# RADIAL SUBSPACES OF BESOV-LIZORKIN-TRIEBEL SPACES

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Abstract. This is a summary of results obtained in collaboration with Leszek Skrzypczak (Poznań) and Jan Vybíral (Linz). We investigate decay and boundedness properties of radial functions belonging to Besov and Lizorkin-Triebel spaces. Our main tools are atomic decompositions in combination with trace theorems.

## 1. Introduction

At the end of the seventies Strauss [43] was the first who observed that there is an interplay between the regularity and decay properties of radial functions. We recall his

**Radial Lemma.** *Let $d \ge 2$. Every radial function $f \in H^1(\mathbb{R}^d)$ is almost everywhere equal to a function $\widetilde{f}$, continuous for $x \neq 0$, such that*

$$|\widetilde{f}(x)| \le c\,|x|^{\frac{1-d}{2}}\,\| f \mid H^1(\mathbb{R}^d)\|, \tag{1}$$

*where $c$ depends only on $d$.*

Strauss stated (1) with the extra condition $|x| \ge 1$, but this restriction is not needed. The *Radial Lemma* contains three different assertions:

(a) the existence of a representative of $f$, which is continuous outside the origin;
(b) the decay of $f$ near infinity;
(c) the limited unboundedness near the origin.

2010 *Mathematics Subject Classification.* Primary 46E35.
*Key words and phrases.* Sobolev spaces; Besov space; Lizorkin-Triebel spaces; atomic decomposition; homogeneous spaces; inhomogeneous spaces; embeddings; compact embeddings; traces.

These three properties do not extend to all functions in $H^1(\mathbb{R}^d)$, of course. In particular, $H^1(\mathbb{R}^d) \not\subset L_\infty(\mathbb{R}^d)$, $d \geq 2$, and consequently, functions in $H^1(\mathbb{R}^d)$ can be unbounded in the neighbourhood of any fixed point $x \in \mathbb{R}^d$. In a series of papers we have investigated the specific regularity and decay properties of radial functions in the general framework of Besov-Lizorkin-Triebel spaces. In our opinion a discussion of these properties in connection with fractional order of smoothness results in a better understanding of the announced interplay of regularity on the one side and local smoothness, decay at infinity and limited unboundedness near the origin on the other side. In the literature there are several approaches to fractional order of smoothness. Probably most popular are Bessel potential spaces $H^s_p(\mathbb{R}^d)$, $s \in \mathbb{R}$, or Slobodeckij spaces $W^s_p(\mathbb{R}^d)$ ($s > 0$, $s \not\in \mathbb{N}$). These scales would be enough to explain the main interrelations. However, for some limiting cases these scales are not sufficient. For that reason we shall discuss generalizations of the *Radial Lemma* in the framework of Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ and Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^d)$. Let us recall that these scales essentially cover the Bessel potential and the Slobodeckij spaces since

- $W^m_p(\mathbb{R}^d) = F^m_{p,2}(\mathbb{R}^d)$, $m \in \mathbb{N}_0$, $1 < p < \infty$;
- $H^s_p(\mathbb{R}^d) = F^s_{p,2}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 < p < \infty$;
- $W^s_p(\mathbb{R}^d) = F^s_{p,p}(\mathbb{R}^d) = B^s_{p,p}(\mathbb{R}^d)$, $s > 0$, $s \not\in \mathbb{N}$, $1 \leq p \leq \infty$,

where all identities have to be understood in the sense of equivalent norms, see, e.g., [45, 2.2.2] and the references given there.

This survey is organized as follows.

At several places, in particular in Subsections 5.1, 5.2, 6.1 and 7.7, we shall give proofs in simplified situations, hoping the reader gets a feeling for the general situations as well. In all other cases detailed references are given.

Besov and Lizorkin-Triebel spaces are discussed at various places, we refer, e.g., to the monographs [28], [32], [45], [46], [48]. We will not give definitions here and refer for this to the quoted literature.

**Notation.** As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the integers and $\mathbb{R}$ the real numbers. If $X$ and $Y$ are two quasi-Banach spaces, then the symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. $X \hookrightarrow\hookrightarrow Y$ means that the embedding is compact. The set of all linear and bounded operators $T\colon X \to Y$, denoted by $\mathcal{L}(X,Y)$, is equipped with the standard quasi-norm. As usual, the symbol $c$ denotes positive constants which depend only on the fixed parameters $s, p, q$ and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. We will write "$A \asymp B$" if there exist constants $c_1, c_2 > 0$ independent of $A$ and $B$ such that $c_1 A \le B \le c_2 A$.

We shall use the following conventions throughout the paper:

- If $E$ denotes a space of functions on $\mathbb{R}^d$ then by $RE$ we mean the subset of radial functions in $E$ and we endow this subset with the same quasi-norm as the original space.
- Inhomogeneous Besov and Lizorkin-Triebel spaces are denoted by $B^s_{p,q}$ and $F^s_{p,q}$, respectively. If there is no reason to distinguish between these two scales we will use the notation $A^s_{p,q}$. Similarly for the radial subspaces.
- If an equivalence class $\{f\}$ (equivalence with respect to coincidence almost everywhere) contains a continuous representative then we call the class continuous and speak of values of $f$ at any point (by taking the values of the continuous representative).
- Throughout the paper $\psi \in C_0^\infty(\mathbb{R}^d)$ denotes a specific radial cut-off function, i.e., $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 3/2$.
- We shall use the abbreviations

$$\sigma_p(d) := d \max\Big(0, \frac{1}{p} - 1\Big) \quad \text{and} \quad \sigma_{p,q}(d) := d \max\Big(0, \frac{1}{p} - 1, \frac{1}{q} - 1\Big). \tag{2}$$

- If not otherwise stated the parameter $q$ varies in $(0, \infty]$, the parameter $p$ varies in $(0, \infty]$ if used in connection with Besov spaces and in $(0, \infty)$ if used in connection with Lizorkin-Triebel spaces and finally $s$ varies in $\mathbb{R}$.

## 2. Atomic decompositions

In many situations one needs descriptions of function spaces which allow some type of localization. Usually this is difficult in connection with characterizations using the Fourier transform. At least for 20 years, say from 1970–1990, the Fourier analytical way to describe Besov-Lizorkin-Triebel spaces on $\mathbb{R}^d$ was the dominating one. The contributions of Frazier and Jawerth [16], [17] to characterizations of these classes by means of atoms have been a breakthrough to local descriptions. Their work has been also the basis for our characterization of radial subspaces, see [35] and [24], which we will recall below.

**2.1. Atomic decompositions of Besov and Lizorkin-Triebel spaces.** In this survey we shall consider several different types of atoms. They are not related to each other (but the philosophy behind is the same). We hope that it will be always clear from the context with which type of atoms we are working. For the following definition of an atom we refer to [16] or [46, 3.2.2]. For an open set $Q$ and $r > 0$ we put $rQ = \{x \in \mathbb{R}^d : \mathrm{dist}(x, Q) < r\}$. Observe that $Q$ is always a subset of $rQ$ whatever $r$ is.

**Definition 1.** Let $s \in \mathbb{R}$ and let $0 < p \le \infty$. Let $L$ and $M$ be integers such that $L \ge 0$ and $M \ge -1$. Let $Q \subset \mathbb{R}^d$ be an open connected set with $\mathrm{diam}\, Q = r$.
(a) A smooth function $a(x)$ is called an $1_L$-atom centered in $Q$ if

$$\operatorname{supp} a \subset \frac{r}{2} Q,$$
$$\sup_{y \in \mathbb{R}^d} |D^\alpha a(y)| \le 1, \quad |\alpha| \le L.$$

(b) A smooth function $a(x)$ is called an $(s,p)_{L,M}$-atom centered in $Q$ if

$$\operatorname{supp} a \subset \frac{r}{2} Q,$$
$$\sup_{y \in \mathbb{R}^d} |D^\alpha a(y)| \le r^{s-|\alpha|-\frac{d}{p}}, \quad |\alpha| \le L,$$
$$\int_{\mathbb{R}^d} a(y) y^\alpha \, dy = 0, \quad |\alpha| \le M.$$

**Remark 1.** If $M = -1$, then the interpretation is that no moment condition is required.

Originally coverings of $\mathbb{R}^d$ by dyadic cubes and related atomic decompositions were considered. We define

$$Q_{j,\ell} := \{x \in \mathbb{R}^d : 2^{-j}\ell_i \le x_i < 2^{-j}(\ell_i + 1),\ i = 1, \ldots, d\}, \quad j \in \mathbb{N}_0,\ \ell \in \mathbb{Z}^d.$$

**Proposition 1.** *Suppose*

$$L \ge \max(0, [s] + 1) \quad \textit{and} \quad M \ge \max([\sigma_p - s], -1)$$

*(here $[\,\cdot\,]$ denotes the integer part). Then any $f \in B^s_{p,q}(\mathbb{R}^d)$ can be represented by*

$$f = \sum_{j=0}^\infty \sum_{\ell=0}^\infty s_{j,\ell} a_{j,\ell} \quad (\textit{convergence in } \mathcal{S}'(\mathbb{R}^d)), \tag{3}$$

*where $a_{0,\ell}$ is an $1_L$-atom centered at the cube $Q_{0,\ell}$, and $a_{j,\ell}$, $j \neq 0$ is an $(s,p)_{L,M}$-atom centered at the cube $Q_{j,\ell}$; $s_{j,\ell}$ are complex numbers satisfying*

$$\| (s_{j,\ell})_{j,\ell} \mid b_{p,q}\| := \Big( \sum_{j=0}^\infty \Big( \sum_{\ell=0}^\infty |s_{j,\ell}|^p \Big)^{q/p}\Big)^{1/q} < \infty . \tag{4}$$

*Furthermore, any distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, given by (3), with (4) is an element of $B^s_{p,q}(\mathbb{R}^d)$. The infimum of (4) with respect to all admissible representations as in (3) yields an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^d)$.*

**Remark 2.** (i) Nowadays characterizations of Besov spaces by wavelets are widely used. The atomic decompositions follow the same philosophy (discretization of the quasi-norm) but allow a greater flexibility. This will be used also in our treatment.

(ii) We shall call a representation (3) optimal in case

$$\| f \mid B^s_{p,q}(\mathbb{R}^d)\| \asymp \Big( \sum_{j=0}^\infty \Big( \sum_{\ell=0}^\infty |s_{j,\ell}|^p \Big)^{q/p}\Big)^{1/q} .$$

It is known how to construct optimal atomic decompositions.

A similar characterization can be given in case of Lizorkin-Triebel spaces. Here $\chi_{j,\ell}$ denotes the characteristic function of the dyadic cube $Q_{j,\ell}$.

**Proposition 2.** *Suppose*

$$L \geq \max(0, [s]+1) \quad \textit{and} \quad M \geq \max([\sigma_{p,q} - s], -1).$$

*Then any $f \in F^s_{p,q}(\mathbb{R}^d)$ can be represented by (3) and this time $s_{j,\ell}$ are complex numbers satisfying*

$$\| (s_{j,\ell})_{j,\ell} \mid f_{p,q}\| := \Big\| \Big( \sum_{j=0}^\infty \sum_{\ell=0}^\infty (2^{jd/p} |s_{j,\ell} \chi_{j,\ell}|)^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| < \infty . \tag{5}$$

*Furthermore, any distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, given by (3), with (5) is an element of $F^s_{p,q}(\mathbb{R}^d)$, The infimum of (5) with respect to all admissible representations as in (3) yields an equivalent quasi-norm in $F^s_{p,q}(\mathbb{R}^d)$.*

**Remark 3.** (i) Proofs of Propositions 1, 2 can be found in [17] and [48, 1.5.1].

(ii) We shall not give definitions of $F^s_{p,q}(\mathbb{R}^d)$ and $B^s_{p,q}(\mathbb{R}^d)$ here. For this and many further equivalent descriptions of these classes of distributions we refer to the literature, see, e.g., [28], [45], [46], [48] or [32]. Formally one could take Proposition 1 and Proposition 2 as definitions.

Now we turn to more general atomic decompositions what concerns the underlying coverings of $\mathbb{R}^d$.

**Definition 2.** We say that the sequence $(\Omega_j)_{j=0}^\infty = ((\Omega_{j,\ell})_{\ell=0}^\infty)_{j=0}^\infty$ of coverings is regular if the following conditions are satisfied:
(a) $\mathbb{R}^d \subset \bigcup_{\ell\in\mathbb{N}} \overline{\Omega}_{j,\ell}$, $j = 0, 1, \ldots$;
(b) there exists some positive number $\varepsilon_0$ such that for all $\varepsilon < \varepsilon_0$ the sequences of coverings $(\varepsilon 2^{-j}\Omega_{j,\ell})_{\ell=0}^\infty$ have finite multiplicity with uniform bound of multiplicity with respect to $\varepsilon$ and $j \in \mathbb{N}_0$;
(c) there exist positive numbers $B_d$ and $C_d$ (depending only on the dimension $d$) such that

$$\operatorname{diam}\Omega_{j,\ell} \le B_d 2^{-j} \quad \text{and} \quad C_d 2^{-jd} \le |\Omega_{j,\ell}|.$$

**Remark 4.** By $\omega_n$ we denote the volume of the unit ball in $\mathbb{R}^d$. Let $A_d = (C_d/\omega_d)^{1/d}$. Then condition (c) implies

$$A_d 2^{-j} \le \operatorname{diam}\Omega_{j,\ell} \le B_d 2^{-j}$$

and

$$C_d 2^{-jd} \le |\Omega_{j,\ell}| \le B_d^d \omega_d 2^{-jd}$$

for all $j$ and all $\ell$.

**Lemma 1.** *Let $(\Omega_j)_{j=0}^\infty$ be a regular sequence of coverings. Then Propositions 1, 2 remain true by replacing the dyadic cube $Q_{j,\ell}$ by $\Omega_{j,\ell}$.*

**Remark 5.** The not very difficult proof of this lemma can be found in [35].

**2.2. Atomic decompositions of radial subspaces.** Later on we shall mainly deal with radial functions but for the moment we prefer a bit greater generality.

**Definition 3.** (i) Let $f \in \mathcal{S}'(\mathbb{R}^d)$. The distribution $f$ is called radial if it is invariant under rotations around the origin, i.e.

$$f(\varphi \circ \Phi) = f(\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^d),$$

for all such rotations $\Phi$.
(ii) $RA^s_{p,q}(\mathbb{R}^d)$ is the collection of all radial distributions $f \in A^s_{p,q}(\mathbb{R}^d)$.

In [35] and [24] we constructed a regular sequence of coverings adapted to the radial situation which we now recall. Consider the annuli (balls if $k = 0$)

$$P_{j,k} := \left\{x \in \mathbb{R}^d : k2^{-j} \le |x| < (k+1)2^{-j}\right\}, \quad j = 0, 1, \ldots, \; k = 0, 1, \ldots.$$

Then there is a sequence $(\Omega_j)_{j=0}^\infty = ((\Omega_{j,k,\ell})_{k,\ell})_{j=0}^\infty$ of coverings of $\mathbb{R}^d$ such that

(a) all $\Omega_{j,k,\ell}$ are balls with center in $x_{j,k,\ell}$ s.t. $x_{j,0,1} = 0$ and $|x_{j,k,\ell}| = 2^{-j}(k + 1/2)$ if $k \geq 1$;
(b) $\operatorname{diam} \Omega_{j,k,\ell} = 12 \cdot 2^{-j}$ for all $k$ and all $\ell$;
(c) $P_{j,k} \subset \bigcup_{\ell=1}^{C(d,k)} \Omega_{j,k,\ell}$, $j = 0, 1, \ldots,$ $k = 0, 1, \ldots,$ where the numbers $C(d,k)$ satisfy the relations $C(d,k) \leq (2k+1)^{d-1}$, $C(d,0) = 1$;
(d) the sums $\sum_{k=0}^\infty \sum_{\ell=1}^{C(d,k)} \chi_{j,k,\ell}(x)$ are uniformly bounded in $x \in \mathbb{R}^d$ and $j = 0, 1, \ldots$ (here $\chi_{j,k,\ell}$ denotes the characteristic function of $\Omega_{j,k,\ell}$);
(e) $\Omega_{j,k,\ell} = \{x \in \mathbb{R}^d : 2^j x \in \Omega_{0,k,\ell}\}$ for all $j$, $k$ and $\ell$;
(f) there exists a natural number $K$ (independent of $j$ and $k$) such that

$$\{(x_1, 0, \ldots, 0) : x_1 \in \mathbb{R}\} \cap \frac{\operatorname{diam}(\Omega_{j,k,\ell})}{2} \Omega_{j,k,\ell} = \emptyset \quad \text{if } \ell > K$$

(with an appropriate enumeration).

