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In: Jiří Rákosník (ed.): Nonlinear Analysis, Function Spaces and Applications, Proceedings of the Spring School held in Prague, May 30-June 6, 2006, Vol. 8. Institute of Mathematics of the Academy of Sciences of the Czech Republic, Praha, 2007. pp. 95--143.

Persistent URL: http://dml.cz/dmlcz/702495

NON-NEWTONIAN FLUIDS AND FUNCTION SPACES

Michael Růžička, Lars Diening

Abstract. In this note we give an overview of recent results in the theory of electrorheological fluids and the theory of function spaces with variable exponents. Moreover, we present a detailed and self-contained exposition of *shifted N-functions* that are used in the studies of generalized Newtonian fluids and problems with p-structure.

1. Introduction

In recent years Lebesgue spaces with variable exponents $L^{p(\cdot)}$ and the corresponding Sobolev spaces have attracted more and more attention. The spaces $L^{p(\cdot)}$ have been studied for the first time already in 1931 by Orlicz [44]. Later these spaces have been investigated in the more general context of generalized Orlicz spaces and modular spaces by Nakano [42], Musielak and Orlicz [40], [41], Hudzik [32] and others. The newer developments in the theory of variable exponents spaces started with the papers by Zhikov [59], Sharapudinov [55] and Kováčik, Rákosník [34]. At the turn of the millenium several factors contributed to start an intensive and systematic study of variable exponent spaces. These factors include that for many problems the *log-Hölder* condition was found to be "correct" and that problems in fluid dynamics and mechanics led naturally to settings with variational exponent spaces (cf. Rajagopal, Růžička [47], [48], [51], Zhikov [60], [61]).

In sections 2–4 of this note we want to give an up-to-date overview of results in the theory of electrorheological fluids and the theory of function

2000 *Mathematics Subject Classification.* 35J60; 46E30; 76D03; 76A05.

Key words and phrases. Electrorheological fluid; generalized Newtonian fluids; existence theory; function spaces with variable exponents; harmonic analysis; Orlicz spaces; shifted N-function.

The authors have been partially supported by the Forschergruppe "Nonlinear Partial Differential Equations: Theoretical and Numerical Analysis". L. Diening is indebted to the Landesstiftung Baden-Württemberg for facilitating the analysis entailed in this paper.

spaces with variable exponents and show how problems arising in these fields have inspired each other. In the last four sections we present a detailed and self-contained exposition of *shifted N-functions*. They are motivated by studies of generalized Newtonian fluids and problems with p-structure. The shifted N-functions are particularly useful when differences of the corresponding operators are investigated (cf. [15], [18], [16], [14]).

2. Modeling

Many *electrorheological fluids* (abbreviated: ERFs) are suspensions consisting of particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field. Winslow [23] is credited with the earliest observations on the change of viscosity in electrorheological materials. For an overview of microscopic models and explanations in electrorheology we refer the reader to [45]. For more details related to the modeling we refer to [48], [51], [52].

In order to get a model where the complex interactions between the electrical, mechanical and magnetic fields are incorporated one starts with the thermo-mechanical balance laws, the Clausius-Duhem inequality and Maxwell's equations. The interaction between the fields is modeled on the basis of the "dipole–current–loop" model. The resulting system is a highly complicated nonlinear system of partial differential equations, which is much to general to describe the behaviour of ERFs. Thus appropriate assumption for the general structure of the constitutive relations are made. Moreover, a non-dimensionalization is performed, which restricts the resulting system to certain but typical situations. This procedure results in the following *electrorheological approximation* describing the isothermal flow of an incompressible ERF[1]

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 0, \\ \operatorname{curl} \mathbf{E} &= \mathbf{0}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \partial_t \mathbf{v} - \operatorname{div} \mathbf{S} + [\nabla \mathbf{v}]\mathbf{v} + \nabla \pi &= \mathbf{f} + \chi^E [\nabla \mathbf{E}]\mathbf{E}, \\ \operatorname{div} \mathbf{v} &= 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \operatorname{div} \mathbf{B} &= 0, \\ \mu_0^{-1} \operatorname{curl} \mathbf{B} + \chi^E \operatorname{curl}(\mathbf{v} \times \mathbf{E}) &= (\varepsilon_0 + \chi^E)\partial_t \mathbf{E}, \end{aligned} \tag{2.3}$$

$$\mathbf{S} \cdot \mathbf{D} \geq 0, \tag{2.4}$$

[1] Here and in the following we use the notation $[\nabla \mathbf{u}]\mathbf{w} = \sum_{j=1}^{3} \big(w_j \frac{\partial u_i}{\partial x_j}\big)_{i=1,2,3}$ for vectors $\mathbf{v}$, $\mathbf{w}$.

where $\mathbf{E}$ is the *electric field*, $\mathbf{v}$ the *velocity*, $\mathbf{S}$ the *extra stress tensor*, π the pressure, $\mathbf{f}$ the *mechanical force density*, χ^E the *dielectric susceptibility*, $\mathbf{B}$ the *magnetic flux density* and ε_0 and μ_0 denote the dielectric constant and the permeability in vacuo, respectively.

