = \begin{cases} -\frac{3x^2}{2} + 2x & x \in [0,1], \\ \frac{x^2}{2} - 2x + 2 & x \in [1,2], \\ 0 & \text{otherwise,} \end{cases}$$

and the corresponding right boundary function is symmetrical with respect to the point 3/2. Above scaling functions satisfy the following refinement equations:

$$\phi(x) = \frac{1}{4}\phi(2x) + \frac{3}{4}\phi(2x-1) + \frac{3}{4}\phi(2x-2) + \frac{1}{2}\phi(2x-3),$$

and

$$\phi_B(x) = \frac{1}{2}\phi_B(2x) + \frac{3}{4}\phi(2x) + \frac{1}{4}\phi(2x-1),$$

respectively.

3. Wavelets

In many applications, it is important not only to have wavelets with short support, with vanishing moments but also with a small condition number. Such wavelets should be as close as possible to some orthonormal wavelets or tight frames, for a given order of regularity or vanishing moments. However, construction of optimally conditioned wavelet bases is still an open question. To construct a compactly supported wavelet, one usually starts with a compactly supported refinable function ϕ with stable shifts. Recall that the shifts of a function ϕ are stable if the sequence formed by whole-number shifts of the function ϕ is a Riesz sequence. Then a compactly supported wavelet is obtained by selecting some finite linear combination of these shifts. For further details on this concept, we refer to [3, 8].

While compactly supported refinable functions with stable shifts can be constructed relatively easily, the construction of compactly supported wavelets generated by B-splines is not straightforward. In [6], Riesz wavelet bases of $L_2(\mathbb{R})$ with m vanishing moments based on B-spline of order m have been proposed. Their wavelets are the shortest supported wavelets of regularity $m-1/2$ with m vanishing moments.

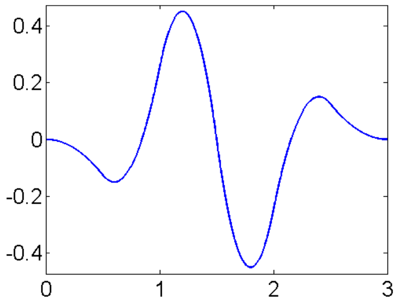

Figure 1: The quadratic wavelet proposed by B. Han and Z. Shen.

The quadratic spline-wavelet constructed by B. Han and Z. Shen is then given by

$$\psi(x) = -\frac{1}{4}\,\phi\,(2x) + \frac{3}{4}\,\phi\,(2x-1) - \frac{3}{4}\,\phi\,(2x-2) + \frac{1}{4}\,\phi\,(2x-3).$$

Its graph is depicted in Figure 1. Now, we would like to adapt it to homogeneous Dirichlet boundary conditions and to keep the number of vanishing moments. First of all, it is not possible to construct boundary wavelets with the same number of vanishing moments as inner wavelets have, and with the same length of support as boundary scaling functions have. They should be supported at least in the interval $[0, 5/2]$. We construct here a boundary wavelet prescribing three vanishing moments, the support in the interval $[0, 5/2]$, homogeneous Dirichlet boundary conditions and finally, it should be from the space spanned by $\{\phi_B(2x),\, \phi(2x-k) : k \in \mathbb{N}_0\}$. The arising wavelet is then given by these conditions up to multiplication by a constant and is determined by

$$\psi_B(x) = -\frac{5}{2}\,\phi_B\,(2x) + \frac{47}{12}\,\phi\,(2x) - \frac{13}{4}\,\phi\,(2x-1) + \phi\,(2x-2).$$

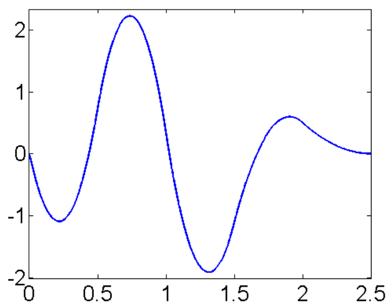

Figure 2: The constructed boundary wavelet.

4. Properties of constructed basis

In this section, we compare selected properties of wavelets introduced in the previous section with wavelets proposed in [1]. We will look at the condition number and the number of nonzero elements for the stiffness matrix corresponding to the equation $u'' = f$ with the Dirichlet boundary conditions $u(0) = u(1) = 0$. We use here also the standard wavelet preconditioning consisting in normalizing all basis function with respect to the arising bilinear form. Further, we solve the above problem corresponding to the exact solution $u = x(1-e^{50x-50})$ which exhibits a steep gradient near the point 1. Results are summarized in Table 1. NZ is the number of nonzero elements in stiffness matrices, COND represents the condition number of diagonally preconditioned stiffness matrices. Achieved approximation error was the same for both bases.

		The proposed basis		CF	
n	$\|\|u_n - u\|\|_{L_2}$	NZ	COND	NZ	COND
8	5.9e-02	58	8.9	-	-
16	1.6e-02	200	10.1	174	12.2
32	2.6e-03	530	10.6	622	12.6
64	3.1e-04	1268	10.8	1822	12.7
128	3.7e-05	2846	10.9	4510	12.8
256	4.5e-06	6128	10.9	10254	12.9
512	5.6e-07	12842	10.9	22190	12.9
1024	7.0e-08	26444	11.0	46590	12.9
2048	8.8e-09	53846	11.0	95998	12.9
4096	1.1e-09	108885	11.0	195502	12.9

Table 1: Obtained numerical results.

5. Conclusion

In this contribution, we proposed new wavelets based on quadratic splines. Due to the shorter support of proposed wavelets, stiffness matrices are sparser than for any known quadratic basis with compactly supported dual wavelets. Moreover, they are slightly better conditioned. Our future aim is to prove that the proposed basis is a Riesz basis and to construct higher order spline-wavelet bases with shorter support than any biorthogonal bases with compactly supported dual wavelets have.

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