We collect some properties of related atomic decompositions. To do this it is convenient to introduce some sequence spaces.

**Definition 4.** Let $\widetilde{\chi}_{j,k}$ denote the characteristic function of the set $P_{j,k}$. Then we define

$$b_{p,q,d} := \Big\{ s = (s_{j,k})_{j,k} : \\ \| s \mid b_{p,q,d} \| = \Big( \sum_{j=0}^\infty \Big( \sum_{k=0}^\infty (1+k)^{d-1} |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} < \infty \Big\}.$$

and

$$f_{p,q,d} := \Big\{ s = (s_{j,k})_{j,k} : \\ \| s \mid f_{p,q,d} \| = \Big\| \Big( \sum_{j=0}^\infty \sum_{k=0}^\infty |s_{j,k}|^q 2^{\frac{jdq}{p}} \widetilde{\chi}_{j,k}(\cdot) \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| < \infty \Big\}$$

with the usual modifications if $p$ and/or $q$ are infinite.

**Remark 6.** Observe $b_{p,p,d} = f_{p,p,d}$ in the sense of equivalent quasi-norms.

More or less as a direct consequence of Lemma 1 and the fact that described sequence $(\Omega_j)_{j=0}^\infty = ((\Omega_{j,k,\ell})_{k,\ell})_{j=0}^\infty$ is regular, one obtains the following description of radial subspaces which will be basic tool for us, see [35] and [24] for all details.

(i) Each $f \in RA^s_{p,q}(\mathbb{R}^d)$ can be decomposed into

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} a_{j,k,\ell} \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^d)), \tag{6}$$

where the functions $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega_{j,k,\ell}$ $(j \ge 1)$, and the functions $a_{0,k,\ell}$ are $1_L$-atoms with respect to $\Omega_{0,k,\ell}$.

(ii) Any formal series $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} a_{j,k,\ell}$ converges in $\mathcal{S}'(\mathbb{R}^d)$ with limit in $A^s_{p,q}(\mathbb{R}^d)$ if the sequence $s = (s_{j,k})_{j,k}$ belongs to $a_{p,q,d}$ and if the $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega_{j,k,\ell}$ $(j \ge 1)$, and the $a_{0,k,\ell}$ are $1_L$-atoms with respect to $\Omega_{0,k,\ell}$. There exists a universal constant such that

$$\Big\| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} a_{j,k,\ell} \, \Big| \, A^s_{p,q}(\mathbb{R}^d) \Big\| \le c \, \| \, s \mid a_{p,q,d} \| \tag{7}$$

holds for all sequences $s = (s_{j,k})_{j,k}$.

(iii) There exists a constant $c$ such that for any $f \in RA^s_{p,q}(\mathbb{R}^d)$ there exists an atomic decomposition as in (6) satisfying

$$\| \, (s_{j,k})_{j,k} \mid a_{p,q,d} \| \le c \, \| \, f \mid A^s_{p,q}(\mathbb{R}^d) \|. \tag{8}$$

(iv) The infimum on the left-hand side in (7) with respect to all admissible representations (6) yields an equivalent quasi-norm on $RA^s_{p,q}(\mathbb{R}^d)$.

**Remark 7.** (i) Here $a = f$ if $A = F$ and $a = b$ if $A = B$, respectively.
(ii) A different approach to atomic decompositions of radial subspaces has been given by Epperson and Frazier [14]. We shall recall these results in connection with homogeneous spaces in Subsection 7.6.

## 3. Traces

Traces of radial subspaces are of interest for its own. However, we shall use them here mainly in connection with the proof of the higher regularity of radial functions outside the origin.

Let $d \geq 2$. Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be a locally integrable radial function. By using a Lebesgue point argument its restriction

$$f_0(t) := f(t, 0, \ldots, 0), \quad t \in \mathbb{R}$$

is well defined a.e. on $\mathbb{R}$. However, this restriction need not be locally integrable. A simple example is given by the function

$$f(x) := \psi(x)|x|^{-1}, \quad x \in \mathbb{R}^d.$$

Furthermore, if we start with a measurable and even function $g\colon \mathbb{R} \to \mathbb{C}$, s.t. $g$ is locally integrable on all intervals $(a, b)$, $0 < a < b < \infty$, then (again using a Lebesgue point argument) the function

$$f(x) := g(|x|), \quad x \in \mathbb{R}^d$$

is well-defined a.e. on $\mathbb{R}^d$ and is radial, of course. In what follows we shall study properties of the associated operators

$$\mathrm{tr}\colon f \mapsto f_0 \quad \text{and} \quad \mathrm{ext}\colon g \mapsto f.$$

Both operators are defined pointwise only. Later on we shall have a short look onto the existence of the trace in the distributional sense, see Subsection 3.4. Probably it would be more natural to deal with functions defined on $[0, \infty)$ in this context. However, that would result in more complicated descriptions of the trace spaces. So, our target spaces will be spaces of even functions defined on $\mathbb{R}$.

**3.1. Traces of radial subspaces with $p = \infty$.** Let $m \in \mathbb{N}_0$. Then $C^m(\mathbb{R}^d)$ denotes the collection of all functions $f\colon \mathbb{R}^d \to \mathbb{C}$ such that all derivatives $D^\alpha f$ of order $|\alpha| \leq m$ exist, are uniformly continuous and bounded. We put

$$\| f \mid C^m(\mathbb{R}^d)\| := \sum_{|\alpha| \leq m} \|D^\alpha f \mid L_\infty(\mathbb{R}^d)\|.$$

**Theorem 1.** *Let $d \geq 2$. For $m \in \mathbb{N}_0$ the mapping* tr *is a linear isomorphism of $RC^m(\mathbb{R}^d)$ onto $RC^m(\mathbb{R})$ with inverse* ext.

**Remark 8.** (i) If we replace uniformly continuous by continuous in the definition of the spaces $C^m(\mathbb{R}^d)$ Theorem 1 remains true with the same proof.

(ii) The proof of Theorem 1 is elementary and may be found in [37].

Using real interpolation it is not difficult to derive the following result for the spaces of Hölder-Zygmund type, see also [37].

**Theorem 2.** *Let* $s > 0$ *and let* $0 < q \le \infty$. *Then the mapping* tr *is a linear isomorphism of* $RB^s_{\infty,q}(\mathbb{R}^d)$ *onto* $RB^s_{\infty,q}(\mathbb{R})$ *with inverse* ext.

**3.2. Traces of radial subspaces with $p < \infty$.** Now we turn to the description of the trace classes of radial Besov and Lizorkin-Triebel spaces with $p < \infty$. Again we start with an almost trivial result. We need a further notation. By $L_p(\mathbb{R}, w)$ we denote the weighted Lebesgue space equipped with the norm

$$\| f \mid L_p(\mathbb{R}, w)\| := \Big(\int_{-\infty}^{\infty} |f(t)|^p w(t)\, dt\Big)^{1/p}$$

with usual modification if $p = \infty$.

**Lemma 2.** *Let* $d \ge 2$.
(i) *Let* $0 < p < \infty$. *Then* $\mathrm{tr}\colon RL_p(\mathbb{R}^d) \to RL_p(\mathbb{R}, |t|^{d-1})$ *is a linear isomorphism with inverse* ext.
(ii) *Let* $p = \infty$. *Then* $\mathrm{tr}\colon RL_\infty(\mathbb{R}^d) \to RL_\infty(\mathbb{R})$ *is a linear isomorphism with inverse* ext.

Based on this elementary lemma we would like to direct the attention of the reader to the following observation. Whenever the Besov-Lizorkin-Triebel space $A^s_{p,q}(\mathbb{R}^d)$ is contained in $L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$, then tr is well-defined on its radial subspace. This is in sharp contrast to the general theory of traces on these spaces. To guarantee that tr is meaningful on $A^s_{p,q}(\mathbb{R}^d)$ one has to require

$$s > \frac{d-1}{p} + \max\Big(0, \frac{1}{p} - 1\Big),$$

cf. e.g. [23], [16], [45, Rem. 2.7.2/4] or [15]. On the other hand we have

$$B^s_{p,q}(\mathbb{R}^d), F^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$$

if $s > d\max\big(0, \frac{1}{p} - 1\big)$, see, e.g., [39]. Since

$$d\max\Big(0, \frac{1}{p} - 1\Big) < \frac{d-1}{p} + \max\Big(0, \frac{1}{p} - 1\Big)$$

we have the existence of tr with respect to $RA^s_{p,q}(\mathbb{R}^d)$ for a wider range of parameters than for $A^s_{p,q}(\mathbb{R}^d)$.

Below we shall develop a description of the traces of the radial subspaces of $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ in terms of atoms. To explain this we need to introduce first an appropriate notion of an atom and second, adapted sequence spaces.

**Definition 5.** Let $L \geq 0$ be an integer. Let $I$ be a set either of the form $I = [-a, a]$ or of the form $I = [-b, -a] \cup [a, b]$ for some $0 < a < b < \infty$. An even function $g \in C^L(\mathbb{R})$ is called an even $L$-atom centered at $I$ if

$$\max_{t \in \mathbb{R}} |g^{(n)}(t)| \leq |I|^{-n}, \quad 0 \leq n \leq L,$$

and if either

$$\operatorname{supp} g \subset \Big[-\frac{3a}{2}, \frac{3a}{2}\Big] \quad \text{in case } I = [-a, a],$$

or

$$\operatorname{supp} g \subset \Big[-\frac{3b-a}{2}, -\frac{3a-b}{2}\Big] \cup \Big[\frac{3a-b}{2}, \frac{3b-a}{2}\Big] \qquad \text{in case } I = [-b, -a] \cup [a, b].$$

**Definition 6.** Let

$$\chi^{\#}_{j,k}(t) := \begin{cases} 1 & \text{if } 2^{-j}k \leq |t| \leq 2^{-j}(k+1), \\ 0 & \text{otherwise}, \end{cases} \qquad t \in \mathbb{R}.$$

Then we define

$$b^s_{p,q,d} := \Big\{ s = (s_{j,k})_{j,k} : \\ \| s \mid b^s_{p,q,d} \| = \Big( \sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \Big( \sum_{k=0}^{\infty} (1+k)^{d-1} |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} < \infty \Big\}$$

and

$$f^s_{p,q,d} := \Big\{ s = (s_{j,k})_{j,k} : \\ \| s \mid f^s_{p,q,d} \| = \Big\| \Big( \sum_{j=0}^{\infty} 2^{jsq} \sum_{k=0}^{\infty} |s_{j,k}|^q \chi^{\#}_{j,k}(\cdot) \Big)^{1/q} \Big| L_p(\mathbb{R}, |t|^{d-1}) \Big\| < \infty \Big\},$$

respectively.

**Remark 9.** Observe $b^s_{p,p,d} = f^s_{p,p,d}$ in the sense of equivalent quasi-norms.

Adapted to these sequence spaces we define now function spaces on $\mathbb{R}$.

**Definition 7.** Let $L \in \mathbb{N}_0$, $A \in \{B, F\}$ and $a \in \{b, f\}$ (with $a = b$ if $A = B$ and $a = f$ if $A = F$). Then $TA^s_{p,q}(\mathbb{R}, L, d)$ is the collection of all functions $g : \mathbb{R} \to \mathbb{C}$ such that there exists a decomposition

$$g(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s_{j,k} g_{j,k}(t) \tag{9}$$

(convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$), where the sequence $(s_{j,k})_{j,k}$ belongs to $a^s_{p,q,d}$ and the functions $g_{j,k}$ are even $L$-atoms centered at either $[-2^{-j}, 2^{-j}]$ if $k = 0$ or at

$$[-2^{-j}(k+1), -2^{-j}k] \cup [2^{-j}k, 2^{-j}(k+1)]$$

if $k > 0$. We put

$$\| g \mid TA^s_{p,q}(\mathbb{R}, L, d)\| := \inf\Big\{\| (s_{j,k}) \mid a^s_{p,q,d}\| : (9) \text{ holds}\Big\}.$$

For a real number $s$ we denote by $[s]$ the integer part, i.e. the largest integer $m$ such that $m \le s$.

**Theorem 3.** *Let $d \ge 2$ and suppose $0 < p < \infty$.*
(i) *Suppose $s > \sigma_p(d)$ and $L \ge [s] + 1$. Then the mapping* tr *is a linear isomorphism of $RB^s_{p,q}(\mathbb{R}^d)$ onto $TB^s_{p,q}(\mathbb{R}, L, d)$ with inverse* ext.
(ii) *Suppose $s > \sigma_{p,q}(d)$ and $L \ge [s] + 1$. Then the mapping* tr *is a linear isomorphism of $RF^s_{p,q}(\mathbb{R}^d)$ onto $TF^s_{p,q}(\mathbb{R}, L, d)$ with inverse* ext.

**Remark 10.** (i) Let $0 < p \le 1 < q \le \infty$. Then the spaces $RB^{\sigma_p}_{p,q}(\mathbb{R}^d)$ contain singular distributions, see [39]. In particular, the Dirac delta distribution belongs to $RB^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$, see, e.g., [32, Rem. 2.2.4/3]. Hence, our pointwise defined mapping tr is not meaningful on those spaces, or, with other words, Theorem 3 does not extend to values $s < \sigma_p(d)$.

(ii) Theorem 3 is proved in [37]. The proof is not very difficult and makes use of standard arguments in the field of atomic decompositions. The interesting point here consists in the fact that our arguments are not restricted to $\mathcal{S}'(\mathbb{R})$, see also Subsection 3.4.

Outside the origin radial distributions are more regular. We shall discuss several examples for this claim. Here is the first one, proved in [37] by using the atomic decompositions of radial distributions recalled in Subsection 2.2.

**Theorem 4.** *Let $d \geq 2$ and $0 < p < \infty$. Suppose $s > \max\big(0, \frac{1}{p} - 1\big)$. Let $f \in RA^s_{p,q}(\mathbb{R}^d)$ s.t. $0 \notin \operatorname{supp} f$. Then $f$ is a regular distribution in $\mathcal{S}'(\mathbb{R}^d)$, in fact, $f \in L_1(\mathbb{R}^d)$.*

**Remark 11.** There is a nice and simple example which explains the sharpness of the restrictions in Theorem 4. We consider the singular distribution $f$ defined by

$$\varphi \mapsto \int_{|x|=1} \varphi(x)\, dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

By using the wavelet characterization of Besov spaces, it is not difficult to prove that the spherical mean distribution $f$ belongs to the spaces $RB^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)$ for all $p$.

One can supplement Theorem 4. Essentially by using the same methods one can prove the following regularity result for the trace of a radial function, see [37].

**Theorem 5.** *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Suppose $s > \max\big(0, \frac{1}{p} - 1\big)$. Let $f \in RA^s_{p,q}(\mathbb{R}^d)$ s.t. $0 \notin \operatorname{supp} f$. Then $f_0 = \operatorname{tr} f$ belongs to $A^s_{p,q}(\mathbb{R})$.*

**Remark 12.** As mentioned above

$$A^s_{p,q}(\mathbb{R}) \hookrightarrow L_1(\mathbb{R}) + L_\infty(\mathbb{R}) \quad \text{if } s > \sigma_p(1) = \max\Big(0, \frac{1}{p} - 1\Big),$$

which shows again that we deal with regular distributions. However, in Theorem 5 some additional regularity is proved.

**3.3. Traces of radial subspaces of Sobolev spaces.** Clearly, one can expect that the description of the traces of radial Sobolev spaces can be given in more elementary terms. We discuss a few examples without having the complete theory.