The system (2.1)–(2.3) is separated. We first solve the quasi-static Maxwell's equations (2.1) for the electric field and then seek for the velocity field by solving (2.2). Knowing $\mathbf{E}$ and $\mathbf{v}$ we can solve (2.3). Maxwell's equations (2.1), (2.3) are widely studied in the literature (cf. the overview article [38]). Thus we shall concentrate on the system (2.2), in which $\mathbf{E}$ is assumed to be a given vector field, having certain regularity properties. In (2.2) it remains to specify a constitutive relation for the extra stress tensor $\mathbf{S}$. ERFs can be modeled with a generalization of a *power-law* or *Carreau* type ansatz

$$\begin{aligned}\mathbf{S} = \alpha_{21}\big((1+|\mathbf{D}|^2)^{\frac{p-1}{2}} - 1\big)\mathbf{E}\otimes\mathbf{E}\\ + (\alpha_{31}+\alpha_{33}|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}\mathbf{D}\\ + \alpha_{51}(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}(\mathbf{D}\mathbf{E}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{D}\mathbf{E}),\end{aligned} \tag{2.5}$$

where α_{ij} are constants and $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^\top)$ denotes the symmetric velocity gradient. The peculiarity of ERFs is that the power-law exponent p depends on the electric field (cf. Halsey, Martin, Adorf [30], Abu-Jdayil, Brunn [1], [2], [3]), i.e. $p = p(|\mathbf{E}|^2)$ is a C^1-function such that

$$1 < p_\infty \le p(|\mathbf{E}|^2) \le p_0. \tag{2.6}$$

Imposing slightly more restrictive conditions on the coefficients α_{ij} than those resulting by the validity of the Clausius–Duhem inequality (2.5) we have that the operator induced by $-\operatorname{div}\mathbf{S}(\mathbf{D},\mathbf{E})$ is *uniformly monotone*, i.e.,

$$\sum_{i,j,k,l}\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}}B_{ij}B_{kl} \ge \gamma_1\big(1+|\mathbf{E}|^2\big)\big(1+|\mathbf{D}|^2\big)^{\frac{p(|\mathbf{E}|^2)-2}{2}}|\mathbf{B}|^2 \tag{2.7}$$

is satisfied for all $\mathbf{B},\mathbf{D} \in X := \{\mathbf{D}\in\mathbb{R}^{3\times3}_{\mathrm{sym}} : \operatorname{tr}\mathbf{D} = 0\}$, and that the following *growth conditions* are satisfied for $i,j,k,l,n = 1,2,3$,

$$\left|\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}}\right| \le \gamma_2\big(1+|\mathbf{E}|^2\big)\big(1+|\mathbf{D}|^2\big)^{\frac{p(|\mathbf{E}|^2)-2}{2}}, \tag{2.8}$$

$$\left|\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial E_n}\right| \le \gamma_3|\mathbf{E}|\big(1+|\mathbf{E}|^2\big)\big(1+|\mathbf{D}|^2\big)^{\frac{p(|\mathbf{E}|^2)-1}{2}}\big(1+\ln(1+|\mathbf{D}|^2)\big). \tag{2.9}$$

Thus the natural function spaces to treat the system (2.2), (2.5)–(2.9) are Lebesgue and Sobolev spaces with variable exponents. From the mathematical point of view the system (2.2), (2.5)–(2.9) is a generalization of *generalized Newtonian fluids*, where the power-law exponent p is constant. These fluids are well studied (cf. MÁLEK, NEČAS, ROKYTA, RŮŽIČKA [36], FREHSE, MÁLEK, STEINHAUER [25], [26], [27], RŮŽIČKA [49], WOLF [58]) and involve techniques like the continuity of Calderón-Zygmund operators, Korn's inequality, passing to divergence free test functions with the help of the divergence equation $\operatorname{div} \mathbf{v} = f$ and others. For a better mathematical understanding of ERFs it is necessary to transfer these techniques to variable exponent spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$.

