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Some Remarks on the Metrical Transitivity of Invariant and Quasi-Invariant Measures

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We consider the behavior of metrical transitivity of invariant and quasi-invariant measures under the product-operation and for the operation of inductive limit of a countable family of σ-finite invariant measures.

In the present paper, an approach to some questions of measure theory is discussed, which is rather useful in certain situations where a measurable space (E, S) is given equipped with a group G of transformations of E and a nonzero σ-finite G-quasi-invariant (G-invariant) measure μ defined on S. In many cases, μ turns out to be metrically transitive with respect to G (see the definition below).

It is well known that the metrical transitivity (or, equivalently, ergodicity) for invariant and quasi-invariant measures plays a significant role in various questions of modern analysis and probability theory. The metrical transitivity of invariant and quasi-invariant measures is an important property for these measures. Moreover, the metrical transitivity of measures is frequently crucial in the process of investigation of many interesting topics of the theory of dynamical systems. For instance, this property is closely connected with the uniqueness property of invariant measures, with nonseparable extensions of invariant and quasi-invariant measures, etc. In this connection, see e.g. [1], [2], [3], [4], [5], [6], [9].

For our further purposes, it is also convenient to introduce the following notation:

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N = the set of all natural numbers.
Q = the set of all rational numbers.
R = the set of all real numbers.
μ' = the completion of a given measure μ.
$dom(\mu)$ = the σ-algebra on which a given measure μ is defined.

Let E be a nonempty set and let G be a group of transformations of E. We say that a G-invariant (G-quasi-invariant) measure μ on E is metrically transitive with respect to G if, for each μ-measurable set X with $\mu(X) > 0$, there exists a countable family $(g_n)_{n\in N} \subset G$, satisfying the equality

$$\mu\Big(E \setminus \bigcup_{n\in N} g_n(X)\Big) = 0 .$$

The following example shows that some σ-algebras S admit nonzero σ-finite quasi-invariant measures, but do not admit nonzero σ-finite invariant measures.

Example 1. Let $E = T$ be the unit circle on the Euclidean plane and let G be a group of transformations of T, such that:

a) each element from G is a diffeomorphism of T;

b) G contains an uncountable subgroup consisting of isometric transformations of T;

c) there exists at least one element from G which is not an isometric transformation of T.

Let S be the σ-algebra of all Lebesgue measurable subsets of T. Then:

(1) the classical Lebesgue measure on S is quasi-invariant with respect to G;

(2) there does not exist a non-zero σ-finite G-invariant measure defined on S.

Note that in this example the metrical transitivity of the Lebesgue measure with respect to G plays an essential role.

It is natural to investigate the question of metrical transitivity for products of σ-finite invariant and quasi-invariant measures, assuming that all them are metrically transitive. In this connection, notice that:

1. If $(E_i, G_i, dom(\mu_i), \mu_i)$ $(1 \le i \le n)$ is a finite family of measurable spaces equipped with σ-finite invariant measures, then the product space

$$\Big(\prod_{1\le i\le n} E_i,\ \prod_{1\le i\le n} G_i,\ \prod_{1\le i\le n} dom(\mu_i),\ \prod_{1\le i\le n} \mu_i\Big)$$

is equipped with a σ-finite $(\prod_{1\le i\le n} G_i)$-invariant measure;

2. If $(E_i, G_i, dom(\mu_i), \mu_i)(i \in I)$ is a family of measurable spaces equipped with probability quasi-invariant measures, then the product space

$$\Big(\prod_{i\in I} E_i,\ \sum_{i\in I} G_i,\ \prod_{i\in I} dom(\mu_i),\ \prod_{i\in I} \mu_i\Big)$$

is equipped with a probability $(\sum_{i\in I} G_i)$-quasi-invariant measure, where $\sum_{i\in I} G_i)$ denotes the direct sum of the groups G_i $(i \in I)$.

Sometimes, a more general concept of metrical transitivity is needed. Namely, let (E, G, S, μ) be again a space equipped with a σ-finite G-invariant (G-quasi-invariant) measure and let H be a subfamily of G. We say that μ is metrically transitive with respect to H if, for each μ-measurable set X with $\mu(X) > 0$, there exists a countable family $(h_n)_{n\in N} \subset H$, satisfying the equality

$$\mu\left(E \setminus \bigcup_{n\in N} h_n(X)\right) = 0 .$$

Example 2. Let $(E, G, dom(\mu), \mu)$ be a measurable space equipped with a σ-finite G-quasi-invariant measure and let H be a subfamily of G. Then the following two assertions are equivalent:

1. μ is metrically transitive with respect to H;
2. if Z is an arbitrary μ-measurable subset of E such that $\mu(h(Z) \triangle Z) = 0$ for all transformations $h \in H$, then we have $\mu(Z) = 0$ or $\mu(E \setminus Z) = 0$.