**Theorem 6.** *Let $d \geq 2$ and $1 \leq p < \infty$.*
(i) *The mapping* tr *is a linear isomorphism (with inverse* ext*) of $RW^1_p(\mathbb{R}^d)$ onto the closure of $RC_0^\infty(\mathbb{R})$ with respect to the norm*

$$\| \, g \mid L_p(\mathbb{R}, |t|^{d-1})\| + \| \, g' \mid L_p(\mathbb{R}, |t|^{d-1})\|.$$

(ii) *The mapping* tr *is a linear isomorphism (with inverse* ext*) of $RW^2_p(\mathbb{R}^d)$ onto the closure of $RC_0^\infty(\mathbb{R})$ with respect to the norm*

$$\| \, g \mid L_p(\mathbb{R}, |t|^{d-1})\| + \| \, g' \mid L_p(\mathbb{R}, |t|^{d-1})\| \\ + \| \, g'/t \mid L_p(\mathbb{R}, |t|^{d-1})\| + \| \, g'' \mid L_p(\mathbb{R}, |t|^{d-1})\|.$$

**Remark 13.** Both statements have elementary proofs, see [37]. However, the complete extension to higher order Sobolev spaces is open.

There are several ways to define Sobolev spaces on $\mathbb{R}^d$. For instance, if $1 < p < \infty$ we have

$$f \in W_p^{2m}(\mathbb{R}^d) \quad \Longleftrightarrow \quad f \in L_p(\mathbb{R}^d) \text{ and } \Delta^m f \in L_p(\mathbb{R}^d).$$

Such an equivalence does not extend to $p = 1$ or $p = \infty$ if $d \geq 2$, see [41, pp. 135/160]. Recall that the Laplace operator $\Delta$ applied to a radial function yields a radial function. In particular, we have

$$\Delta f(x) = D_r f_0(r) := f_0''(r) + \frac{d-1}{r} f_0'(r), \quad r = |x|,$$

in case that $f$ is radial and $\operatorname{tr} f = f_0$. Obviously, if $f \in RC_0^\infty(\mathbb{R}^d)$, then

$$\begin{aligned} &\| f \mid L_p(\mathbb{R}^d)\| + \| \Delta^m f \mid L_p(\mathbb{R}^d)\| \\ &\qquad = \Big(\frac{\pi^{d/2}}{\Gamma(d/2)}\Big)^{1/p} \big(\| f_0 \mid L_p(\mathbb{R}, |t|^{d-1})\| + \| D_r^m f_0 \mid L_p(\mathbb{R}, |t|^{d-1})\|\big). \end{aligned}$$

This proves the next characterization.

**Theorem 7.** *Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then the mapping* tr *yields a linear isomorphism (with inverse* ext*) of $RW_p^{2m}(\mathbb{R}^d)$ onto the closure of $RC_0^\infty(\mathbb{R})$ with respect to the norm*

$$\| f_0 \mid L_p(\mathbb{R}, |t|^{d-1})\| + \| D_r^m f_0 \mid L_p(\mathbb{R}, |t|^{d-1})\|.$$

**Remark 14.** By means of Hardy-type inequalities one can simplify the terms $\| D_r^m f_0 \mid L_p(\mathbb{R}, |t|^{d-1})\|$ to some extent, see Theorem 6(ii) for a comparison. We do not go into detail.

**3.4. The trace in $\mathcal{S}'(\mathbb{R})$.** Many times applications of traces are connected with boundary value problems. In such a context the continuity of tr considered as a mapping into $S'$ is essential. Again we consider the simple situation of the $L_p$-spaces first.

**Lemma 3.** *Let $d \geq 2$ and let $0 < p < \infty$. Then $RL_p(\mathbb{R}, |t|^{d-1}) \subset S'(\mathbb{R})$ if, and only if, $d < p$.*

From the known embedding relations of $RA^s_{p,q}(\mathbb{R}^d)$ into $L_u$-spaces one obtains one half of the proof of the following general result.

**Theorem 8.** *Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(a) *Let $s > \sigma_p(d)$ and $L \geq [s]+1$. Then the following assertions are equivalent:*

(i) *The mapping* tr *maps $RB^s_{p,q}(\mathbb{R}^d)$ into $S'(\mathbb{R})$.*
(ii) *The mapping* $\mathrm{tr}\colon RB^s_{p,q}(\mathbb{R}^d) \to S'(\mathbb{R})$ *is continuous.*
(iii) *We have $TB^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$.*
(iv) *We have either $s > d\big(\frac{1}{p} - \frac{1}{d}\big)$ or $s = d\big(\frac{1}{p} - \frac{1}{d}\big)$ and $q \leq 1$.*

(b) *Let $s > \sigma_{p,q}(d)$ and $L \geq [s]+1$. Then following assertions are equivalent:*

(i) *The mapping* tr *maps $RF^s_{p,q}(\mathbb{R}^d)$ into $S'(\mathbb{R})$.*
(ii) *The mapping* $\mathrm{tr}\colon RF^s_{p,q}(\mathbb{R}^d) \to S'(\mathbb{R})$ *is continuous.*
(iii) *We have $TF^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$.*
(iv) *We have either $s > d\big(\frac{1}{p} - \frac{1}{d}\big)$ or $s = d\big(\frac{1}{p} - \frac{1}{d}\big)$ and $0 < p \leq 1$.*

**Remark 15.** Theorem 8 is proved in [37].

**3.5. The trace in $\mathcal{S}'(\mathbb{R})$ and weighted function spaces of Besov and Lizorkin-Triebel type.** Weighted function spaces of Besov and Lizorkin-Triebel type, denoted by $B^s_{p,q}(\mathbb{R}, w)$ and $F^s_{p,q}(\mathbb{R}, w)$, respectively, are a well-developed subject in the literature, we refer to [7], [8], [33]. Fourier analytic definitions as well as characterizations by atoms are given under various restrictions on the weights, see e.g. [6], [7], [8], [20], [21], [34]. In this subsection we are interested in these spaces with respect to the weights $w_{d-1}(t) := |t|^{d-1}$, $t \in \mathbb{R}$, $d \geq 2$. Of course, these weights belong to the Muckenhoupt class $\mathcal{A}_\infty$, more exactly $w_{d-1} \in \mathcal{A}_r$ for any $r > d$, see [42].

**Theorem 9.** *Let $d \geq 2$ and $0 < p < \infty$.*
(i) *Suppose $s > \sigma_p(d)$ and let $L \geq [s]+1$. If $TB^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$ (see Theorem 8), then $TB^s_{p,q}(\mathbb{R}, L, d) = RB^s_{p,q}(\mathbb{R}, w_{d-1})$ in the sense of equivalent quasi-norms.*
(ii) *Suppose $s > \sigma_{p,q}(d)$ and let $L \geq [s]+1$. If $TF^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$ (see Theorem 8), then $TF^s_{p,q}(\mathbb{R}, L, d) = RF^s_{p,q}(\mathbb{R}, w_{d-1})$ in the sense of equivalent quasi-norms.*

**Remark 16.** (i) We add some statements concerning the regularity of the most prominent singular distribution, namely $\delta\colon \varphi \to \varphi(0)$, $\varphi \in S(\mathbb{R}^d)$. This tempered distribution has the following regularity properties:

- First we deal with the situation on $\mathbb{R}^d$. We have $\delta \in RB^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$ (but $\delta \notin RB^{\frac{d}{p}-d}_{p,q}(\mathbb{R}^d)$ for $q < \infty$ and $\delta \notin RF^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$), see, e.g., [32, Rem. 2.2.4/3].

- Now we turn to the situation on $\mathbb{R}$. By using more or less the same arguments as on $\mathbb{R}^d$ one can show $\delta \in B^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}, w_{d-1})$ (but $\delta \notin B^{\frac{d}{p}-1}_{p,q}(\mathbb{R}, w_{d-1})$ for any $q < \infty$ and $\delta \notin F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}, w_{d-1})$).

(ii) For the simple proof of Theorem 9 we refer to [37]. It consists in a comparison of the atomic characterizations of $TA^s_{p,q}(\mathbb{R}, L, d)$ and of $RA^s_{p,q}(\mathbb{R}, w_{d-1})$, respectively. For the latter we refer to [20].

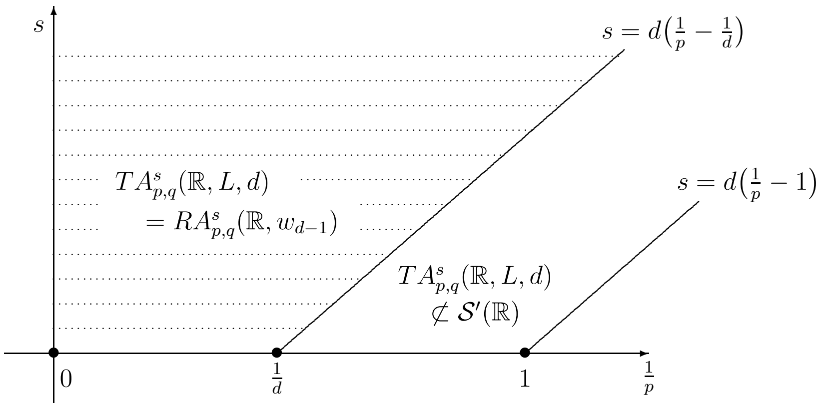


Figure 1.

## 4. The regularity of radial functions outside the origin. I. Inhomogeneous spaces

First assertions concerning the regularity of radial functions we have already given in Theorems 4, 5. In this section we continue these investigations.

**4.1. Some inequalities for radial functions.** Let $f$ be a radial function such that $\operatorname{supp} f \subset \{x \in \mathbb{R}^d : |x| \geq \tau\}$ for some $\tau > 0$. Then the following inequality is obvious:

$$\| f_0 \mid L_p(\mathbb{R})\| \leq \tau^{-(d-1)/p} \Big(\frac{\Gamma(d/2)}{\pi^{d/2}}\Big)^{1/p} \| f \mid L_p(\mathbb{R}^d)\|.$$

An extension to first or second order Sobolev spaces can be done by using Theorem 6. However, an extension to all spaces under consideration here is less obvious. Comparing the atomic decompositions of Theorem 3 with the known atomic or wavelet characterizations of $B^s_{p,q}(\mathbb{R})$ and $F^s_{p,q}(\mathbb{R})$ one obtains the following, see [37].

**Proposition 3.** *Let $d \geq 2$, $0 < p < \infty$, and $\tau > 0$.*
(i) *We suppose $s > \sigma_p(d)$. If $f \in RB^s_{p,q}(\mathbb{R}^d)$ such that*

$$\operatorname{supp} f \subset \{x \in \mathbb{R}^d : |x| \geq \tau\} \tag{10}$$

*then its trace $f_0$ belongs to $B^s_{p,q}(\mathbb{R})$. Furthermore, there exists a constant $c$ (not depending on $f$ and $\tau$) such that*

$$\| f_0 \mid B^s_{p,q}(\mathbb{R})\| \leq c\tau^{-(d-1)/p} \| f \mid B^s_{p,q}(\mathbb{R}^d)\|$$

*holds for all such functions $f$ and all $\tau > 0$.*
(ii) *We suppose $s > \sigma_{p,q}(d)$. If $f \in RF^s_{p,q}(\mathbb{R}^d)$ such that* (10) *holds, then its trace $f_0$ belongs to $F^s_{p,q}(\mathbb{R})$. Furthermore, there exists a constant $c$ (not depending on $f$ and $\tau$) such that*

$$\| f_0 \mid F^s_{p,q}(\mathbb{R})\| \leq c\tau^{-(d-1)/p} \| f \mid F^s_{p,q}(\mathbb{R}^d)\|$$

*holds for all such functions $f$ all $\tau > 0$.*

Proposition 3 has a partial inverse.

**Proposition 4.** *Let $d \geq 2$, $0 < p < \infty$, and $0 < a < b < \infty$.*
(i) *We suppose $s > \sigma_p(d)$. If $g \in RB^s_{p,q}(\mathbb{R})$ such that*

$$\operatorname{supp} g \subset \{x \in \mathbb{R} : a \leq |x| \leq b\} \tag{11}$$

*then the radial function $f := \operatorname{ext} g$ belongs to $RB^s_{p,q}(\mathbb{R}^d)$ and there exist positive constants $A$, $B$ such that*

$$A\| g \mid B^s_{p,q}(\mathbb{R})\| \leq \| f \mid B^s_{p,q}(\mathbb{R}^d)\| \leq B\| g \mid B^s_{p,q}(\mathbb{R})\|.$$

(ii) *We suppose $s > \sigma_{p,q}(d)$. If $g \in RF^s_{p,q}(\mathbb{R})$ such that* (11) *holds, then the radial function $f := \operatorname{ext} g$ belongs to $RF^s_{p,q}(\mathbb{R}^d)$ and there exist positive constants $A, B$ such that*

$$A\| g \mid F^s_{p,q}(\mathbb{R})\| \leq \| f \mid F^s_{p,q}(\mathbb{R}^d)\| \leq B\| g \mid F^s_{p,q}(\mathbb{R})\|.$$

**4.2. The continuity of radial functions outside the origin.** For our next result we need Hölder-Zygmund spaces. Recall, that the Hölder classes $C^s(\mathbb{R}^d)$ are special cases of Besov spaces if $s > 0$, but not a natural number. In fact $C^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d)$ in the sense of equivalent norms if $s \not\in \mathbb{N}_0$. Of course, also the spaces $B^s_{\infty,\infty}(\mathbb{R}^d)$ with $s \in \mathbb{N}$ allow a characterization by differences. These are the Zygmund-type spaces. We refer to [45, 2.2.2, 2.5.7] and [46, 3.5.3]. We shall use the abbreviation

$$\mathcal{Z}^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d), \quad s > 0.$$

Taking into account the well-known embedding relations for Besov as well as for Lizorkin-Triebel spaces, defined on $\mathbb{R}$, Theorem 5 implies in particular:

**Corollary 1.** *Let $d \geq 2$, $0 < p < \infty$ and $s > 1/p$. Let $\varphi$ be a smooth radial function, uniformly bounded together with all its derivatives, and such that $0 \notin \operatorname{supp} \varphi$. If $f \in RA^s_{p,q}(\mathbb{R}^d)$, then $\varphi f \in \mathcal{Z}^{s-1/p}(\mathbb{R}^d)$.*

**Remark 17.** P. L. Lions [26] has proved the counterpart of Corollary 1 for first order Sobolev spaces. We also dealt in [35] with these problems.

Of particular importance for us will be the continuity of radial functions. Again by combining Theorem 5 with the known embedding relations for Besov-Lizorkin-Triebel spaces on $\mathbb{R}$ we obtain:

**Corollary 2.** *Let $d \geq 2$, $0 < p < \infty$ and $\tau > 0$.*
(i) *If either $s > 1/p$ or $s = 1/p$ and $q \leq 1$ then $f \in RB^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on the set $|x| \geq \tau$.*
(ii) *If either $s > 1/p$ or $s = 1/p$ and $p \leq 1$ then $f \in RF^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on the set $|x| \geq \tau$.*

By looking at the restrictions in Corollary 2 we introduce the following set of parameters.

**Definition 8.** (i) We say $(s, p, q)$ belongs to the set $U(B)$ if $(s, p, q)$ satisfies the restrictions in part (i) of Corollary 2.
(ii) The triple $(s, p, q)$ belongs to the set $U(F)$ if $(s, p, q)$ satisfies the restrictions in part (ii) of Corollary 2.

**Remark 18.** (a) The abbreviation $(s, p, q) \in U(A)$ will be used with the obvious meaning.

(b) Let $1 \leq p = p_0 < \infty$ be fixed. Then there is always a largest space in the set

$$\{B^s_{p_0,q}(\mathbb{R}^d) : (s, p_0, q) \in U(B)\} \cup \{F^s_{p_0,q}(\mathbb{R}^d) : (s, p_0, q) \in U(F)\}.$$

This space is given either by $F^1_{1,\infty}(\mathbb{R}^d)$ if $p_0 = 1$ or by $B^{1/p_0}_{p_0,1}(\mathbb{R}^d)$ if $1 < p_0 < \infty$. If $p_0 < 1$, then obviously $B^{1/p_0}_{p_0,1}(\mathbb{R}^d)$ is the largest Besov space and $F^{1/p_0}_{p_0,\infty}(\mathbb{R}^d)$ is the largest Lizorkin-Triebel space in the above family. However, these spaces are incomparable.