3. Variable exponent spaces

Let us now introduce the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some fundamental properties of these spaces, which can be found in the literature mentioned above. Hereby $\Omega \subseteq \mathbb{R}^n$ denotes a domain with sufficiently smooth boundary. For a measurable and almost everywhere finite function $p : \mathbb{R}^n \to [1, \infty)$ (called the *exponent*) we define $L^{p(\cdot)}(\Omega)$ to consist of measurable functions $f : \Omega \to \mathbb{R}$ such that the *modular*

$$\rho_p(f) := \int_\Omega |f(x)|^{p(x)}\, dx$$

is finite. If $p^+ := \sup p < \infty$ (called a *bounded exponent*), then the expression

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\}$$

defines a norm on $L^{p(\cdot)}(\Omega)$. This makes $L^{p(\cdot)}(\Omega)$ a Banach space. Moreover, one can show that $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and that $L^{p(\cdot)}(\Omega)$ is separable. If $p^- := \inf p > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex and the dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. Further, let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions $f : \Omega \to \mathbb{R}$ such that f and the distributional derivative ∇f are in $L^{p(\cdot)}$. The norm $\|f\|_{1,p(\cdot)} := \|f\|_{p(\cdot)} + \|\nabla f\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. By $W_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

The spaces $L^{p(\cdot)}$ have some undesired properties. For example the translation operator is in general not continuous on $L^{p(\cdot)}$. Especially, for every non-constant exponent p there exists $f \in L^{p(\cdot)}$ and a translation τ_h such that $\tau_h f \notin L^{p(\cdot)}$ (cf. [34], [11]). As a consequence the convolution with a function $g \in L^1$ is in general not continuous; in fact, Young's inequality $\|f * g\|_{p(\cdot)} \le C\|g\|_1\|f\|_{p(\cdot)}$ holds if and only if p is constant (cf. [11]).

Surprisingly, it turns out that under rather weak conditions on the exponent p many of the results from the classical Lebesgue and Sobolev spaces can be recovered. The crucial condition is the so-called *log-Hölder* continuity of the exponent p, i.e.,

$$|p(x) - p(y)| \le \frac{C}{\big|\ln|x-y|\big|} \tag{3.1}$$

for all $|x - y| < 1/2$. If Ω is unbounded, then (3.1) is supplemented by the condition that there exists the limit $p(\infty) := \lim_{x\to\infty} p(x)$ and

$$|p(x) - p(\infty)| \le \frac{C}{\ln(e + |x|)}. \tag{3.2}$$

A breakthrough in the theory of variable exponent spaces was the observation by Diening [11] that the Hardy–Littlewood maximal operator M is continuous in $L^{p(\cdot)}$ if p satisfies (3.1) and is constant outside some large ball. Later the result was refined with (3.2) by Nekvinda [43] and Cruz-Uribe, Fiorenza, Neugebauer [10].

Theorem 3.1. *Suppose that the bounded exponent p is log-Hölder continuous and satisfies* (3.2) *and* $1 < p^-$. *Then the Hardy–Littlewood maximal operator M is continuous from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$.*

We say that the exponent belongs to the class $\mathcal{P}$ if the Hardy–Littlewood maximal operator M is continuous from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. Note that the conditions in the above theorem are optimal in the sense of the modulus of continuity (cf. [46], [10]). As consequences of Theorem 3.1 one obtains that it is possible to mollify functions in $L^{p(\cdot)}$ with mollifying kernels $\omega \in C_0^\infty(\mathbb{R}^n)$ (cf. [53], [11], [56]), that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ for domains Ω with Lipschitz-continuous boundary and that the Riesz operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (cf. [54], [12]). Moreover, one can show that the sharp maximal operator $M^\sharp$ provides an equivalent norm on $L^{p(\cdot)}(\mathbb{R}^n)$. Recently, Diening [13] gave a characterization of exponents belonging to class $\mathcal{P}$. One says that the exponent p is of class $\mathcal{A}$ if the averaging operator $T_{\mathcal{Q}} : f \to \sum_{Q\in\mathcal{Q}} \chi_Q \fint_Q f\,dx$ is uniformly bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ with respect to all families $\mathcal{Q}$ of disjoint cubes Q.

Theorem 3.2. *Let p be a bounded exponent with* $1 < p^-$. *The following assertions are equivalent:*

(i) *$p(\cdot)$ is of class $\mathcal{A}$;*
(ii) *M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, i.e. $p \in \mathcal{P}$;*

(iii) $M_q f := (M(|f|^q))^{1/q}$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$ *for some* $q > 1$ *("left-openness");*
(iv) M *is bounded on* $L^{p(\cdot)/q}(\mathbb{R}^n)$ *for some* $q > 1$ *("left-openness");*
(v) M *is bounded on* $L^{p'(\cdot)}(\mathbb{R}^n)$.