Let us consider one more example which yields a purely topological characterization of the metrical transitivity of the classical Lebesgue measure and plays an essential role in studyings various properties of this measure.

Example 3. Let E be a finite-dimensional Euclidean space, let H be a subgroup of the group of all isometric transformations of E, and λ denote the Lebesgue measure on E. Then the following two assertions are equivalent.

a) λ is metrically transitive with respect to H;

b) for each point $x \in E$, the orbit $H(x)$ is uncountable and dense everywhere in E.

For the proof, see, e.g., [2] or [6].

Now, we can formulate the following statement.

Theorem 1. *Let $(E_i, G_i, \mu_i)(1 \leq i \leq n)$ be a finite family of measurable spaces equipped with σ-finite invariant (respectively, quasi-invariant) measures. If each measure μ_i $(1 \leq i \leq n)$ is metrically transitive with respect to a countable subgroup $H_i \subset G_i(1 \leq i \leq n)$, then the product measure $\prod_{1\leq i\leq n} \mu_i$ is metrically transitive with respect to the group $\prod_{1\leq i\leq n} H_i$. In particular, the measure $\prod_{1\leq i\leq n} \mu_i$ is metrically transitive with respect to $\prod_{1\leq i\leq n} G_i$.*

The proof of Theorem 1 can be found in [7]. Actually, the argument used in the proof yields a more general result. Namely, if $(E_i, G_i, \mu_i)(1 \leq i \leq n)$ is a finite family of σ-finite invariant (respectively, quasi-invariant) measures such that every measure μ_i $(1 \leq i \leq n-1)$ is metrically transitive with respect to some countable group $H_i \subset G_i(1 \leq i \leq n-1)$ and μ_n is metrically transitive with respect to G_n, then the product measure $\prod_{1 \le i \le n} \mu_i$ is metrically transitive with respect to $\prod_{1 \le i \le n} G_i$.

As already mentioned, the notion of metrical transitivity of an invariant measure is closely connected with the uniqueness property of the same measure.

Let again E be a nonempty set, G be a group of transformations of E and let M be a class of σ-finite G-invariant measures on E (let us remark at once that the domains of measures from M can differ from each other). We say that a measure $\mu \in M$ has the uniqueness property with respect to M if, for every measure $\nu \in M$ such that $dom(\mu) = dom(\nu)$, there exists a coefficient $t \in R^+$ for which the equality $\nu = t \cdot \mu$ is fulfilled.

Evidently, if $M = \{\nu\colon \nu$ is a σ-finite G-invariant measure defined on $dom(\mu)\}$, then the previous definition reduces to the usual definition of the uniqueness property of invariant measures (see, e.g., [5], [6], [7]).

Now, let us formulate an auxiliary statement, which we need in our further considerations.

Lemma 1. *Let (E, G) be a space with a transformation group and let μ be a nonzero σ-finite G-quasi-invariant measure on E. Suppose that G contains an uncountable subgroup H acting freely in E. Then there exists a subset of E nonmeasurable with respect to μ.*

Lemma 2. *Let E be a nonempty set, G be a group of transformations of E containing an uncountable subgroup H acting freely in E, and let μ be a complete σ-finite G-invariant measure on E. Then the following two assertions are equivalent:*

1. μ has the uniqueness property;

2. μ is metrically transitive.

The proofs of Lemmas 1 and 2 can be found in [5].

From Lemma 2 and from [7] we deduce the next statement.

Theorem 2. *Let $(E_i, G_i, \mu_i)(i \in I)$ be an arbitrary family of measurable spaces equipped with complete invariant probability measures. If each measure $\mu_i(i \in I)$ is metrically transitive with respect to the group $G_i(i \in I)$, then the product measure $\prod_{i \in I} \mu_i$ is metrically transitive with respect to the product group $\prod_{i \in I} G_i$.*

In the space R^N there exists a σ-finite measure χ satisfying the following relations:

(1) χ is invariant with respect to some vector subspace of R^N which is everywhere dense in R^N;

(2) χ is metrically transitive with respect to the direct sum $\sum_{i \in N} Q_i$, where $Q_i = Q$ for all $i \in N$.