## 5. Decay and boundedness properties of radial functions. I. Inhomogeneous spaces

We deal with improvements of Strauss' *Radial Lemma*. Decay can only be expected if we measure smoothness in function spaces built on $L_p(\mathbb{R}^d)$ with $p < \infty$.

**5.1. Decay and boundedness properties of radial functions in $W_1^1(\mathbb{R}^d)$.** It is instructive to have a short look onto the case of first order Sobolev spaces. Let $f = g(r(x)) \in RC_0^\infty(\mathbb{R}^d)$. Then

$$\frac{\partial f}{\partial x_i}(x) = g'(r)\frac{x_i}{r}, \quad r = |x| > 0,\ i = 1, \ldots, d.$$

Hence,

$$\| \, |\nabla f(x)| \mid L_p(\mathbb{R}^d)\| = c_d \| \, g' \mid L_p(\mathbb{R}, |t|^{d-1})\|,$$

where $1 \le p < \infty$. Next we apply the identity

$$g(r) = -\int_r^\infty g'(t)\, dt$$

and obtain

$$|g(r)| \le \int_r^\infty |g'(t)| dt \le r^{-(d-1)} \int_r^\infty t^{d-1} |g'(t)|\, dt.$$

This extends to all functions in $RW_1^1(\mathbb{R}^d)$ by a density argument. On this elementary way we have proved the inequality

$$\begin{aligned} |x|^{d-1} |f(x)| = r^{d-1} |g(r)| &\le \frac{1}{c_d} \int_{|x|>r} |\nabla f(x)|\, dx \\ &\le \frac{1}{c_d} \|\nabla f(x)\|_1. \end{aligned} \tag{12}$$

This inequality can be interpreted in several ways:

- The possible unboundedness in the origin is limited.
- There is some decay, uniformly in $f$, if $|x|$ tends to $+\infty$.
- We have $\lim_{|x|\to\infty} |x|^{d-1} |f(x)| = 0$ for all $f \in RW_1^1(\mathbb{R}^d)$.
- It makes sense to switch to homogeneous function spaces, since in (12) only the norm of the homogeneous Sobolev space occurs.

Now we discuss the extension of these assertions to fractional order of smoothness.

**5.2. The behaviour of radial functions near infinity.** Suppose that $(s, p, q) \in U(A)$. Then $f \in RA^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous near infinity and belongs to $L_p(\mathbb{R}^d)$. This implies $\lim_{|x|\to\infty} |f(x)| = 0$. However, much more is true.

**Theorem 10.** *Let $d \geq 2$ and $0 < p < \infty$.*
(i) *Suppose $(s,p,q) \in U(A)$. Then there exists a constant $c$ s.t.*

$$|x|^{(d-1)/p} |f(x)| \leq c \| f \mid A^s_{p,q}(\mathbb{R}^d)\| \tag{13}$$

*holds for all $|x| \geq 1$ and all $f \in RA^s_{p,q}(\mathbb{R}^d)$.*
(ii) *Suppose $(s,p,q) \in U(A)$. Then*

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{p}} |f(x)| = 0$$

*holds for all $f \in RA^s_{p,q}(\mathbb{R}^d)$.*
(iii) *Suppose $(s,p,q) \in U(A)$. Then there exists a constant $c > 0$ such that for all $x$, $|x| > 1$, there exists a smooth radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \mid A^s_{p,q}(\mathbb{R}^d)\| = 1$, s.t.*

$$|x|^{\frac{d-1}{p}} |f(x)| \geq c. \tag{14}$$

(iv) *Suppose $(s,p,q) \notin U(A)$ and $\frac{1}{p} > \sigma_p(d)$. We assume also that $\frac{1}{p} > \sigma_q(d)$ in the $F$-case. Then, for all sequences $(x^j)_{j=1}^\infty \subset \mathbb{R}^d \setminus \{0\}$ s.t. $\lim_{j\to\infty} |x^j| = \infty$, there exists a radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \mid A^s_{p,q}(\mathbb{R}^d)\| = 1$, s.t. $f$ is unbounded in any neighbourhood of $x^j$, $j \in \mathbb{N}$.*

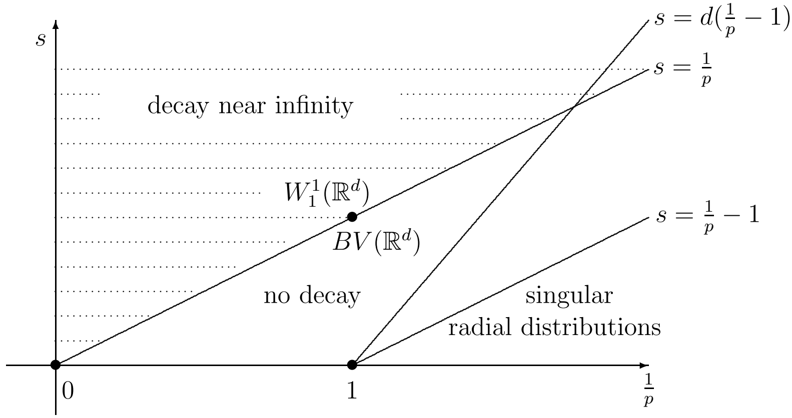


Figure 2.

**Remark 19.** (i) For $p = 1$ there are two other spaces for which such a decay estimate as in (13) hold. As we have seen above, it remains true for radial

functions in $W^1_1(\mathbb{R}^d)$. In addition, it remains correct for the larger class of radial functions of bounded variation, see [37].
(ii) Increasing $s$ (for fixed $p$) is not improving the decay rate. In the case of Banach spaces, i.e., $p,q \ge 1$, the additional assumptions in point (iv) are always fullfiled. Hence, the largest spaces, guaranteeing the decay rate $(d-1)/p$, are spaces with $s = 1/p$, see Remark 18.
(iii) Observe that in part (iii) of the theorem the function depends on $|x|$. There is no function in $RA^s_{p,q}(\mathbb{R}^d)$ such that (14) holds for all $x$, $|x| \ge 1$, simultaneously. The naive construction $f(x) := (1-\psi(x))|x|^{\frac{1-d}{p}}$, $x \in \mathbb{R}^d$, does not belong to $L_p(\mathbb{R}^d)$.
(iv) Of course, formula (13) generalizes the estimate (1). Also Coleman, Glazer and Martin [10] have dealt with (1). P. L. Lions [26] proved a $p$-version of the *Radial Lemma*.
(v) In case $A = B$ the theorem has been proved in [35]. The general case was treated in [37].

**A partial proof of Theorem 10.** We shall partly prove (i) and (iii) of Theorem 10.
*Step 1.* To demonstrate the usefulness of the atomic decompositions of radial functions, described in Subsection 2.2, we give a partial proof of the inequality (13). We concentrate on Besov spaces. It will be enough to consider the case $s = 1/p$ and $q = 1$.

Let $|x| \ge 1$. Let $f \in RB^{1/p}_{p,1}(\mathbb{R}^d)$. There exists an optimal atomic decomposition of this radial distribution

$$f = \sum_{j=0}^{\infty} s_{j,0}\, a_{j,0} + \sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\ell=1}^{C_{d,k}} s_{j,k} a_{j,k,\ell},$$

such that

$$\begin{aligned} \| \, (s_{j,k})_{j,k} \mid b_{p,1,d}\| &= \Big(\sum_{j=0}^{\infty}\Big(\sum_{k=0}^{\infty}(1+k)^{d-1}|s_{j,k}|^p\Big)^{q/p}\Big)^{1/q} \\ &\le c\| \, f \mid B^{1/p}_{p,1}(\mathbb{R}^d)\|, \end{aligned} \tag{15}$$

see (8). Observe, that for all $j \ge 0$ there exists a natural number $k_j$ such that

$$k_j 2^{-j} \le |x| < (k_j+1)2^{-j}. \tag{16}$$

Then the main part of $f$ near $(|x|, 0, \ldots, 0)$ is given by the function

$$f^M(y) = \sum_{j=0}^{\infty} s_{j,k_j} a_{j,k_j,0}(y).$$

Indeed, $f(x)$ is a finite sum of functions of type

$$\sum_{j=0}^{\infty} s_{j,k_j+r_j} a_{j,k_j+r_j,\ell_j+t_j}(y),$$

and $|r_j|$ and $|t_j|$ are uniformly bounded. This follows from properties (a), (b), (e) and (f) of our sequence of coverings $((\Omega_{j,k,\ell})_{k,\ell})_j$ and the support conditions fulfiled by the atoms. For convenience we give an estimate of the main part $f^M$ only. Because of (16) and the normalization of the atoms we obtain

$$\begin{aligned} |f^M(x)| &\le \sum_{j=0}^{\infty} |s_{j,k_j}| 2^{-j(s-\frac{d}{p})} \\ &\le \sum_{j=0}^{\infty} k_j^{\frac{1-d}{p}} 2^{-j(\frac{1}{p}-\frac{d}{p})} \Big( \sum_{k=1}^{\infty} k^{d-1} |s_{j,k}|^p \Big)^{1/p} \\ &\le 2^{\frac{d-1}{p}} |x|^{\frac{1-d}{p}} \sum_{j=0}^{\infty} \Big( \sum_{k=1}^{\infty} k^{d-1} |s_{j,k}|^p \Big)^{1/p} \\ &\le c|x|^{\frac{1-d}{p}} \| f \mid B^{1/p}_{p,1}(\mathbb{R}^d)\|, \end{aligned}$$

where we used (15) in the last step. The constant $c$ does not depend on $x$ and $f$. This proves Theorem 10 for radial Besov spaces.

*Step 2.* Let $A = B$. We suppose $s = 1/p$ and $q > 1$. According to Lemma 6(i) below there exists a compactly supported function $g_0$ which belongs to $RB^{1/p}_{p,q}(\mathbb{R})$ and is unbounded near the origin. By multiplying with a smooth cut-off function if necessary we can make the support of this functions as small as we want. For the given sequence $(x^j)_j$ we define

$$g(t) := \sum_{j=1}^{\infty} \frac{1}{\max(|x^j|, j)^{\alpha}} g_0(t - |x^j|), \quad t \in \mathbb{R},$$

where one has to choose $\alpha > 0$ in dependence on $p$ large enough. Obviously, the function $g$ is unbounded near $|x^j|$ and radial. For a proof of $g \in RB^{1/p}_{p,q}(\mathbb{R}^d)$ we refer to [37].

The remaining parts of the proof can be found in [37].

**5.3. The behaviour of radial functions near the origin.** Radial functions have also some extra properties near the origin. Under certain restrictions on the regularity (in terms of our spaces $RA^s_{p,q}(\mathbb{R}^d)$) the possible unboundedness near the origin is limited.

**5.3.1. Embeddings into $L_\infty(\mathbb{R}^d)$.** At first we mention that the embedding relations with respect to $L_\infty(\mathbb{R}^d)$ do not change when we switch from $A^s_{p,q}(\mathbb{R}^d)$ to its radial subspace $RA^s_{p,q}(\mathbb{R}^d)$.

**Lemma 4.** (i) *The embedding $RB^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ holds if, and only if, either $s > d/p$ or $s = d/p$ and $q \le 1$.*
(ii) *The embedding $RF^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ holds if, and only if, either $s > d/p$ or $s = d/p$ and $p \le 1$.*

The sufficiency part of the Lemma 4 is classical. Of interest is the necessity. Here we recall some results of Bourdaud [5], for the special case $B^s_{p,p}(\mathbb{R}^d)$ see also Triebel [47]. For $(\alpha,\sigma) \in \mathbb{R}^2$ we define the extremal functions

$$f_{\alpha,\sigma}(x) := \psi(x)\big|\log|x|\big|^{\alpha}\Big|\log\big|\log|x|\big|\Big|^{-\sigma}, \quad x \in \mathbb{R}^d. \tag{17}$$

Furthermore we define a set $U_t \subset \mathbb{R}^2$ as follows:

$$U_t := \begin{cases} (\alpha = 0 \text{ and } \sigma > 0) \text{ or } \alpha < 0 & \text{if } t = 1, \\ (\alpha = 1 - \frac{1}{t} \text{ and } \sigma > \frac{1}{t}) \text{ or } \alpha < 1 - \frac{1}{t} & \text{if } 1 < t < \infty, \\ (\alpha = 1 \text{ and } \sigma \ge 0) \text{ or } \alpha < 1 & \text{if } t = \infty, \end{cases}$$

**Lemma 5.** (i) *Let $0 < p \le \infty$ and $1 < q \le \infty$. Then $f_{\alpha,\sigma}$ belongs to $RB^{d/p}_{p,q}(\mathbb{R}^d)$ if, and only if, $(\alpha,\sigma) \in U_q$.*
(ii) *Let $1 < p < \infty$. Then $f_{\alpha,\sigma}$ belongs to $RF^{d/p}_{p,q}(\mathbb{R}^d)$ if, and only if, $(\alpha,\sigma) \in U_p$.*

Hence, unboundedness can only happen in case $s \le d/p$. We split our considerations into the cases $s < d/p$ and $s = d/p$.

**5.3.2. Controlled unboundedness near the origin in case $s < d/p$.** Again by making use of the atomic decompositions of radial functions one can prove the following, see [37].

**Theorem 11.** *Let $d \ge 2$ and $0 < p < \infty$.*
(i) *Suppose $(s,p,q) \in U(A)$ and $s < \frac{d}{p}$. Then there exists a constant $c$ s.t.*

$$|x|^{\frac{d}{p}-s}|f(x)| \le c\| f \mid A^s_{p,q}(\mathbb{R}^d)\| \tag{18}$$

*holds for all $0 < |x| \le 1$ and all $f \in RA^s_{p,q}(\mathbb{R}^d)$.*

(ii) *Let $\sigma_p(d) < s < d/p$. There exists a constant $c > 0$ such that for all $x$, $0 < |x| < 1$, there exists a smooth radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \mid A^s_{p,q}(\mathbb{R}^d)\| = 1$, s.t.*

$$|x|^{\frac{d}{p}-s}|f(x)| \geq c. \tag{19}$$

In the following figure we summarize our knowledge.

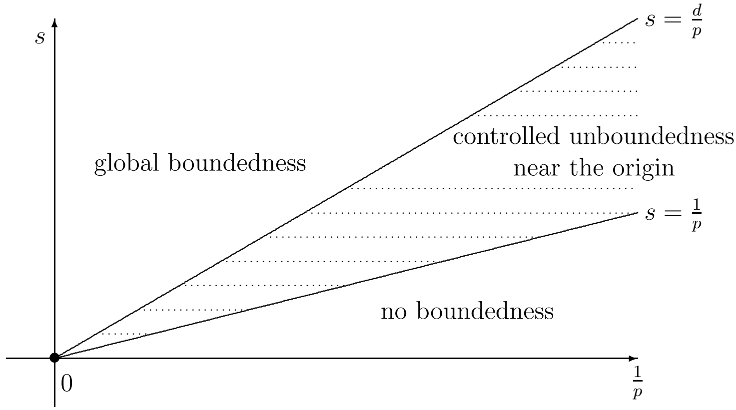


Figure 3.

**Remark 20.** (i) In case of $RB^s_{p,\infty}(\mathbb{R}^d)$ we have a function which realizes the extremal behaviour for all $|x| < 1$ simultaneously. It is well-known, see e.g. [32, Lem. 2.3.1/1], that the function

$$f(x) := \psi(x)|x|^{\frac{d}{p}-s}, \quad x \in \mathbb{R}^d,$$

belongs to $RB^s_{p,\infty}(\mathbb{R}^d)$, as long as $s > \sigma_p(d)$. This function does not belong to $RB^s_{p,q}(\mathbb{R}^d)$, $q < \infty$. Since it is also not contained in $RF^s_{p,q}(\mathbb{R}^d)$, $0 < q \leq \infty$ we conclude that in these cases there is no function, which realizes this upper bound for all $x$ simultaneously. In these cases the function $f$ in (19) has to depend on $x$.
(ii) In the literature one can find various types of further inequalities for radial functions. Many times preference is given to a homogeneous context, see for example the inequality (12). This will be discussed in Section 7. Sometimes also decay estimates are proved by replacing on the right-hand side the norm in the space $A^s_{p,q}(\mathbb{R}^d)$ by products of norms, e.g., $\| f \mid L_p(\mathbb{R}^d)\|^{1-\Theta} \| f \mid A^s_{p,q}(\mathbb{R}^d)\|^{\Theta}$ for some $\Theta \in (0,1)$, see [26], [27], [31] and [9]. Here we will not deal with those modifications (improvements).