Another important issue is the extension of the theory of singular operators to variable exponent spaces. The first results in this direction, proving the boundedness of singular operators and extending the Calderón–Zygmund theory, can be found in Diening, Růžička [19]. We say that an operator T is associated with a kernel k if

$$(Tf)(x) := \int_{\mathbb{R}^n} k(x,y) f(y)\, dy$$

holds for $f \in C_0^\infty(\mathbb{R}^n)$and for a.e. $x \notin \operatorname{supp}(f)$, where the kernel k is a locally integrable function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \operatorname{diag}$. A truncated kernel k_ε, $\varepsilon > 0$, is defined through $k_\varepsilon(x,y) = k(x,y)$ if $|x-y| > \varepsilon$ and otherwise zero. Using the results from [19] and [13] we have the following:

Theorem 3.3. *Let T be an operator associated with a kernel k which satisfies*

$$\begin{aligned} |k(x,y)| &\le A\,|x-y|^{-n}, \\ |k(x,y) - k(z,y)| &\le A\,|x-z|^{\delta} |x-y|^{-n-\delta} \end{aligned} \tag{3.3}$$

for all $x, y, z \in \mathbb{R}^n$ with $x \neq y$ and $|x-z| < \frac{1}{2}|x-y|$. Suppose that T extends to a bounded operator from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Let $p \in \mathcal{P}(\mathbb{R}^n)$ be a bounded exponent with $p^- > 1$. Then T is a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$.

A kernel k is called *Calderón–Zygmund kernel* if and only if $N(x,z) := k(x, x-z)$ is homogeneous in z of degree $-n$ and $\int_{|z|=1} N(x,z)\, dz = 0$, $\int_{|z|=1} |N(x,z)|^{q'}\, dz \le C$ for some $q > 1$ (cf. Calderón–Zygmund [7]). We denote by T_ε the operator associated with the truncated kernel k_ε.

Theorem 3.4. *Let k be a Calderón–Zygmund kernel which satisfies* (3.3) *and*

$$|k(y,x) - k(y,z)| \le A\,|x-z|^{\delta} |x-y|^{-n-\delta}.$$

Let $p \in \mathcal{P}(\mathbb{R}^n)$ be a bounded exponent with $p^- > 1$. Then the operators T_ε,

$$(T_\varepsilon f)(x) := \int_{\mathbb{R}^n} k_\varepsilon(x,y) f(y)\, dy,$$

are uniformly bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ *with respect to* $\varepsilon > 0$. *Moreover,*

$$Tf(x) := \lim_{\varepsilon \to 0^+} T_\varepsilon f(x)$$

exists almost everywhere and $\lim_{\varepsilon \to 0^+} T_\varepsilon f = Tf$ *in the* $L^{p(\cdot)}(\mathbb{R}^n)$*-norm. In particular,* T *is continuous on* $L^{p(\cdot)}(\mathbb{R}^n)$.

Using these results one can obtain optimal $W^{2,p(\cdot)}(\mathbb{R}^n)$ estimates for solutions of linear elliptic equations and systems. For corresponding optimal estimates in the halfspace one can generalize the AGMON, DOUGLIS, NIRENBERG theory (cf. [4]) to variable exponent spaces, which is done in DIENING, RŮŽIČKA [20], [21].

A completely different approach to the boundedness of operators in variable exponent spaces is provided by CRUZ-URIBE, FIORENZA, MARTELL, PÉREZ [9] via norm estimates in classical weighted Lebesgue spaces L^p_ω, where ω is a Muckenhoupt weight (cf. [39]). The main result in [9] can be formulated as follows.

Theorem 3.5. *Let* $\mathcal{F}$ *be a family of ordered pairs of non-negative, measurable functions* (f, g) *and* $\Omega \subset \mathbb{R}^n$ *be an open set. Let* p *be a bounded exponent with* $p^- > 1$ *and let* $1 < q < p^-$. *Suppose that*

$$\int_{\mathbb{R}^n} f(x)^q \omega(x)\, dx \le K(\omega) \int_{\mathbb{R}^n} g(x)^q \omega(x)\, dx$$

holds for all $(f, g) \in \mathcal{F}$ *and all Muckenhoupt weights* $\omega \in A_1$ *with an* A_1*-consistent constant* $K(\omega)$. *Here we suppose that the left-hand side is finite. If* $p/q \in \mathcal{P}$, *then for all* $(f, g) \in \mathcal{F}$ *such that* $f \in L^{p(\cdot)}(\Omega)$,

$$\|f\|_{p(\cdot)} \le C \|g\|_{p(\cdot)}.$$

From this theorem one can deduce that in variable exponent spaces singular integral operators associated with symmetric kernels are bounded, that commutator estimates hold, that fractional integrals and fractional maximal operators are bounded and that certain extension results hold (cf. [9]).

Moreover, one can extend the results of BOGOVSKII [5], [6] for the divergence equation to variable exponent spaces (cf. DIENING, RŮŽIČKA [19], HUBER [31]).