Note that $\sum_{i \in N} Q_i$ is countable, which enables us to apply Theorem 1. More detailed information about χ is presented in [5].

In an analogous manner we can construct a σ-finite invariant measure for the countable product of spaces equipped with nonzero σ-finite invariant measures (see, e.g. [10]). Let $(E_i, G_i, \mu_i)(i \in N)$ be a countable family of measurable spaces with σ-finite invariant measures. Let us assume that there exist sets $\triangle_i \subset E_i(i \in N)$, such that

$$\mu_i(\triangle_i) = 1$$

We put

$$A_n = E_1 \times E_2 \times \ldots \times E_n \times \Big(\prod_{i>n} \triangle_i\Big).$$

For an arbitrary $n \in N$ let χ_n denote the measure defined by the formula

$$\chi_n = \Big(\prod_{1 \le i \le n} \mu_i\Big) \times \Big(\prod_{i>n} \mu_i\Big),$$

and let $\overline{\chi}_n$ be the measure on the space $\prod_{i \in N} E_i$ defined by

$$(\forall X)\Big(X \in \prod_{i \in N} dom(\mu_i) \Rightarrow \overline{\chi}_n(X) = \chi_n(X \cap A_n)\Big).$$

For an arbitrary set $X \in \prod_{i \in N} dom(\mu_i)$, there exists a limit

$$\chi(X) = \lim_{n \to \infty} \overline{\chi}(X).$$

Moreover, the functional χ is a σ-finite $(\sum_{i \in N} G_i)$-invariant measure on the σ-algebra $\prod_{i \in N} dom(\mu_i)$ in the space $\prod_{i \in N} E_i$. For this measure we have

$$\chi\Big(\Big(\prod_{i \in N} E_i\Big) \setminus \Big(\bigcup_{i \in N} A_i\Big)\Big) = 0.$$

In other words, the set $\bigcup_{i \in N} A_i$ is a support of χ. In fact, χ can be considered as an inductive limit of the given family of measures $(\mu_i)_{i \in N}$.

The following notion is (in a certain sense) a local version of the uniqueness property. This notion was first introduced and discussed in [8].

Let E be a nonempty set, G be a group of transformations of E, S be a G-invariant σ-algebra of subsets of E and let M be a class of G-invariant measures defined on S. We say that a measure $\mu \in M$ has the uniqueness property on a set $Z \in S$, if for each measure $\nu \in M$, the relation

$$(\forall X)(X \in S \Rightarrow \nu(X \cap Z) = \mu(X \cap Z))$$

is satisfied.

Lemma 3. *If every measure* μ_i $(i \in N)$ *is metrically transitive with respect to a countable subgroup* $H_i \subset G_i (i \in N)$, *then the measure* χ' *has the uniqueness property on the set* $\bigcup_{n\in N} A_n$ *with respect to the class* M *of all* σ*-finite* $(\sum_{i\in N} G_i)$*-invariant measures* μ *such that* $\mu(\prod_{i\in N} \triangle_i) = 1$.

For the proof see, e.g., [7]. Using this lemma, we can prove the next statement.

Theorem 3. *If every measure* $\mu_i (i \in N)$ *is metrically transitive with respect to a countable subgroup* $H_i \subset G_i (i \in N)$, *then the measure* χ' *is metrically transitive with respect to* $(\sum_{i\in N} G_i)$.

The proof of Theorem 2 essentially relies on the fact that the property of invariance of a measure can be transferred from an algebra to the generated σ-algebra. In general, this fails to be true for the property of quasi-invariance of a measure. It is worth noting that the metrical transitivity of measures also cannot be transferred from an algebra to the generated σ-algebra. Indeed, let us consider the Cantor discontinuum

$$2^Z = \prod_{k\in Z} \{0;1\}_k,$$

where Z is the set of all integers. Furthermore, for $x \in [0;1]$ let us denote by $\mu_k^{(x)}$ a measure on $\{0;1\}_k$ such that

$$\mu_k^{(x)}(\{0\}) = x, \mu_k^{(x)}(\{1\}) = 1 - x$$

and let

$$\mu^{(x)} = \prod_{k\in Z} \mu_k^{(x)}.$$

Let $P(Z)$ be the group of all permutations of Z and let $\bar{g}(a) = (a_{g(k)})_{k\in Z}$, where