Finally, we have to investigate $s \leq 1/p$ and $(s,p,q) \notin U(A)$.

**Lemma 6.** *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Suppose $(s,p,q) \notin U(A)$ and $\sigma_p(d) < 1/p$. Moreover let $\sigma_q(d) < 1/p$ in the $F$-case. Then there exists a radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \mid RA^s_{p,q}(\mathbb{R}^d)\| = 1$, and a sequence $(x^j)_j \subset \mathbb{R}^d \setminus \{0\}$ s.t. $\lim_{j\to\infty} |x^j| = 0$ and $f$ is unbounded in a neighbourhood of all $x^j$.*

**Remark 21.** One can use the same type of counterexamples as used in proof of Theorem 10.

**5.3.3. The behaviour of radial functions near the origin – borderline cases.** Now we turn to the remaining limiting situation. We shall show that there is also controlled unboundedness near the origin if $s = d/p$ and $RA^{d/p}_{p,q}(\mathbb{R}^d) \not\subset L_\infty(\mathbb{R}^d)$.

**Theorem 12.** *Let $d \geq 2$, $0 < p < \infty$ and suppose $s = d/p$.*
(i) *Let $1 < q \leq \infty$. Then there exists constant $c$ s.t.*

$$(-\log|x|)^{-1/q'}|f(x)| \leq c\| f \mid B^{d/p}_{p,q}(\mathbb{R}^d)\|$$

*holds for all $0 < |x| \leq 1/2$ and all $f \in RB^{d/p}_{p,q}(\mathbb{R}^d)$.*
(ii) *Let $1 < p < \infty$. Then there exists constant $c$ s.t.*

$$(-\log|x|)^{-1/p'}|f(x)| \leq c\| f \mid F^{d/p}_{p,q}(\mathbb{R}^d)\|$$

*holds for all $0 < |x| \leq 1/2$ and all $f \in RF^{d/p}_{p,q}(\mathbb{R}^d)$.*

**Remark 22.** Comparing Lemma 6 and Theorem 12 we find the following. For the case $q = \infty$ in Theorem 12(i) the function $f_{1,0}$, see (17), realizes the extremal behaviour. In all other cases there remains a gap of order $\log\log$ to some power.

## 6. Compact embeddings on $\mathbb{R}^d$. Part I

It is elementary to see that an embedding $A^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow A^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ can not be compact. To see this one may take such a smooth cut-off function $\psi$. This function belongs to all spaces $A^{s_0}_{p_0,q_0}(\mathbb{R}^d)$, whatever the parameters are. Then we define

$$\psi_\lambda(x) := \psi(x-\lambda) \quad x \in \mathbb{R}^d,\ \lambda \in \mathbb{Z}^d.$$

Then, by translation invariance of the quasi-norm $\| \cdot \mid A^{s_0}_{p_0,q_0}(\mathbb{R}^d)\|$ it follows

$$\| \psi_\lambda \mid A^{s_0}_{p_0,q_0}(\mathbb{R}^d)\| = \| \psi \mid A^{s_0}_{p_0,q_0}(\mathbb{R}^d)\| < \infty.$$

Hence, with $\lambda := (4, 0, \ldots, 0)$ and

$$F := \{\psi_{n\lambda} : n \in \mathbb{N}\}$$

we obtain a bounded subset of $A^{s_0}_{p_0,q_0}(\mathbb{R}^d)$. This subset $F$ of $A^{s_0}_{p_0,q_0}(\mathbb{R}^d)$ does not contain a convergent subsequence in $A^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ since

$$\begin{aligned} \| \psi_{n\lambda} - \psi_{m\lambda} \mid A^{s_1}_{p_1,q_1}(\mathbb{R}^d)\| &\asymp \| \psi_{n\lambda} \mid A^{s_1}_{p_1,q_1}(\mathbb{R}^d)\| + \| \psi_{m\lambda} \mid A^{s_1}_{p_1,q_1}(\mathbb{R}^d)\| \\ &= 2 \| \psi \mid A^{s_1}_{p_1,q_1}(\mathbb{R}^d)\| > 0, \quad n \neq m, \end{aligned}$$

where we applied the disjointness of the supports of $\psi_{n\lambda}$ and $\psi_{m\lambda}$.

The situation changes dramatically when we switch to the radial subspaces. Before we formulate the general result on compactness of embeddings of radial Besov-Lizorkin-Triebel spaces we treat a special situation with an elementary proof.

**6.1. The compactness of the embedding $RH^1(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d)$.** We are going to prove

$$RH^1(\mathbb{R}) \hookrightarrow\hookrightarrow L_q(\mathbb{R}), \quad 2 < q < \begin{cases} \frac{2d}{d-2} & d > 2, \\ \infty & d = 2. \end{cases} \tag{20}$$

**Proof.** Let $(f_j)_j$ be a bounded sequence in $RH^1(\mathbb{R})$, say

$$\sup_j \| f_j \mid H^1(\mathbb{R})\| \leq 1.$$

Let

$$2 \leq p < \begin{cases} \frac{2d}{d-2} & d > 2, \\ \infty & d = 2. \end{cases}$$

By $B_r$ we denote the ball with radius $r$ and centre in the origin. The embedding $H^1(B_r) \hookrightarrow\hookrightarrow L_p(B_r)$ is compact for any $r > 0$. This implies the existence of a function $f \in L_p^{\rm loc}(\mathbb{R})$ and a subsequence $(f_{j_\ell})_\ell$ s.t.

$$\Big( \int_{B_\ell} |f(x) - f_{j_\ell}(x)|^p \, dx \Big)^{1/p} < \frac{1}{\ell}.$$

From the continuity of the embedding $RH^1(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d)$ we derive $\| f_j \mid L_p(\mathbb{R}^d)\| \leq C$ and therefore $f \in L_p(\mathbb{R}^d)$ and $\| f \mid L_p(\mathbb{R}^d)\| \leq C$. We put $B^r := \{x \in \mathbb{R} : |x| > r\}$. Let $q$ be as in (20) and define $\Theta := 1 - 2/q$.

Then our decay estimate (13) (recall $H^1(\mathbb{R}^d) = F^1_{2,2}(\mathbb{R}^d) = B^1_{2,2}(\mathbb{R}^d)$ in the sense of equivalent norms) implies

$$\| f_j \mid L_q(B^r)\| \le \| f_j \mid L_2(B^r)\|^{1-\Theta} \|f_j \mid L_\infty(B^r)\|^{\Theta} \le c r^{-\frac{(d-1)\Theta}{2}} \tag{21}$$

Because of the a.e. convergence of a subsequence to $f$ the limit function has the same decay properties. Therefore, (21) holds with $f$ as well. For arbitrary $r > 0$ we find

$$\| \psi(\,\cdot/r)(f - f_{j_\ell}) \mid L_q(\mathbb{R})\| \to 0 \quad \text{as } \ell \to \infty.$$

Furthermore

$$\| (1-\psi(\,\cdot/r))(f - f_{j_\ell}) \mid L_q(\mathbb{R})\| \le c r^{-\frac{(d-1)\Theta}{2}} \to 0 \quad \text{as } r \to \infty,$$

independent of $\ell$. This proves the compactness. □

**Remark 23.** The proof given above is also typical for the general situation. A few references will be given in Remark 25 below.

**6.2. The compactness of the embedding $RA^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_u(\mathbb{R}^d)$.** To begin with we recall some embedding relations for Besov and Lizorkin-Triebel spaces.

**Proposition 5.** (i) *The embedding $B^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ holds if, and only if, $p_0 \le p_1$ and either*

$$s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1} \tag{22}$$

*or*

$$s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1} \quad \textit{and} \quad q_0 \le q_1.$$

(ii) *The embedding $F^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ holds if, and only if, either $p_0 < p_1$ and $s_0 - \frac{d}{p_0} \ge s_1 - \frac{d}{p_1}$ holds or $p_0 = p_1$ and $s_0 > s_1$ or $p_0 = p_1$, $s_0 = s_1$ and $q_0 \le q_1$.*

**Remark 24.** (i) For $A^s_{p,q}(\mathbb{R}^d)$ the quantity $s - d/p$ is called the associated differential dimension.
(ii) Proposition 5 can be found in [39]. However, sufficiency of the conditions is proved in many places.

Now we turn to the question which of these embeddings becomes compact when switching from $A^s_{p,q}(\mathbb{R}^d)$ to its radial subspace. Here we have a final result, originally proved in [35].

**Theorem 13.** *Let* $A, \mathcal{A} \in \{B, F\}$.
(i) *Let* $d \geq 2$. *Then the embedding*

$$RA^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow R\mathcal{A}^{s_1}_{p_1,q_1}(\mathbb{R}^d)$$

*is compact if, and only if,* $p_0 < p_1$ *and*

$$s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}.$$

(ii) *Let* $d = 1$. *Then for all pairs of triples* $(s_0, p_0, q_0)$ *and* $(s_1, p_1, q_1)$ *the space* $RA^{s_0}_{p_0,q_0}(\mathbb{R})$ *is not compactly embedded into* $R\mathcal{A}^{s_1}_{p_1,q_1}(\mathbb{R})$.

Observe, that the microscopic indices $q_0$ and $q_1$ do not influence the conditions. By means of the elementary embeddings

$$B^0_{p,1}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^d)$$

we immediately obtain the following corollary.

**Corollary 3.** *Let* $1 \leq p_1 \leq \infty$.
(i) *Let* $d \geq 2$. *Then the embedding*

$$RA^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow L_{p_1}(\mathbb{R}^d)$$

*is compact if, and only if,* $p_0 < p_1$ *and*

$$s_0 > d\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big).$$

(ii) *Let* $d = 1$. *Then for all pairs of triples* $(s_0, p_0, q_0)$ *and* $(s_1, p_1, q_1)$ *the space* $RA^{s_0}_{p_0,q_0}(\mathbb{R})$ *is not compactly embedded into* $L_{p_1}(\mathbb{R})$.

**Remark 25.** (i) In case of first order Sobolev spaces the Corollary 3 (sufficiency part) has been known for a long time, we refer to Berestycki and Lions [2], Coleman, Glazer and Martin [10], Strauss [43], and Lions [26]. In case of radial Sobolev spaces this result can also be found in the monograph by Kuzin and Pohozaev [25, 2.8] and in the lecture note by Hebey [19, 5.3]. Cho and Ozawa [9] discussed this problem for fractional order of smoothness, but with $p = 2$. Necessity of these conditions in case of Sobolev spaces has been observed by Ebihara and Schonbeck [13].
(ii) Quite recently Cwickel and Tintarev [11] gave a different, elegant and short proof of Corollary 3 restricted to Besov spaces and with some additional restrictions for $p_0$ and $q_0$.

## 7. Regularity, decay and boundedness properties of radial functions in homogeneous spaces

As we have seen in (12), Subsection 5.1, sometimes decay estimates can be proved by using homogeneous norms instead of the inhomogeneous ones. Here in this section we shall investigate this problem in greater detail. For homogeneous spaces we prefer to start with the Fourier analytic approach.

**7.1. Distribution spaces modulo polynomials.** General references for homogeneous Besov and Lizorkin-Triebel spaces are [16], [17], [18], [28], [45]. For convenience of the reader we recall the definition and a few properties of these spaces.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be a radial function such that $\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^d : 1/4 \le |\xi| \le 4\}$ and $\varphi(\xi) = 1$ if $1/2 \le |\xi| \le 2$. Then we define

$$\varphi_j(\xi) := \varphi(2^{-j+1}\xi), \quad \xi \in \mathbb{R}^d,\ j \in \mathbb{Z}.$$

This leads to a specific homogeneous smooth dyadic decomposition of unity since

$$\sum_{j=-\infty}^{\infty} \varphi_j(\xi) = 1, \quad \xi \neq 0.$$

We shall identify tempered distributions modulo polynomials. In fact, we consider the classes

$$[f] := \{f + p : p \text{ polynomial over } \mathbb{R}^d\}, \quad f \in S'(\mathbb{R}^d).$$

**Definition 9.** (i) Let $0 < p \le \infty$. Then the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ is the collection of all classes $[f]$ such that

$$\| [f] \mid \dot{B}^s_{p,q}(\mathbb{R}^d)\| := \Big( \sum_{j=-\infty}^{\infty} 2^{jsq} \| \mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)](\,\cdot\,) \mid L_p(\mathbb{R}^d)\|^q \Big)^{1/q} < \infty.$$

(ii) Let $0 < p < \infty$. Then the homogeneous Lizorkin-Triebel space $\dot{F}^s_{p,q}(\mathbb{R}^d)$ is the collection of all classes $[f]$ such that

$$\| [f] \mid \dot{F}^s_{p,q}(\mathbb{R}^d)\| := \Big\| \Big( \sum_{j=-\infty}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)](\,\cdot\,)|^q \Big)^{1/q} \Bigm| L_p(\mathbb{R}^d)\Big\| < \infty.$$

**Remark 26.** (i) The definition makes sense since

$$\mathcal{F}^{-1}[\varphi_j\, \mathcal{F}(f+p)] = \mathcal{F}^{-1}[\varphi_j\, \mathcal{F}f]$$

for all polynomials $p$, all $f \in S'(\mathbb{R}^d)$, and all $j \in \mathbb{Z}$. Moreover, the spaces $\dot{B}^s_{p,q}(\mathbb{R}^d)$ and $\dot{F}^s_{p,q}(\mathbb{R}^d)$ are independent of the resolution of unity up to equivalence of quasi-norms. Furthermore, we always have

$$\sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}(\varphi_j \mathcal{F} g) \in [f] \quad \forall g \in [f].$$

(ii) The spaces $\dot{B}^s_{p,q}(\mathbb{R}^d)$ and $\dot{F}^s_{p,q}(\mathbb{R}^d)$ are quasi-Banach spaces.
(iii) Let $1 < p < \infty$. Define $\dot{H}^s_p(\mathbb{R}^d)$ as the collection of all classes $[f]$ such that $\mathcal{F}^{-1}[|\xi|^s \mathcal{F} f(\xi)](\,\cdot\,) \in L_p(\mathbb{R}^d)$ equipped with the induced norm. Usually $\dot{H}^s_p(\mathbb{R}^d)$ are called homogeneous potential spaces. Then $\dot{H}^s_p(\mathbb{R}^d)$ coincides with $\dot{F}^s_{p,2}(\mathbb{R}^d)$ in the sense of equivalent norms.

**7.2. Radial classes of distributions.** Following [14] we use the following definition of radiality in the homogeneous context.

**Definition 10.** Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then we call the class $[f]$ radial if $[f]$ contains a radial distribution $g$.

Of some use will be the following simple observation. Let $[f] \in \dot{A}^s_{p,q}(\mathbb{R}^d)$ be radial. Let $g$ be one of the radial elements in $[f]$. Let $(\varphi_j)_j$ be the smooth, homogeneous, dyadic and radial decomposition of unity, defined in the previous Subsection 7.1. Then the distribution

$$\sigma(g) := \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F} g]$$

is radial as well, since the Fourier transform of a radial function is radial. However, the right-hand side does not depend on the particular element $g$ in $[f]$. Hence, we may write

$$\sigma([f]) = \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F} h], \tag{23}$$

where $h \in [f]$ is arbitrary. The mapping $[f] \mapsto \sigma([f])$ has some further nice properties which we are going to recall now.

Some properties of Besov spaces are partly easier described in terms of Lorentz spaces than in terms of Lebesgue spaces. For a measurable function $f\colon \mathbb{R}^d \to \mathbb{C}$ its non-increasing rearrangement is denoted by $f^*$, i.e.,

$$f^*(t) := \inf\bigl\{\lambda : |\{x \in \mathbb{R}^d : |f(x)| > \lambda\}| \le t\bigr\}, \quad 0 < t < \infty.$$

Then the Lorentz space $L_{p,q}(\mathbb{R}^d)$ is the collection of all functions $f$ s.t.

$$\| f \mid L_{p,q}(\mathbb{R}^d)\| := \begin{cases} \left(\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{0<t<\infty} t^{1/p} f^*(t) & \text{if } q = \infty \end{cases}$$

is finite. We refer to [1, Chap. 4], [3, 1.3] or [44, 1.18.6] for the basic properties of these spaces. They represent natural refinements of the Lebesgue spaces $L_p(\mathbb{R}^d)$ in view of the identity $L_p(\mathbb{R}^d) = L_{p,p}(\mathbb{R}^d)$. Let $C_0(\mathbb{R}^d)$ be the space of all uniformly continuous functions vanishing at infinity.