Theorem 3.6. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain with Lipschitz-continuous boundary and let* $p, q \in \mathcal{P}$ *be bounded exponents with* $p^-, q^- > 1$. *Then for all* $\mathbf{f} \in$

$W^{1,p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega)$ *there exists a solution* $\mathbf{v} \in W_0^{1,p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega)$ *of the equation* $\operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{f}$ *satisfying the estimates*

$$\begin{aligned} \|\nabla \mathbf{v}\|_{p(\cdot)} &\le C \|\operatorname{div} \mathbf{f}\|_{p(\cdot)}, \\ \|\mathbf{v}\|_{q(\cdot)} &\le C \|\mathbf{f}\|_{q(\cdot)}. \end{aligned}$$

4. Application to ERFs

In this section we shortly summarize how the above theory of variable exponent spaces can be used in the mathematical theory of ERFs. Using basic properties of the variable exponent spaces and the theory of pseudomonotone operators it is shown in Růžička [51], Růžička, Ettwein [24] that the steady problem, i.e. when the time derivative $\partial_t \mathbf{v}$ in (2.2) is neglected, has solutions.

Theorem 4.1. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with Lipschitz-continuous boundary. Let* $\mathbf{E} \in W^{1,\infty}(\Omega)$ *and* $\mathbf{f} \in \big(W_0^{1,p(|\mathbf{E}|^2)}(\Omega)\big)'$ *and let* $\mathbf{S}$ *satisfy* (2.5)–(2.9). *Then there exists a weak solution* $\mathbf{v} \in W_0^{1,p(|\mathbf{E}|^2)}(\Omega)$ *of the steady problem* (2.2) *equipped with homogeneous Dirichlet boundary conditions whenever*

$$p_\infty > 9/5.$$

Moreover, the solution satisfies for all $U \subset\subset \Omega$

$$\int_U \big(1 + |\mathbf{D}(\mathbf{v})|^2\big)^{\frac{p(|\mathbf{E}|^2)-2}{2}} |\mathbf{D}(\nabla \mathbf{v})|^2 \, dx < \infty.$$

For the unsteady problem equipped with space periodic boundary conditions and initial condition $\mathbf{v}(0) = \mathbf{v}_0$ one can prove (cf. Růžička [50]):

Theorem 4.2. *Let* $\Omega = (0, L)^3$ *be a given cube,* $I = [0, T]$ *a given time interval and assume that* $\mathbf{v}_0 \in W_0^{1,2}$, $\operatorname{div} \mathbf{v}_0 = 0$, $\mathbf{E} \in L^\infty(I, W^{1,\infty}(\Omega))$, *and* $\mathbf{f} \in L^r(I \times \Omega)$, $r = \max\{p'_\infty, 2\}$, *are given. Assume that* $\mathbf{S}$ *satisfies* (2.5)–(2.9). *Whenever*

$$9/5 < p_\infty \le p(|\mathbf{E}|^2) \le p_0 < p_\infty + 1,$$

there exists a weak solution $\mathbf{v} \in L^\infty(I, L^2(\Omega)) \cap L^{p_\infty}(I, W_0^{1,p_\infty}(\Omega))$ *of the problem* (2.2) *such that* $\mathbf{D}(\mathbf{v}) \in L^{p(|\mathbf{E}|^2)}(I \times \Omega)$. *Moreover, if*

$$11/5 < p_\infty \le p(|\mathbf{E}|^2) \le p_0 < p_\infty + 4/3,$$

there exists a unique solution of the problem (2.2) *with the additional property*

$$\int_I \int_\Omega \left(1 + |\mathbf{D}(\mathbf{v})|^2\right)^{\frac{p(|\mathbf{E}|^2)-2}{2}} |\mathbf{D}(\nabla \mathbf{v})|^2 \, dx \, dt < \infty.$$

In the case of Dirichlet boundary conditions one has to require more restrictive conditions on $p(|\mathbf{E}|^2)$ to ensure the validity of a similar theorem (cf. Růžička [51]).

The idea of using L^∞-testfunctions presented in [25], [49] was extended to the treatment of ERFs by Huber [31].

Theorem 4.3. *Under the assumptions of Theorem 4.1 there exists a weak solution of the steady problem* (2.2) *whenever*

$$p_\infty > 3/2.$$

Also the Lipschitz approximation technique from [27] can be applied to the treatment of ERFs. Diening, Málek, Steinhauer [17] further extended the previous theorem to the following:

