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A Simple Geometrie Proof of a Theorem for Starshaped Unions of Convex Sets

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Let $\mathcal{F}$ be a family of $n + 1$ convex sets in $\mathbb{R}^d$, each n of which have a point in common, such that $\bigcup \mathcal{F}$ is starshaped. If either all members of $\mathcal{F}$ are closed or all members of $\mathcal{F}$ are open, then $\bigcap \mathcal{F}$ is nonempty. This result which strengthens a theorem by V. Klee follows from a topological theorem of C. D. Horvath and M. Lassonde. We present a simple geometrie proof in the spirit of Klee's proof. This immediately provides an alternative proof of a Helly type theorem due to M. Breen. An abstract vector space variant of the above result is given, too.

Dedicated to the memory of V. Klee

The following theorem was proved by V. Klee [4] in 1951, and then independently by C. Berge [1] in 1959:

Theorem KB. *Let n, d be positive integers. Suppose that $C_0, \ldots, C_n$ are closed convex subsets of $\mathbb{R}^d$, each n of which have a point in common, and that $\bigcup_0^n C_i$ is convex. Then the intersection $\bigcap_0^n C_i$ is nonempty.*

In 1997, C. D. Horvath and M. Lassonde [3] proved a topological theorem, equivalent to Brouwer's fixed point theorem, which implies that Theorem KB remains valid if we suppose that $\bigcup_0^n C_i$ is only starshaped (instead of convex). The aim of our paper is to give a simple geometric proof of this stronger version of Theorem KB (see Theorem 1), based on Klee's original argument from [4].

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Theorem 1 immediately gives an alternative proof of a generalization, due to M. Breen [2], of the famous Helly's theorem. We also state a vector space variant of Theorem 1.

All vector spaces in the present paper are assumed to be real. Recall that a subset A of a vector space is called *starshaped* if there exists $x \in A$ such that, for any $y \in A$, the (closed) segment $[x, y]$ is contained in A. The (convex) set of all such points x is called the *kernel* of A.

Theorem 1. *Let $C_0, \ldots, C_n$ be convex subsets of $\mathbb{R}^d$, each n of which have a point in common, such that $\bigcup_0^n C_i$ is starshaped. Suppose that either each C_i is closed or each C_i is open. Then the intersection $\bigcap_0^n C_i$ is nonempty.*

Proof. If all C_i's are closed, we may assume that they are compact (by intersecting them with a sufficiently large closed ball). Let us proceed by induction with respect to n. For $n = 1$ the theorem follows from the fact that each starshaped set is obviously connected. Now suppose it holds for $n = k - 1$ (and every d) and consider the case $n = k$. Let z_0 be a point from the kernel of the starshaped set $\bigcup_0^k C_i$. We may assume that $z_0 \in C_0$. If $\bigcap_0^k C_i = \emptyset$ then C_0 and $P = \bigcap_1^k C_i$ are nonempty disjoint compact (resp., open) convex sets, so they can be separated by a hyperplane H disjoint from both of them. Let $D_i = C_i \cap H$ $(1 \leq i \leq k)$. For an arbitrary $i_0 \in I = \{1,\ldots,k\}$, let $Q = \bigcap_{i \in I \setminus \{i_0\}} C_i$. Since each k of the C_i's have a point in common, Q intersects C_0. And since furthemore $P \subset Q$, Q must intersect H and hence $\bigcap_{i \in I \setminus \{i_0\}} D_i \neq \emptyset$. Once we show that $\bigcup_1^k D_i$ is starshaped, the theorem will be proved. Indeed, since D_i's are compact (resp., open) in H which is isomorphic to $\mathbb{R}^{d-1}$, it will follow from the inductive hypothesis that $\bigcap_1^k D_i \neq \emptyset$. But this contradicts the fact that $P \cap H = \emptyset$.

Fix an arbitrary point $p \in P$. Since $z_0 \in C_0$, the segment $[p, z_0]$ intersects H at a point z_1. Let x be an arbitrary point of $\bigcup_1^k D_i$. Then the segment $[p, x]$ is contained in $\bigcup_1^k C_i$. By the definition of z_0, $[y, z_0] \subset \bigcup_0^k C_i$ for each $y \in [p, x]$. Consequently, the triangle conv $\{p,x,z_0\} = \bigcup_{y \in [p,x]} [y, z_0]$ is contained in $\bigcup_0^k C_i$. In particular, the segment $[z_1, x]$ is contained in $(\bigcup_0^k C_i) \cap H = (\bigcup_1^k C_i) \cap H = \bigcup_1^k D_i$. This proves that $\bigcup_1^k D_i$ is starshaped, as we needed. □

As already observed in [3], Theorem 1 immediately implies the following strengthening of Helly's theorem, due (for closed sets) to M. Breen [2].

Corollary 2. *Let $\mathcal{F}$ be a family of nonempty convex sets in $\mathbb{R}^d$ such that every subfamily of $\mathcal{F}$ consisting of $d + 1$ or fewer sets has a stashaped union. Suppose that at least one of the following three conditions is satisfied:*

(a) *$\mathcal{F}$ is finite and its members are closed;*

(b) *$\mathcal{F}$ is finite and its members are open;*

(c) *the members of $\mathcal{F}$ are closed and at least one of them is compact.*

Then the intersection $\bigcap \mathcal{F}$ is nonempty.

Proof. Let n be the largest integer such that $n \leq d + 1$ and any n elements of $\mathcal{F}$ have a point in common. Observe that $n \geq 2$ by connectedness. Now, $n = d + 1$ since otherwise Theorem 1 would lead to a contradiction with the maximality of n. Apply Helly's theorem (see, e.g., [5, Theorems 6.2, 6.3]). □

Let us conclude with a more general version of Theorem 1 (see Corollary 4 below). A subset of a vector space X is *algebraically open* if its intersection with any line in X is an open subset of the line. An *algebraically closed set* is a set whose complement is algebraically open (or equivalently, a set whose intersection with any line is closed in the line).

We shall need the following known fact. It follows, e.g., from [6, Theorems 1.16, 1.17] via the well-known fact that any finite-dimensional convex set has a nonempty relative interior. Recall that the *relative interior* of a convex set C, denoted by ri C, is the interior of C with respect to its affine hull aff C.

Fact 3. *Let C be a convex subset of $\mathbb{R}^d$. If C is algebraically open (respectively, algebraically closed), then it is open (resp., closed).*

Now we are ready to state the promised form of Theorem 1.

Corollary 4. *Let X be a vector space, n a positive integer. Let $C_0, \ldots, C_n$ be convex subsets of X, each n of which have a point in common, such that $\bigcup_0^n C_i$ is starshaped. Suppose that either each C_i is algebraically closed or each C_i is algebraically open. Then the intersection $\bigcap_0^n C_i$ is nonempty.*

Proof. Let z be any point of the kernel of the starshaped set $\bigcup_0^n C_i$. For each $k \in I = \{0, \ldots, n\}$ fix an arbitrary $x_k \in \bigcap_{i \in I \setminus \{k\}} C_i$, and denote $Y = \text{aff}\{z, x_0, \ldots, x_n\}$. Consider the sets $D_i = C_i \cap Y (0 \leq i \leq n)$. By Fact 3, they are closed (resp., open) in Y, each n of them have a point in common and their union is starshaped. By Theorem 1, $\bigcap_0^n D_i \neq \emptyset$. □

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