**Lemma 7.** *Let* $0 < p < \infty$.
(i) *Let* $\sigma_p(d) < s < d/p$. *Then*

$$\sigma \in \mathcal{L}(\dot{B}^s_{p,q}(\mathbb{R}^d), L_{t,q}(\mathbb{R}^d)), \quad t := \frac{d}{\frac{d}{p} - s}$$

*and*

$$\sigma \in \mathcal{L}(\dot{F}^s_{p,q}(\mathbb{R}^d), L_t(\mathbb{R}^d)), \quad t := \frac{d}{\frac{d}{p} - s}$$

(ii) *Let* $s = d/p$. *Then*

$$\sigma \in \mathcal{L}(\dot{B}^{d/p}_{p,1}(\mathbb{R}^d), C_0(\mathbb{R}^d)).$$

(iii) *Let* $0 < p \le 1$ *and* $s = d/p$. *Then*

$$\sigma \in \mathcal{L}(\dot{F}^{d/p}_{p,q}(\mathbb{R}^d), C_0(\mathbb{R}^d)).$$

**Remark 27.** (i) Essentially, Lemma 7 is well-known. We refer to Peetre [28, Thm. 7, Chap. 11, p. 242] for the first part (inhomogeneous case) and to Bourdaud [4] for the second. Also in [36] we gave a detailed proof.

(ii) The limiting situation $s = \sigma_p(d)$ is investigated in Peetre [28, Thm. 7, Chap. 11, p. 242] and in Vybíral [49].

**7.3. Radial subspaces of homogeneous Besov-Lizorkin-Triebel spaces.** Lemma 7 gives us the possibility to describe the subsets of homogeneous Besov-Lizorkin-Triebel spaces we are interested in, in a much more transparent way. By $\sigma(\dot{A}^s_{p,q}(\mathbb{R}^d))$ we denote the set of all images under the mapping $\sigma$ equipped with the quasi-norm

$$\| \sigma([f]) \mid \sigma(\dot{A}^s_{p,q}(\mathbb{R}^d))\| := \| [f] \mid \dot{A}^s_{p,q}(\mathbb{R}^d)\|.$$

**Lemma 8.** *Let* $0 < p < \infty$.
(i) *Let* $\sigma_p < s < d/p$. *Then*

$$\sigma(R\dot{A}^s_{p,q}(\mathbb{R}^d)) = \dot{A}^s_{p,q}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d), \quad t := \frac{d}{\frac{d}{p} - s}.$$

(ii) *Let* $s = d/p$. *Then*

$$\sigma(R\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)) = \dot{B}^{d/p}_{p,1}(\mathbb{R}^d) \cap RC_0(\mathbb{R}^d).$$

(iii) *Let* $0 < p \le 1$ *and* $s = d/p$. *Then*

$$\sigma(R\dot{F}^{d/p}_{p,q}(\mathbb{R}^d)) = \dot{F}^{d/p}_{p,q}(\mathbb{R}^d) \cap RC_0(\mathbb{R}^d).$$

**Remark 28.** (i) It is not difficult to see that Lemma 8 can be supplemented by

$$\sigma(R\dot{B}^s_{p,q}(\mathbb{R}^d)) = \dot{B}^s_{p,q}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d)), \quad t := \frac{d}{\frac{d}{p} - s}, \ q \le t,$$

and

$$\sigma(R\dot{F}^s_{p,q}(\mathbb{R}^d)) = \dot{F}^s_{p,q}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d)), \quad t := \frac{d}{\frac{d}{p} - s}, \ 0 < q \le \infty.$$

(ii) Cho and Ozawa [9] have used the above identity to introduce radial subspaces of $\dot{H}^s(\mathbb{R}^d) = \dot{B}^s_{2,2}(\mathbb{R}^d) = \dot{F}^s_{2,2}(\mathbb{R}^d)$. The general statement has been proved in [36].

**7.4. Homogeneous versus inhomogeneous spaces.** For convenience of the reader we recall the relations between the homogeneous spaces $\dot{A}^s_{p,q}(\mathbb{R}^d)$ and their inhomogeneous counterparts $A^s_{p,q}(\mathbb{R}^d)$. If $0 < p < \infty$, $0 < q \le \infty$ and $s > \sigma_p$, then

$$A^s_{p,q}(\mathbb{R}^d) = L_p(\mathbb{R}^d) \cap \dot{A}^s_{p,q}(\mathbb{R}^d) \tag{24}$$

and

$$\| \, g \mid A^s_{p,q}(\mathbb{R}^d)\| \asymp \| \, g \mid L_p(\mathbb{R}^d)\| + \| \, [g] \mid \dot{A}^s_{p,q}(\mathbb{R}^d)\|. \tag{25}$$

Formula (24) has to be interpreted in the following way:

- If inside the class $[f] \in \dot{A}^s_{p,q}(\mathbb{R}^d)$ is one representative $g$ belonging to $L_p(\mathbb{R}^d)$, then this function $g$ belongs to the inhomogeneous space $A^s_{p,q}(\mathbb{R}^d)$ and the quasi-norm equivalence (25) holds.

- On the other hand, if $g \in A^s_{p,q}(\mathbb{R}^d)$ then the associated class $[g]$ belongs to $\dot{A}^s_{p,q}(\mathbb{R}^d)$ and the quasi-norm equivalence (25) holds.

By means of such an interpretation it is clear that under the given restrictions the homogeneous spaces are larger than its inhomogeneous counterparts. Hence, decay and boundedness properties of elements of radial subspaces of homogeneous spaces can be quite different from those of of radial subspaces of inhomogeneous spaces. It is instructive to look at the following family of test functions. For $\alpha > 0$ and $\delta \geq 0$ we define

$$g_{\alpha,\delta}(x) := (1+|x|^2)^{-\alpha/2}(\log(\mathrm{e}+|x|^2))^{-\delta} \tag{26}$$

Elementary calculations yield the following.

**Lemma 9.** *Let $1 \leq p < \infty$ and $0 < q < \infty$.*
(i) *The function $g_{\alpha,\delta}$ belongs to $L_{p,q}(\mathbb{R}^d)$ if, and only if, either $\alpha > d/p$ or $\alpha = d/p$ and $\delta q > 1$.*
(ii) *Let $m \in \mathbb{N}$. Then $g_{\alpha,\delta} \in \dot{W}^m_p(\mathbb{R}^d)$ if, and only if, either $\alpha + m > d/p$ or $\alpha + m = d/p$ and $\delta > 1/p$.*

**Remark 29.** Comparing (i) and (ii) it becomes obvious that the conditions for belonging to the space $\dot{W}^m_p(\mathbb{R}^d)$ are weaker than those for belonging to $L_p(\mathbb{R}^d)$.

These assertions extend in a natural way to fractional order of smoothness. For a proof we refer to [36].

**Proposition 6.** *Let $\delta > 0$.*
(i) *We suppose $\sigma_{p,q} < s < d/p$. Then $g_{\alpha,\delta}$ belongs to $\dot{F}^s_{p,q}(\mathbb{R}^d)$ if either $\alpha > \frac{d}{p} - s$ ($\delta \geq 0$ arbitrary) or $\alpha = \frac{d}{p} - s$ and $\delta > 1/p$.*
(ii) *We suppose $\sigma_p < s < d/p$. Then $g_{\alpha,\delta}$ belongs to $\dot{B}^s_{p,q}(\mathbb{R}^d)$ if either $\alpha > \frac{d}{p} - s$ or $\alpha = \frac{d}{p} - s$ and $\delta > 1/q$.*

**Remark 30.** Let either $\sigma_{p,q} < s < d/p$ (if $A = F$) or $\sigma_p < s < d/p$ (if $A = B$). Then the embeddings

$$A^s_{p,q}(\mathbb{R}^d) \hookrightarrow \big(\dot{A}^s_{p,q}(\mathbb{R}^d) \cap L_{t,\infty}(\mathbb{R}^d)\big), \quad t := \frac{d}{\frac{d}{p} - s},$$

are strict.

**7.5. The regularity of radial functions outside the origin. II. Homogeneous spaces.** Now we turn to the problems we have already dealt with in the context of inhomogeneous spaces, see Subsection4.2, namely the continuity of radial functions outside the origin. The extension to homogeneous spaces is more or less obvious. Let $[f] \in R\dot{A}^s_{p,q}(\mathbb{R}^d)$. We use (23). For any element $g$ of $[f]$ we have

$$g = p + \sigma([f]) = p + f_0 + f_1, \tag{27}$$

where $p$ denotes an appropriate polynomial and

$$f_0 := \sum_{j=-\infty}^{-1} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \quad \text{and} \quad f_1 := \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f). \tag{28}$$

By classical properties of the Fourier transform (the Fourier transform of a radial function is radial) and because of the fact, that also the functions $\varphi_j$ are radial, both $f_0$ and $f_1$ are radial. The first sum $f_0$ is an entire analytic function of exponential type, whereas the second sum $f_1$ belongs to the inhomogeneous space $RA^s_{p,q}(\mathbb{R}^d)$. Thus, the local smoothness depends on the second sum and therefore it is the same as in case of radial inhomogeneous spaces.

**Lemma 10.** *Let $[f] \in R\dot{A}^{s_0}_{p_0,q_0}(\mathbb{R}^d)$. Then, for any $g \in [f]$, we have $g \in A^{s_1,\mathrm{loc}}_{p_1,q_1}(\mathbb{R}^d)$ if $f_1 \in RA^{s_1}_{p_1,q_1}(\mathbb{R}^d)$.*

Now we simply refer to Subsection 4.2. The outcome are the following corollaries.

**Corollary 4.** *Let $d \geq 2$, $0 < p < \infty$, and $s > 1/p$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be a radial function such that $0 \notin \operatorname{supp} \varphi$. If $[f] \in R\dot{A}^s_{p,q}(\mathbb{R}^d)$, then for all $g \in [f]$ we have $\varphi g \in \mathcal{Z}^{s-1/p}(\mathbb{R}^d)$.*

Some limiting cases are collected in the next corollary.

**Corollary 5.** *Let $d \geq 2$ and $0 < p < \infty$.*
(i) *If $[f] \in R\dot{B}^{1/p}_{p,1}(\mathbb{R}^d)$, then all $g \in [f]$ are continuous outside the origin.*
(ii) *Let $0 < p \leq 1$. If $[f] \in R\dot{F}^{1/p}_{p,\infty}(\mathbb{R}^d)$, then all $g \in [f]$ are continuous outside the origin.*

**Remark 31.** Let $s = d/p$. Then it follows from Lemma 7 that $[f] \in R\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)$ implies that all $g \in [f]$ are continuous on $\mathbb{R}^d$. Similarly, if $[f] \in R\dot{F}^{d/p}_{p,\infty}(\mathbb{R}^d)$, $0 < p \leq 1$, then all $g \in [f]$ are continuous on $\mathbb{R}^d$.

**7.6. Atomic decompositions in homogeneous spaces.** Again atomic decompositions represent the main tool in dealing with decay properties. We recall a construction of EPPERSON and FRAZIER [14]. We will do that with certain detail but with a different normalization.

Let $J_\nu$ denote the Bessel function of order $\nu$, $\nu \ge -\frac{1}{2}$, defined by

$$J_\nu(t) := \begin{cases} \frac{(t/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_{-1}^{1} (1-y^2)^{\nu-1/2} \mathrm{e}^{ity}\, dy & \text{if } \nu > -\frac{1}{2}, \\ \left(\frac{2}{\pi t}\right)^{1/2} \cos t & \text{if } \nu = -\frac{1}{2}, \end{cases} \qquad t \in \mathbb{R}.$$

Let $\mu_{\nu,0} < \mu_{\nu,1} < \ldots$ be the positive zeros of $J_\nu$. We put $\mu_{\nu,-1} := 0$. Then

$$\mu_{\nu,k} = \pi\Big(k + \frac{\nu}{2}\Big) + O\Big(\frac{1}{k+1}\Big)$$

and

$$\mu_{\nu,k} - \mu_{\nu,k-1} = \pi + O\Big(\frac{1}{k+1}\Big).$$

For $k = 0, 1, 2, \ldots$ we introduce associated annuli (balls, if $k = 0$)

$$A_{j,k} := \{x \in \mathbb{R}^d : 2^{-j}\mu_{\nu,k-1} \le |x| \le 2^{-j}\mu_{\nu,k}\}, \quad j \in \mathbb{Z}$$
$$\widetilde{A}_{j,k} := \{x \in \mathbb{R}^d : 2^{-j}(\mu_{\nu,k-1} - 1) \le |x| \le 2^{-j}(\mu_{\nu,k} + 1)\}, \quad j \in \mathbb{Z}.$$

From now on we fix $\nu = \frac{d-2}{2}$ and drop it in notation.

Next we recall the definition of smooth radial atoms from [14].

**Definition 11.** Let $s \in \mathbb{R}$ and $0 < p < \infty$. A radial function $a$ is called a smooth radial atom associated to $A_{j,k}$ if it satisfies the following conditions:

$$\operatorname{supp} a \subset \widetilde{A}_{j,k}, \tag{29}$$

$$\int a(x)\, dx = 0,$$

$$\sup_{x \in \mathbb{R}^d} |D^\gamma a(x)| \le c_\gamma 2^{-j\left(s - |\gamma| - \frac{d}{p}\right)} \quad \forall \gamma \in \mathbb{N}_0^d. \tag{30}$$

Here $c_\gamma := 1$ if $|\gamma| \le s+1$ and $c_\gamma$ must be independent of $j$ and $k$ if $|\gamma| > s+1$.

As usual one has to introduce associated sequence spaces as well. Let $\chi_{A_{j,k}}$ denote the characteristic function of the set $A_{j,k}$. Then we define $\widetilde{\chi}_{j,\ell}^{(p)} := 2^{\frac{jd}{p}} \chi_{A_{j,\ell}}$. The announced sequence spaces are then given by

$$\dot{b}_{p,q} := \Big\{ (s_{j,k})_{j,k} : s_{j,k} \in \mathbb{C},$$
$$\| (s_{j,k})_{j,k} \mid \dot{b}_{p,q} \| := \Big( \sum_{j \in \mathbb{Z}} \Big\| \sum_{k=0}^{\infty} s_{j,k} \widetilde{\chi}_{j,k}^{(p)} \Bigm| L_p(\mathbb{R}^d) \Big\|^q \Big)^{1/q} < \infty \Big\}$$

and

$$\dot{f}_{p,q} := \Big\{ (s_{j,k})_{j,k} : s_{j,k} \in \mathbb{C},$$
$$\| (s_{j,k})_{j,k} \mid \dot{f}_{p,q}\| := \Big\| \Big( \sum_{j\in\mathbb{Z}} \sum_{k=0}^{\infty} \big| s_{j,k} \widetilde{\chi}^{(p)}_{j,k}(\cdot)\big|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d)\Big\| < \infty \Big\}.$$

Again we will use these notation with $a$ in place of $b$ or $f$ if there is no need to distinguish these cases. Now we are in position to formulate the result of Epperson and Frazier [14], see Theorem 4.1 and the comments in Section 5.