Theorem 4.4. *Under the assumptions of Theorem 4.1 there exists a weak solution of the steady problem* (2.2) *whenever*

$$p_\infty > 6/5.$$

5. Shifted Orlicz functions

For a better understanding of the structure of the elliptic part of the problem (2.2), (2.5) it is useful to study the model case

$$\begin{aligned} -\operatorname{div}(\mathbf{A}(\nabla \mathbf{v})) = \mathbf{f} \quad & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} \quad & \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

where $\mathbf{A}(\mathbf{P}) := |\mathbf{P}|^{p-2}\mathbf{P}$, $1 < p < \infty$, for $\mathbf{P} \in \mathbb{R}^{N \times n}$. The natural energy space for this problem is the Sobolev space $W_0^{1,p}(\Omega)$, since

$$\mathbf{A}(\mathbf{P}) \cdot \mathbf{P} = |\mathbf{P}|^p.$$

By standard methods one obtains the existence of a unique weak solution $\mathbf{v} \in W_0^{1,p}(\Omega)$ of (5.1) under appropriate assumptions on $\mathbf{f}$. However, it turns out that the Sobolev space $W^{1,p}(\Omega)$ is not the correct setting for a more detailed investigation of this solution such as its regularity properties and error estimates for an FEM approximation. In fact, many of such results are based on the behaviour of the quantity

$$(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q})$$

which is equivalent (see below) to

$$(|\mathbf{P}| + |\mathbf{P} - \mathbf{Q}|)^{p-2} |\mathbf{P} - \mathbf{Q}|^2. \tag{5.2}$$

This information can not be expressed in terms of a Lebesgue space, however it leads to an Orlicz space. In fact, (5.2) can be written as $\mathbf{P} - \mathbf{Q} \in L^{\varphi_{|\mathbf{P}|}}(\Omega)$, with the N-function $\varphi_a(t) \approx (a+t)^{p-2} t^2$, $a, t \geq 0$. Such functions are called *shifted N-functions* and will be defined precisely below. Thus already the pure p-Laplace problem leads naturally to the study of Orlicz and Orlicz–Sobolev spaces. This property becomes even more evident if one investigates (5.1) with the operator $\mathbf{A}(\mathbf{P}) := (\kappa + |\mathbf{P}|)^{p-2}\mathbf{P}$, $\kappa \geq 0$, $1 < p < \infty$, $\mathbf{P} \in \mathbb{R}^{N \times n}$. The natural energy space for this problem is $W_0^{1,\varphi_\kappa}(\Omega)$. Motivated by this we present some results related to the functional setting of the problem (5.1) where the operator $\mathbf{A}$ has N-potential φ (cf. Section 6) or φ-structure for some N-function φ (cf. Section 7). In Section 8 the modifications for the treatment of problems from fluid mechanics are discussed.

Before we define shifted N-functions we recall some basic facts about N-functions. For more details we refer to Krasnosel'skii, Rutickii[35]. A function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is called *N-function* if it is continuous, convex and such that $\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$, $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$, and $\varphi(t) > 0$ for $t > 0$. In the following we use the convention that $\frac{\varphi(0)}{0} := 0$. An N-function φ possesses a right derivative, denoted by φ', which is right continuous, non-decreasing, and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi'(t) = \infty$. Moreover, the representation

$$\varphi(t) = \int_0^t \varphi'(s)\, ds \tag{5.3}$$

holds. Formula (5.3) for a function φ' with the above properties could be taken as an equivalent definition of an N-function. The right inverse of φ' is denoted by $(\varphi')^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ and defined through

$$(\varphi')^{-1}(t) := \sup\{u \in \mathbb{R}^{\geq 0} \mid \varphi'(u) \leq t\}.$$

The complementary function φ^* of φ is defined by

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\, ds. \tag{5.4}$$

It is easily seen that φ^* is again an N-function and that for $t \geq 0$ holds

$$(\varphi^*)'(t) = (\varphi')^{-1}(t).$$

Note that $(\varphi^*)^* = \varphi$. Since $(\varphi')^{-1}$ is the right inverse, we have

$$\begin{aligned} (\varphi^*)'(\varphi'(t) - \varepsilon) \leq t \leq (\varphi^*)'(\varphi'(t)), \\ \varphi'((\varphi^*)'(t) - \varepsilon) \leq t \leq \varphi'((\varphi^*)'(t)) \end{aligned} \tag{5.5}$$

for all $t > 0$ and all sufficiently small $\varepsilon > 0$. If φ' is strictly increasing, than $(\varphi')^{-1}$ is the inverse function of φ'. The complementary function φ^* of φ could equivalently be defined by

$$\varphi^*(t) := \sup\{ut - \varphi(u) \mid u \in \mathbb{R}^{\geq 0}\}. \tag{5.6}$$