$$a = (a_k)_{k\in Z} \in 2^Z, \quad g \in P(Z), \quad g(k) = k + 1.$$

If G_Z is the group generated by all transformation $\bar{g}$, then in view of the Birkhoff ergodic theorem it turns out that the measure $\mu^{(x)}$ is metrically transitive with respect to the group G_Z. Take now $y = 1 - x$ and define a G_Z-invariant measure μ by the formula

$$\mu = x\mu^{(x)} + y\mu^{(y)}.$$

It is possible to prove that the measure μ is not metrically transitive (for details, see [6]), but its restriction to the algebra of elementary sets of 2^Z is metrically transitive.

Notice that a certain analog of Lemma 2 for quasi-invariant measures is valid if the uniqueness property is replaced by the equivalence of these measures. Using Lemma 1, we can establish the following statement concerning the behaviour of quasi-invariant measures under the operation of their product.

Theorem 4. *Let* $(E_i, G_i, \mu_i)(i \in I)$ *be a family of probability* G_i*-quasi-invariant measures. If the group* $(\sum_{i\in I} G_i)$ *contains an uncountable subgroup acting freely in the space* $E = \prod_{i\in I} E_i$, *and* ν *is another* σ*-finite* $(\sum_{i\in I} G_i)$*-quasi-invariant measure on* E *such that* $dom(\nu) = dom(\prod_{i\in I}\mu_i)'$, *then* ν *is absolutely continuous with respect to* $(\prod_{i\in I}\mu_i)'$, *i.e. for each set* $X \in dom(\prod_{i\in I}\mu_i)'$, *we have the implication*

$$\left(\prod_{i\in I}\mu_i\right)'(X) = 0 \Rightarrow \nu(X) = 0.$$

Since ν is absolutely continuous with respect to $(\prod_{i\in I}\mu_i')$, we can apply the classical Radon-Nikodym theorem to these two measures. Hence there exists a $(\prod_{i\in I}\mu_i)'$-measurable function

$$\psi : \prod_{i\in I} E_i \to R^+$$

such that the equality

$$\nu(X) = \int_X \psi d\left(\prod_{i\in I}\mu_i\right)'$$

is fulfilled for every $(\prod_{i\in I}\mu_i)'$-measurable set X. Clearly, if the function ψ is strictly positive, then the measures ν and $(\prod_{i\in I}\mu_i)'$ are equivalent.

Remark. Assume that each group $G_i(i \in I)$ acts freely in the space E_i $(i \in I)$ and is nontrivial, i.e. G_i contains at least two transformations. If the set I isuncountable, then the group $\sum_{i\in I} G_i$ is uncountable, too, and acts freely in the space $\prod_{i\in I} E_i$. Therefore, Theorem 4 can be applied to this situation.

Rererences

[1] HEWITT, E. AND ROSS, K., *Abstract harmonic analysis*, vol. 1, Springer-Verlag, Berlin, 1963.

[2] HALMOS, P., *Measure Theory*. Princeton, Van Nostrand, 1950.

[3] HALMOS, P., *Lectures on ergodic theory*, MSJ, 1956.

[4] CORNFELD, I. P., SINAJ, J. G., FOMIN, S. V., *Ergodic Theory*. Izd. Nauka, Moscow, 1980 (in Russian).

[5] KHARAZISHVILI, A. B., *Transformation groups and invariant measures*, World Scientific Publishing Co., London-Singapore, 1998.

[6] KHARAZISHVILI, A. B., *Invariant extensions of Lebesgue measure*, Tbilisi University Press, Tbilisi, 1983 (in Russian).

[7] Kirtadze, A., *On the uniqueness property for invariant measures*. Georgian Math. Journal, vol. 12, no. 3, 2005.

[8] Kirtadze, A., Pantsulaia, G., *On the substantial uniqueness property of invariant measures in vector spaces*, Bulletin of the Academy of Sciences of the Georgian SSR, vol. 127, no. 1, 1987 (in Russian).

[9] Kharazishvili, A. B., *Ergodic components and nonseparable extensions of invariant measures*, Proc. A. Razmadze Math. Inst. 145, 2007.

[10] Kharazishvili, A. B., *On invariant measures in the Hilbert space*, Bulletin of the Academy of Sciences of the Georgian SSR, vol. 114, no. 1, 1984 (in Russian).