**Theorem 14.** *Let either $s > \sigma_{p,q}(d) - 1$ if $A = F$ or $s > \sigma_p(d) - 1$ if $A = B$. For $[f] \in R\dot{A}^s_{p,q}(\mathbb{R}^d)$ there exist smooth radial atoms $a_{j,k}$ associated to $A_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, and a sequence $(s_{j,k})_{j,k} \in \dot{a}_{p,q}$, such that*

$$\sum_{j\in\mathbb{Z}} \sum_{k=0}^{\infty} s_{j,k}\, a_{j,k} \in [f] \tag{31}$$

*and*

$$\| [f] \mid \dot{A}^s_{p,q}(\mathbb{R}^d)\| \asymp \| (s_{j,k})_{j,k} \mid \dot{a}_{p,q}\|. \tag{32}$$

**Remark 32.** The identity (31) should be interpreted in the following way. The sequence $(f_n)_n$, where

$$f_n = \sum_{j=-n}^{n} \sum_{k=0}^{\infty} s_{j,k} a_{j,k},$$

converges to some $g \in [f]$ with respect to the quasi-norm in $\dot{A}^s_{p,q}(\mathbb{R}^d)$ as $n$ tends to infinity, if $q < \infty$, and in $S'(\mathbb{R}^d)/\mathcal{P}$ if $q = \infty$.

We need another result of Epperson and Frazier, see Theorem 3.1 and the comments in Section 5 in [14].

**Lemma 11.** *Let either $s > \sigma_{p,q}(d)$ if $A = F$ or $s > \sigma_p(d)$ if $A = B$. Then there exists a positive constant $c$ s.t. for any sequence $(a_{j,k})_{j\in\mathbb{Z}}$, $k \in \mathbb{N}_0$ of radial functions satisfying the conditions* (29), (30) *(restricted to values of $\gamma$ s.t. $|\gamma| \le s+1$) and any sequence $(s_{j,k})_{j,k} \in \dot{a}_{p,q}$ the inequality*

$$\Big\| \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} s_{j,k} a_{j,k} \Big| R\dot{A}^s_{p,q}(\mathbb{R}^d)\Big\| \le c \| (s_{j,k})_{j,k} \mid \dot{a}_{p,q}\|$$

*holds.*

**Remark 33.** Radial subspaces of homogeneous Besov spaces have been characterized in a wavelet-style by Rauhut [29] and Rauhut and Rösler [30]. These methods could be used here as well.

**7.3. Decay properties of radial functions – non-limiting cases.** Next we want to deal with the problems already touched in Section 5. Here we shall investigate the elements of the spaces $\sigma(\dot{A}^s_{p,q}(\mathbb{R}^d))$, hence true functions. Below we shall use the reformulation of this condition given in Lemma 8. In the first part we deal with smoothness $s$ taken from the interval $(1/p, d/p)$. Recall, for fixed $s$ and $p$, the Besov space $R\dot{B}^s_{p,\infty}(\mathbb{R}^d)$ is the largest within the scale $R\dot{A}^s_{p,q}(\mathbb{R}^d)$, $0 < q \le \infty$. Applying essentially the same arguments as in proof of Theorem 10, see Subsection 5.2, but now with the atomic decomposition from the previous subsection, we have proved in [36] the following.

**Theorem 15.** *Let $d \ge 2$, $0 < p < \infty$, $s > \sigma_p(d)$ and in addition $1/p < s < d/p$.*
(i) *There exists a constant $c > 0$ s.t.*

$$|x|^{\frac{d}{p}-s}|g(x)| \le c\| \, [g] \mid \dot{B}^s_{p,\infty}(\mathbb{R}^d)\| \tag{33}$$

*holds for all $g \in \dot{B}^s_{p,\infty}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d)$, $t = d/\big(\frac{d}{p} - s\big)$, and all $x \neq 0$.*
*(ii) There exist a positive constant $c > 0$ and a function $g \in RL_{t,\infty}(\mathbb{R}^d)$, $t = d/\big(\frac{d}{p} - s\big)$, s.t. $[g] \in \dot{B}^s_{p,\infty}(\mathbb{R}^d)$ and*

$$|x|^{\frac{d}{p}-s}|g(x)| \ge c\| \, [g] \mid \dot{B}^s_{p,\infty}(\mathbb{R}^d)\|$$

*holds for all $x \neq 0$.*

For the spaces $\dot{B}^s_{p,q}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d)$, $q < \infty$ and $\dot{F}^s_{p,q}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d)$ there is only a weaker estimate in (ii).

**Lemma 12.** *Let $d, p, q$ and $s$ be as in Theorem 15. Then there exists a positive constant $c > 0$ s.t. for all $x \neq 0$ there exists a nontrivial function $g \in C_0^\infty(\mathbb{R}^d)$, $[g] \in R\dot{B}^s_{p,q}(\mathbb{R}^d)$, and*

$$|x|^{\frac{d}{p}-s}|g(x)| \ge c\| \, [g] \mid \dot{B}^s_{p,q}(\mathbb{R}^d)\|.$$

*If in addition $s > \sigma_{p,q}$, then the assertion remains true if we replace $B$ by $F$.*

**Remark 34.** There are more explicit functions which realize the extremal behaviour up to logarithmic terms. For example, let

$$g(x) := \psi(x)|x|^{s-\frac{d}{p}}(-\log|x|)^{-\delta} + (1+|x|^2)^{-(\frac{d}{p}-s)/2}(\log(\mathrm{e}+|x|^2))^{-\delta}, \ \ x \in \mathbb{R}^d.$$

Under the restrictions of Theorem 15 the class $[g]$ belongs to $\dot{B}^s_{p,\infty}(\mathbb{R}^d)$ if $\delta > 0$, see [32, Lemma 2.3.1] and Proposition 6.

There are some more results explaining the sharpness of (33). The first one is a consequence of the homogeneity of the quasi-norms.

**Lemma 13.** *Let $0 < p < \infty$, $0 < q \le \infty$, and $s \le d/p$. Let $\varrho\colon (0,\infty) \to (0,\infty)$ be continuous. Let us assume that there exists a constant c s.t.*

$$\varrho(|x|)|x|^{\frac{d}{p}-s}|\sigma([f])(x)| \le c\|\,[f]\mid \dot{A}^s_{p,q}(\mathbb{R}^d)\| \tag{34}$$

*holds for all $[f] \in R\dot{A}^s_{p,q}(\mathbb{R}^d)$ and all $x \ne 0$. Then $\varrho$ must be bounded.*

**Proof.** Let $\lambda > 0$. With a function $f$ also $f(\lambda\,\cdot)$ belongs to $R\dot{A}^s_{p,q}(\mathbb{R}^d)$ or more exactly, the corresponding classes. Furthermore, we have

$$\|\,[f(\lambda\,\cdot)]\mid \dot{A}^s_{p,q}(\mathbb{R}^d)\| \asymp \lambda^{s-d/p}\|\,[f]\mid \dot{A}^s_{p,q}(\mathbb{R}^d)\|, \tag{35}$$

see e.g. [28] or [45, Rem. 5.1.3/4]. Clearly $\sigma(f(\lambda\,\cdot)) = \sigma(f)(\lambda\,\cdot)$. We apply the inequality (34) with $f(\lambda\,\cdot)$ and obtain for the particular choice $\lambda = |x|^{-1}$

$$\varrho(|x|)|\sigma(f)(1,0,\ldots,0)| \le c\|\,[f]\mid R\dot{A}^s_{p,q}(\mathbb{R}^d)\|.$$

Choosing $f$ s.t. $\sigma(f)(1,0,\ldots,0) \ne 0$ we obtain the boundedness of $\varrho$. □

**Remark 25** *Comparison with the inhomogeneous situation***.** The behaviour near the origin is unchanged by switching from the inhomogeneous to the larger homogeneous spaces.The decay rate at infinity is different. It is worse in case of homogeneous spaces in comparison with the smaller inhomogeneous spaces.

**7.8. Decay of radial functions – limiting cases.** In this subsection we deal with the limiting situations, i.e., $s = \frac{1}{p}$ and $s = \frac{d}{p}$. To begin with we state positive results, first for Besov spaces, second for Lizorkin-Triebel spaces.

**Theorem 16.** *Let $d \ge 2$ and $0 < p < \infty$.*
(i) *Let in addition $p > 1 - \frac{1}{d}$. Then there exists a constant c s.t.*

$$|x|^{\frac{d-1}{p}}|g(x)| \le c\|\,[g]\mid \dot{B}^{1/p}_{p,1}(\mathbb{R}^d)\|$$

*holds for all $g \in \dot{B}^{1/p}_{p,1}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d)$, $t = dp/(d-1)$.*
(ii) *Let $s = d/p$. There exists a constant c s.t.*

$$|g(x)| \le c\|\,[g]\mid \dot{B}^{d/p}_{p,1}(\mathbb{R}^d)\|$$

*holds for all $g \in \dot{B}^{d/p}_{p,1}(\mathbb{R}^d) \cap RC_0(\mathbb{R}^d)$.*

**Remark 36.** Part (ii) follows from Lemma 7.

**Theorem 17.** *Let* $d \geq 2$ *and* $0 < p \leq 1$.
(i) *Let in addition* $p > 1 - \frac{1}{d}$. *There exists a constant* $c$ *s.t.*

$$|x|^{\frac{d-1}{p}} |g(x)| \leq c \| \, [g] \mid \dot{F}^{1/p}_{p,\infty}(\mathbb{R}^d) \|$$

*holds for all* $g \in \dot{F}^{1/p}_{p,\infty}(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d)$, $t = dp/(d-1)$.
(ii) *There exists a constant* $c$ *s.t.*

$$|g(x)| \leq c \| \, [g] \mid \dot{F}^{d/p}_{p,\infty}(\mathbb{R}^d) \|$$

*holds for all* $g \in \dot{F}^{d/p}_{p,\infty}(\mathbb{R}^d) \cap RC_0(\mathbb{R}^d)$.

**Remark 37.** (i) Part (ii) follows from Lemma 7.
(ii) Since the inhomogeneous spaces $RA^s_{p,q}(\mathbb{R}^d)$ are subspaces of $\sigma(R\dot{A}^s_{p,q}(\mathbb{R}^d))$, the negative results, obtained in Subsection 5.2, carry over.
(iii) We comment on the case $s = d/p$. If $1 < q \leq \infty$, then the inhomogeneous space $RB^{d/p}_{p,q}(\mathbb{R}^d)$ contains functions which are unbounded in a neighbourhood of the origin, see [5]. Hence, if $g$ is such a function, the class $[g]$ contains elements which are all unbounded in a neighbourhood of the origin. Similar arguments apply to the cases $RF^{d/p}_{p,q}(\mathbb{R}^d)$, $1 < p < \infty$, $0 < q \leq \infty$.

For convenience of the reader we formulate the consequences for potential and Sobolev spaces.

**Corollary 6.** *Let* $1 < p < \infty$. *Then the following assertions are equivalent.*

- *There exists a constant* $c$ *s.t.*

$$|x|^{\frac{d}{p}-s} |g(x)| \leq c \| \, [g] \mid \dot{H}^s_p(\mathbb{R}^d) \| \tag{36}$$

  *holds for all* $g \in \dot{H}^s_p(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d)$, $t = d/\big(\frac{d}{p} - s\big)$, *and all* $x \neq 0$.
- *We have* $1/p < s < d/p$.

**Remark 38.** (i) The equivalence follows from the identity

$$\dot{H}^s_p(\mathbb{R}^d) = \dot{F}^s_{p,2}(\mathbb{R}^d),$$

see Remark 26(iii), Theorem 15 and Corollary 6.

(ii) For $p = 2$ the inequality (36) has been proved in a simpler way by Cho and Ozawa [9]. They used tools from Fourier analysis. By explicit counterexamples they disproved (36) in case $\sigma(R\dot{H}^{d/2}_2(\mathbb{R}^d))$.

**Corollary 7.** *Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then the following assertions are equivalent.*

- *There exists a constant $c$ s.t.*
$$|x|^{\frac{d}{p}-m}|g(x)| \le c\| \, [g]_m \mid \dot{W}^m_p(\mathbb{R}^d)\| \tag{37}$$
*holds for all $g \in \dot{W}^m_p(\mathbb{R}^d) \cap RL_t(\mathbb{R}^d)$, $t = d/\big(\frac{d}{p} - s\big)$, and all $x \neq 0$.*
- *We have $1 \le m < d/p$.*

**Remark 39.** As proved in Subsection 5.1 the inequality (37) remains true in case $m = p = 1$.

## 8. Compact embeddings on $\mathbb{R}^d$. Part II

We continue our investigations from Section 6. The naive extension of Corollary 3(i), i.e., replacing $RA^{s_0}_{p_0,q_0}(\mathbb{R}^d)$ by $\sigma(R\dot{A}^{s_0}_{p_0,q_0}(\mathbb{R}^d)$, does not lead to compact embeddings.

**Lemma 14.** *Let $d \ge 2$, $1 < p_1 < \infty$ and $A \in \{B, F\}$. Then $\sigma(R\dot{A}^{s_0}_{p_0,q_0}(\mathbb{R}^d))$ is never compactly embedded into $L_{p_1}(\mathbb{R}^d)$.*

**Remark 40.** The main idea of the proof consists in the following observation. On the one hand,
$$s_0 - \frac{d}{p_0} = 0 - \frac{d}{p_1} \tag{38}$$
is a necessary condition for the embedding $\sigma(R\dot{A}^{s_0}_{p_0,q_0}(\mathbb{R}^d)) \hookrightarrow L_{p_1}(\mathbb{R}^d)$, see (35). On the other hand, we know $RA^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow \sigma(R\dot{A}^{s_0}_{p_0,q_0}(\mathbb{R}^d))$ and hence, the compactness of $RA^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow L_{p_1}(\mathbb{R}^d)$ is another necessary condition. Corollary 3 tells us, that this is in contradiction with (38).

In what follows we shall deal with two less obvious situations where we still have compact embeddings on unbounded domains.

**8.1. Compactness of embeddings into sums of Lebesgue spaces.** Let $X$ and $Y$ be Banach spaces. For simplicity we assume that $X, Y \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. By $X + Y$ we denote the space of all tempered distributions $f$, which can be represented as a sum $f = f_1 + f_2$, where $f_1 \in X$ and $f_2 \in Y$. $X + Y$ becomes a Banach space if equipped with the norm
$$\| \, f \mid X + Y\| := \inf\big\{\| \, f_1 \mid X\| + \| \, f_2 \mid Y\| : f = f_1 + f_2\big\}.$$

Before we turn to compactness of embeddings we shall investigate continuity of embeddings.

**Lemma 15.** *Let $d \ge 2$, $0 < p < \infty$, $1 \le p_1 \le p_2 \le \infty$ and*

$$\sigma_p(d) < s < \frac{d}{p}.$$

(i) *Then $\sigma(R\dot{F}^s_{p,q}(\mathbb{R}^d))$ is continuously embedded into $L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)$ if*

$$p \le p_1 \le \frac{d}{\frac{d}{p} - s} \le p_2 \le \infty.$$

(ii) *Then $\sigma(R\dot{B}^s_{p,q}(\mathbb{R}^d))$ is continuously embedded into $L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)$ if*

$$p \le p_1 < \frac{d}{\frac{d}{p} - s} < p_2 \le \infty.$$

*If in addition $q \le \frac{d}{\frac{d}{p}-s}$, then $p_1 = \frac{d}{\frac{d}{p}-s}$ and $p_2 = \frac{d}{\frac{d}{p}-s}$ become admissible.*

**Remark 41.** (i) For the existence of the embedding

$$\sigma(R\dot{A}^s_{p,q}(\mathbb{R}^d)) \hookrightarrow L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)$$

under the condition (39) the relation

$$p_1 \le \frac{d}{\frac{d}{p} - s} \le p_2$$

is necessary. This can be seen as follows. First, let us assume $p_2 < d/\big(\frac{d}{p} - s\big)$. Consider our test function $g_{\alpha,\delta}$ with $\alpha = d/p - s$ and $\delta > 1/q$. Then $g_{\alpha,\delta} \in \sigma(\dot{B}^s_{p,q}(\mathbb{R}^d))$, see Proposition 6. Obviously $g_{\alpha,\delta} \notin L_{p_2}(\mathbb{R}^d)$. Because of $|g_{\alpha,\delta}(x)| \le 1$ for all $x$ this implies $g_{\alpha,\delta} \notin (L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d))$. Second, we assume $p_1 > d/\big(\frac{d}{p} - s\big)$. This time we consider the test function $f_{\alpha,\delta}$ given by

$$f_{\alpha,\delta}(x) := \psi(x)|x|^{-\alpha}(\log(\mathrm{e} + |x|^2))^{-\delta}$$

with $\alpha = \frac{d}{p} - s$ and $\delta > 1/q$. Then

$$f_{\alpha,\delta} \in B^s_{p,q}(\mathbb{R}^d) \hookrightarrow \sigma(\dot{B}^s_{p,q}(\mathbb{R}^d)),$$

see [32, Lemma 2.3.1/1]. Obviously $f_{\alpha,\delta} \notin L_{p_1}(\mathbb{R}^d)$. Because $|f_{\alpha,\delta}(x)| \ge 1$ for $|x| \le 1$, this implies $f_{\alpha,\delta} \notin (L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d))$. This proves the claim for $A = B$. In case $A = F$ we apply the elementary embedding $\sigma(R\dot{B}^s_{p,\min(p,q)}(\mathbb{R}^d)) \hookrightarrow \sigma(R\dot{F}^s_{p,q}(\mathbb{R}^d))$ and argue as above.