From (5.6) follows immediately Young's inequality, i.e. for all $t, u \geq 0$,

$$tu \leq \varphi(t) + \varphi^*(u). \tag{5.7}$$

Choosing $\varphi(t) = \varphi^*(u) = v$ in this inequality, we obtain

$$\varphi^{-1}(v)\, (\varphi^*)^{-1}(v) \leq 2v. \tag{5.8}$$

Since φ' is non-decreasing it follows from (5.3) that

$$\varphi(t) \leq t\varphi'(t) \tag{5.9}$$

for $t \geq 0$ with strict inequality for $t > 0$, which yields

$$(\varphi')^{-1}\Big(\frac{\varphi(t)}{t}\Big) \leq t$$

for $t \geq 0$ by the properties of $(\varphi')^{-1}$ (cf. (5.5)). From this and (5.9) we obtain

$$\varphi^*\Big(\frac{\varphi(t)}{t}\Big) = \int_0^{\frac{\varphi(t)}{t}} (\varphi')^{-1}(s)\, ds \leq \frac{\varphi(t)}{t} (\varphi')^{-1}\Big(\frac{\varphi(t)}{t}\Big) \leq \varphi(t).$$

Choosing $\varphi(t) = u$ we get

$$u \leq \varphi^{-1}(u)\, (\varphi^*)^{-1}(u).$$

Further we have

$$\varphi(2t) \geq \int_t^{2t} \varphi'(s)\, ds \geq t\varphi(t).$$

If we denote $v = \varphi(t)$, then we deduce from (5.8) $(\varphi^*)^{-1}(\varphi(t)) \leq 2\varphi(t)/t$, which implies

$$\varphi(t) \leq \varphi^*\Big(\frac{2\varphi(t)}{t}\Big).$$

Thus we proved:

Lemma 5.1. *Let φ be an N-function and φ^* its complementary function. Then we have for all $t \geq 0$*

$$t \leq \varphi^{-1}(t)\,(\varphi^*)^{-1}(t) \leq 2t, \tag{5.10}$$

$$\varphi(t) \leq t\varphi'(t) \leq \varphi(2t), \tag{5.11}$$

$$\varphi^*\Big(\frac{\varphi(t)}{t}\Big) \leq \varphi(t) \leq \varphi^*\Big(\frac{2\varphi(t)}{t}\Big). \tag{5.12}$$

Note that the second inequalities in (5.10), (5.12) turn into equalities for $\varphi(t) = \frac{1}{2}t^2$, while the first inequalities in (5.10), (5.11), (5.12) are optimal in the limit $p \searrow 1$ for $\varphi(t) = \frac{1}{p}t^p$. The last inequality in (5.11) is optimal in the limit $t \searrow 1$ and $\varepsilon \searrow 0$ for $\varphi(t) = \frac{\varepsilon}{2}t^2 + (t-1)_+$.

Using (5.11) for φ^* and (5.5) we obtain for sufficiently small $\varepsilon > 0$ that

$$\varphi^*(\varphi'(t)-\varepsilon) \leq (\varphi'(t)-\varepsilon)\,(\varphi^*)'(\varphi'(t)-\varepsilon) \leq (\varphi'(t)-\varepsilon)t \leq \varphi(2t) - \varepsilon t,$$

which in the limit $\varepsilon \to 0$ implies

$$\varphi^*(\varphi'(t)) \leq \varphi(2t). \tag{5.13}$$

An important subclass of N-functions are those satisfying the Δ_2-*condition*, i.e. functions φ satisfying for all $t \geq 0$ the estimate $\varphi(2t) \leq K\varphi(t)$ with a constant $K \geq 2$. The Δ_2-constant of φ is the smallest constant K having this property. Since φ is increasing, for N-functions satisfying the Δ_2-condition we have

$$\varphi(t) \leq \varphi(2t) \leq K\varphi(t). \tag{5.14}$$

From (5.11) it follows that φ' also satisfies the Δ_2-condition since

$$\varphi'(2t) \leq \frac{\varphi(4t)}{2t} \leq \frac{K^2}{2}\frac{\varphi(t)}{t} \leq \frac{K^2}{2}\varphi'(t). \tag{5.15}$$

The inequalities (5.11), (5.14) imply

$$\varphi(t) \leq t\varphi'(t) \leq K\varphi(t) \tag{5.16}$$

for φ satisfying the Δ_2-condition. For φ and φ^* satisfying the Δ_2-condition we deduce from (5.12), (5.11) and (5.13) that

$$\frac{1}{K_*}\,\varphi(t) \leq \frac{1}{K_*}\,\varphi^*\Big(\frac{2\varphi(t)}{t}\Big) \leq \varphi^*(\varphi'(t)) \leq \varphi(2t) \leq K\varphi(t). \tag{5.17}$$

In the following we denote by K, K', K_* and K'_* the Δ_2-constants of of φ, φ', φ^* and $(\varphi^*)'$, respectively. We can now get a useful version of Young's inequality if φ or φ^* satisfies the Δ_2-condition. If φ^* satisfies the Δ_2-condition, then

$$tu \leq \delta\varphi(t) + K_*^M \varphi^*(u)$$

for all $\delta \in (0, 1)$ and $M \in \mathbb{N}$ such that $\delta^{-1} \leq 2^M$. Here we used that, by the convexity of φ,

$$\varphi(\delta t) \leq \delta\varphi(t) \tag{5.18}$$

for $\delta \in [0, 1]$ and all $t \geq 0$. Analogously, for φ satisfying the Δ_2-condition we obtain that for all $\delta \in (0, 1)$,

$$tu \leq \delta\varphi^*(u) + K^M \varphi(t)$$

with $M \in \mathbb{N}$ such that $\delta^{-1} \leq 2^M$.