(ii) The proof uses the decomposition in (27) and is given in [36].

As in case of inhomogeneous radial subspaces, see Corollary 3, in comparison with the conditions for continuous embeddings, we have to exclude some limiting cases and the remaining are compact embeddings, see [36].

**Theorem 18.** *Let $d \geq 2$, $0 < p < \infty$, $1 \leq p_1 \leq p_2 \leq \infty$ and $s$ as in* (39)*. Then $\sigma(R\dot{A}^s_{p,q}(\mathbb{R}^d))$ is compactly embedded into $L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)$ if*

$$p < p_1 < \frac{d}{\frac{d}{p} - s} < p_2 \leq \infty.$$

**8.2. Compactness of embeddings – exterior domains.** We consider spaces defined on the complement of a ball with center in the origin. For simplicity we choose $\Omega := \mathbb{R}^d \setminus \{x : |x| < 1\}$. Let $[f] \in R\dot{A}^s_{p,q}(\mathbb{R}^d)$. Under the restrictions of Lemma 7, $\sigma(f)$ is a radial function. By $\tau(f)$ we denote the restriction of this function to $\Omega$. We define

$$R\dot{A}^s_{p,q}(\Omega) := \big\{\tau(f) : [f] \in R\dot{A}^s_{p,q}(\mathbb{R}^d)\big\},$$
$$\|\,\tau(f) \mid R\dot{A}^s_{p,q}(\Omega)\| := \inf\big\{\|\,[g] \mid R\dot{A}^s_{p,q}(\mathbb{R}^d)\| : \tau(f) = \tau(g)\big\}.$$

These restrictions can be understood in the pointwise sense, since the elements in $R\dot{A}^s_{p,q}(\mathbb{R}^d)$ are continuous outside the origin, see Corollary 4. Of course, the restrictions of radial functions in $A^s_{p,q}(\mathbb{R}^d)$ to $\Omega$ belong to $R\dot{A}^s_{p,q}(\Omega)$, but this embedding is proper, see Proposition 6. By using a similar notation for the inhomogeneous spaces a direct consequence of Corollary 3 is the following: the embedding $RA^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow L_{p_1}(\Omega)$ is compact if $p_0 < p_1$ and $s_0 > d\big(\frac{1}{p_0} - \frac{1}{p_1}\big)$. For those exterior domains this can be partly improved. Let $C(\mathbb{R}^d)$ be the space of all uniformly continuous functions equipped with the supremum norm.

**Theorem 19.** *Let $d \geq 2$, $0 < p < \infty$,*

$$\max\Big(\sigma_p(d), \frac{1}{p}\Big) < s < \frac{d}{p} \quad \text{and} \quad \frac{d}{\frac{d}{p} - s} < p_1 < \infty. \tag{40}$$

*Then $R\dot{B}^s_{p,\infty}(\Omega)$ is compactly embedded into $L_{p_1}(\Omega) \cap RC(\Omega)$.*

**Remark 42.** (i) The proof relies on Theorem 15 and the Arzela-Ascoli Theorem, see [36].
(ii) Let $A \in \{B, F\}$. From the elementary embeddings for Besov-Lizorkin-Triebel spaces it follows that under the conditions of Theorem 19 the embedding $R\dot{A}^s_{p,q}(\Omega) \hookrightarrow L_{p_1}(\Omega) \cap RC(\Omega)$ is compact.

## 9. Some final comments

There are many open questions in this field. We would like to mention a few of them.

- A more general type of symmetry leading to compactness of embeddings, so-called cylindrical symmetry (sometimes also called block-radial symmetry) has been investigated in P. L. Lions [26], Ding Yi [12], Hebey [19], Kuzin and Pohozaev [25, 17.2] in the framework of Sobolev spaces and by Skrzypczak [40] in the general case of Besov-Lizorkin-Triebel spaces. Which parts of the theory presented here extend to cylindrical symmetry?
- Cho and Ozawa [9] have introduced function spaces which are sufficiently close to $RH^s(\mathbb{R}^d)$ s.t. the decay properties near infinity remain unchanged. They use spherical coordinates $r, \theta_1, \ldots, \theta_{d-1}$ and a notion of regularity with respect the angles $\theta_1, \ldots, \theta_{d-1}$ based on the use of the Laplace-Beltrami operator. Does this method extend to all $p$?
- Find function spaces which are larger than $B^{1/p}_{p,1}(\mathbb{R}^d)$ s.t. the elements of its radial subspace have the same decay properties then those of $B^{1/p}_{p,1}(\mathbb{R}^d)$. An obvious example is given by

$$M_p(\mathbb{R}^d) := \big(RBV(\mathbb{R}^d), RC(\mathbb{R}^d)\big)_{\Theta,\infty}, \quad \Theta = 1 - \frac{1}{p},\ 1 < p < \infty,$$

  where $(\,\cdot\,,\,\cdot\,)_{\Theta,q}$ denotes the real interpolation method and $BV(\mathbb{R}^d)$ denotes the collection of all functions of bounded variation. However, a simple description of these spaces $M_p(\mathbb{R}^d)$ seems to be not known.
- Describe the radial subspaces $RA^s_{p,q}(\mathbb{R}^d)$ by differences. The classical notion of smoothness is connected with derivatives and differences. So this problem has its roots in the origin of Besov spaces. A first attempt has been made in [SSV3] but restricted to values of $s$ less than 1 (first order differences). An extension seems to be less obvious.

## References

[1] C. Bennett and R. Sharpley: *Interpolation of operators.* New York: Academic Press, 2nd edition, 2003. Zbl 0647.46057, MR0928802.

[2] B. Berestycki and P. L. Lions: *Existence of a ground state in nonlinear equations of the type Klein-Gordon.* In: Variational inequalities and complementarity problems (Proc. Internat. School, Erice, 1978), pp. 35-51, Chichester: Wiley, 1980. Zbl 0707.35143, MR0578738.

[3] J. Bergh and J. Löfström: *Interpolation spaces. An Introduction.* Grundlehren der Mathematischen Wissenschaften, **223**. Berlin: Springer, 1976. Zbl 0344.46071, MR0482275.

[4] G. Bourdaud: *Réalisations des espaces de Besov homogènes.* Ark. Mat. **26** (1988), no. 1, 41–54. MR0948279.

[5] G. Bourdaud: *A sharpness result for powers of Besov functions.* J. Funct. Spaces Appl. **2** (2004), no. 3, 267–277. Zbl 1071.46025, MR2094590.

[6] M. Bownik and Kwok-Pun Ho: *Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces.* Trans. Amer. Math. Soc. **358** (2006), no. 4, 1469–1510. Zbl 1083.42016, MR2186983.

[7] H.-Q. Bui: *Weighted Besov and Triebel spaces: Interpolation by the real method.* Hiroshima Math. J. **12** (1982), no. 3, 581–605. Zbl 0525.46023, MR0676560.

[8] H.-Q. Bui: *Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures.* J. Funct. Anal. **55** (1984), no. 1, 39–62. Zbl 0638.46029, MR0733032.

[9] Y. Cho and T. Ozawa: *Sobolev inequalities with symmetry.* Commun. Contemp. Math. **11** (2009), no. 3, 355–365. Zbl 1184.46035, MR2538202.

[10] S. Coleman, V. Glazer and A. Martin: *Action minima among solutions to a class of Euclidean scalar field equations.* Comm. Math. Phys. **58** (1978), no. 2, 211–221. MR0468913.

[11] M. Cwickel and K. Tintarev: *On interpolation of cocompact imbeddings.* arXiv: 1008.4371v2 (2010), 20 pp.

[12] Ding Yi: *Non-radial solutions of semilinear elliptic equation.* Math. Acta Sci. **10** (1990), no. 2, 229–239. Zbl 0713.35034,MR1070826.

[13] Y. Ebihara, T. P. Schonbek: *On the (non)compactness of the radial Sobolev spaces.* Hiroshima Math. J. **16** (1986), no. 3, 665–669. Zbl 0617.46040, MR0867590.

[14] J. Epperson and M. Frazier: *An almost orthogonal radial wavelet expansion for radial distributions.* J. Fourier Anal. Appl. **1** (1995), no. 3, 311–353. Zbl 0887.42031, MR1353543.

[15] W. Farkas, J. Johnsen and W. Sickel: *Traces of anisotropic Besov–Lizorkin–Triebel spaces – a complete treatment of the borderline cases.* Math. Bohemica **125** (2000), no. 1, 1–37. Zbl 0970.46019, MR1752077.

[16] M. Frazier and B. Jawerth: *Decomposition of Besov spaces.* Indiana Univ. Math. J. **34** (1985), no. 4, 777–799. Zbl 0551.46018, MR0808825.

[17] M. Frazier and B. Jawerth: *A discrete transform and decomposition of distribution spaces.* J. Funct. Anal. **93** (1990), no. 1, 34–170. Zbl 0716.46031, MR1070037.

[18] M. Frazier, B. Jawerth and G. Weiss: *Littlewood-Paley theory and the study of function spaces.* CBMS Regional Conference Series in Mathematics, **79**. Providence, RI: Amer. Math. Soc., 1991. Zbl 0757.42006, MR1107300.

[19] E. Hebey: *Sobolev spaces on Riemannian Manifolds.* Lecture Notes in Mathematics, **1635** Berlin: Springer, 1996. Zbl 0866.58068, MR1481970.

[20] D. Haroske and I. Piotrowska: *Atomic decompositions of function spaces with Muckenhoupt weights, and some relations to fractal geometry.* Math. Nachr. **281** (2008), no. 10, 1476–1494. Zbl 1160.46023, MR2454945.

[21] M. Izuki and Y. Sawano: *Wavelet bases in the weighted Besov and Triebel-Lizorkin spaces with $A_p^{\mathrm{loc}}$-weights.* J. Approx. Theory **161** (2009), no. 2, 656–673. Zbl 1183.42037, MR2563074.

[22] B. Jawerth: *Some observations on Besov and Lizorkin-Triebel spaces.* Math. Scand. **40** (1977), no. 1, 94–104. Zbl 0358.46023, MR0454618.

[23] B. Jawerth: *The trace of Sobolev and Besov spaces if* $0 < p < 1$. Studia Math. **62** (1978), no. 1, 65–71. Zbl 0423.46022, MR0482141.

[24] T. Kühn, H.-G. Leopold, W. Sickel, and L. Skrzypczak: *Entropy numbers of Sobolev embeddings of radial Besov spaces.* J. Approx. Theory **121** (2003), mo. 2, 244–268. Zbl 1047.47016, MR1971771.

[25] I. Kuzin, S. Pohozaev: *Entire solutions of semilinear elliptic equations.* Progress in Nonlinear Differential Equations and their Applications, **33**. Basel: Birkhäuser, 1997. Zbl 0882.35003, MR1479168.

[26] P. L. Lions: *Symétrie et compacité dans les espaces de Sobolev.* J. Funct. Anal. **49** (1982), no. 3, 315–334. Zbl 0501.46032, MR0683027.

[27] W. M. Ni: *A nonlinear Dirichlet problem on the unit ball and its applications.* Indiana Univ. Math. J. **31** (1982), no. 6, 801–807. Zbl 0515.35033, MR0674869.

[28] J. Peetre: *New thoughts on Besov spaces.* Duke University Mathematics Series, **1**. Durham: Duke Univ. Press, 1976. Zbl 0356.46038, MR0461123.

[29] H. Rauhut: *Time-frequency and wavelet analysis of functions with symmetry properties.* Berlin: Logos-Verlag, 2005. Zbl 1094.42030.

[30] H. Rauhut and M. Rösler: *Radial multiresolution in dimension three.* Constr. Approx. **22** (2005), no. 2, 193-218. Zbl 1097.42027, MR2148531.

[31] W. Rother: *Some existence results for the equation* $\Delta U + K(x)U^p = 0$. Commun. Partial Differ. Equations **15** (1990), no. 10, 1461–1473. Zbl 0729.35045, MR1077474.

[32] T. Runst and W. Sickel: *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations.* Berlin: de Gruyter, 1996. Zbl 0873.35001, MR1419319.

[33] V.S. Rychkov: *Littlewood-Paley theory and function spaces with $\mathcal{A}_p^{\mathrm{loc}}$ weights.* Math. Nachr. **224** (2001), 145–180. Zbl 0984.42011, MR1821243.

[34] H.-J. Schmeisser and H. Triebel: *Topics in Fourier analysis and function spaces.* A Wiley-Interscience Publication. Chichester: Wiley, 1987. Zbl 0661.46025, MR0891189.

[35] W. Sickel and L. Skrzypczak: *Radial subspaces of Besov and Lizorkin-Triebel spaces: extended Strauss lemma and compactness of embeddings.* J. Fourier Anal. Appl. **106** (2000), no. 6, 639–662. Zbl 0990.46020, MR1790248.

[36] W. Sickel and L. Skrzypczak: *On the interplay of regularity and decay in case of radial functions. II. Homogeneous spaces.* Submitted.

[37] W. Sickel, L. Skrzypczak and J. Vybiral: *On the interplay of regularity and decay in case of radial functions. I. Inhomogeneous spaces.* Commun. Contemp. Math. (to appear).

[38] W. Sickel, L. Skrzypczak and J. Vybiral: *Radial subspaces of Besov- and Lizorkin-Triebel spaces – complex interpolation and characterization by differences.* (Work in progress), Jena, Poznań, Linz 2010.

[39] W. Sickel and H. Triebel: *Hölder inequalities and sharp embeddings in function spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type.* Z. Anal. Anwend. **14** (1995), no. 1, 105–140. Zbl 0820.46030, MR1327495.

[40] L. Skrzypczak: *Rotation invariant subspaces of Besov and Triebel-Lizorkin spaces: compactness of embeddings, smoothness and decay properties.* Rev. Mat. Iberoamericana **18** (2002), no. 2, 267–299. Zbl 1036.46028, MR1949829.

[41] E. M. Stein: *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, **30**. Princeton, N.J.: Princeton Univ. Press, 1970. Zbl 0207.13501, MR0290095.

[42] E. M. Stein: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy.* Princeton Mathematical Series, **43**. Princeton, N.J.: Princeton Univ. Press, 1993. Zbl 0821.42001, MR1232192.

[43] W. A. Strauss: *Existence of solitary waves in higher dimensions.* Commun. Math. Phys. **55** (1977), no. 2, 149–162. Zbl 0356.35028,MR0454365.

[44] H. Triebel: *Interpolation theory, function spaces, differential operators.* North-Holland Mathematical Library, **18**. Amsterdam: North-Holland, 1978. Zbl 0387.46032, MR0503903.

[45] H. Triebel: *Theory of function spaces.* Monographs in Mathematics, **78**. Basel: Birkhäuser, 1983. Zbl 0546.46027, MR0781540.

[46] H. Triebel: *Theory of function spaces. II.* Monographs in Mathematics, **84**. Basel: Birkhäuser, 1992. Zbl 0763.46025, MR1163193.

[47] H. Triebel: *Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces.* Proc. London Math. Soc. **66** (1993), no. 3, 589–618. Zbl 0792.46024, MR1207550.

[48] H. Triebel: *Theory of function spaces. III.* Monographs in Mathematics, **100**. Basel: Birkhäuser, 2006. Zbl 1104.46001, MR2250142.

[49] J. Vybíral: *On sharp embeddings of Besov and Triebel-Lizorkin spaces in the subcritical case.* Proc. Amer. Math. Soc. **138** (2010), no. 1, 141–146. Zbl 1204.46023, MR2550178.