Lemma 5.2. *Let the N-function φ satisfy the Δ_2-condition. Then we have*

$$K \leq 2K' \leq K^2. \tag{5.19}$$

Proof. The last inequality is proved in (5.15), while the first one follows from

$$\varphi(2t) = \int_0^{2t} \varphi'(s)\, ds = \int_0^t \varphi'(2s)\, 2ds \leq 2K' \int_0^t \varphi'(s)\, ds = 2K'\varphi(t).$$

□

Note that in the first inequality in (5.19) equality holds for $\varphi(t) = \frac{1}{p}t^p$. Moreover, (5.19) implies that $K' \geq 1$, since $K \geq 2$. The proof of Lemma 5.2 also implies that an N-function φ, for which φ' satisfies the Δ_2-condition, also satisfies the Δ_2-condition.

Now we will define *shifted N-functions* φ_a. For a given N-function φ we set

$$\varphi_a'(t) = (\varphi_a)'(t) := \frac{\varphi'(t + a)}{t + a}\, t \tag{5.20}$$

for $a, t \geq 0$ and define $\varphi_a : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ through

$$\varphi_a(t) := \int_0^t \varphi_a'(s)\, ds. \tag{5.21}$$

Lemma 5.3. *Let the N-function φ satisfy the Δ_2-condition. Then for all $a \geq 0$ the functions φ_a defined in* (5.21) *are N-functions satisfying the Δ_2-condition. The Δ_2-constants K_a and K'_a of φ_a and φ'_a, respectively, satisfy the inequalities*

$$K_a \leq 2K'_a \leq 4K' \leq 2K^2. \tag{5.22}$$

Proof. We have $\varphi_0 = \varphi$ and thus we can assume in the sequel that $a > 0$. From the definition of $\varphi_a(t)$ in (5.20) it is clear that it is a right continuous function. Moreover, $\varphi'(t+a)$ and $\frac{t}{t+a}$ are non-decreasing with respect to t and thus also φ'_a is non-decreasing. The properties of φ' imply immediately $\varphi'_a(0) = 0$, $\varphi'_a(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi'_a(t) = \infty$. From

$$\varphi'_a(2t) = \frac{\varphi'(a+2t)2t}{a+2t} \leq \frac{\varphi'(2(a+t))2t}{a+t} \leq 2K'\varphi'_a(t)$$

it follows $K'_a \leq 2K'$, which with the help of (5.19) implies (5.22). □

In view of the above lemma we call φ_a the *shifted N-function.*

We need some inequalities between the shifted N-function φ_a and the N-function φ itself.

Lemma 5.4. *Let φ be an N-function and let $M \in \mathbb{N}$. Then for all $t \geq a(2^M - 1)^{-1}$,*

$$\frac{1}{2^M}\varphi'(t) \leq \varphi'_a(t) \leq \varphi'(2^M t),$$

where the second inequality is strict for $a > 0$.

Proof. Since φ' is non-decreasing, we have

$$\varphi'_a(t) = \frac{\varphi'(a+t)}{a+t}\,t \leq \frac{\varphi'(2^M t)}{t}\,t = \varphi'(2^M t),$$
$$\varphi'_a(t) = \frac{\varphi'(a+t)}{a+t}\,t \geq \frac{\varphi'(t)}{2^M t}\,t = \frac{1}{2^M}\varphi'(t),$$

which proves the assertion. □

Lemma 5.5. *Let φ be an N-function, $a > 0$, and let $M \in \mathbb{N}$. Then for all $0 \leq t \leq a(2^M - 1)$,*

$$\frac{1}{2^M}\frac{\varphi'(a)}{a}\,t \leq \varphi'_a(t) \leq \frac{\varphi'(2^M a)}{a}\,t,$$

where the second inequality is strict if $t > 0$.

Proof. We have

$$\varphi_a'(t) = \frac{\varphi'(a+t)}{a+t}\,t \le \frac{\varphi'(2^M a)}{a}\,t,$$
$$\varphi_a'(t) = \frac{\varphi'(a+t)}{a+t}\,t \ge \frac{\varphi'(a)}{2^M a}\,t,$$

which proves the assertion. Note that for $t > 0$ it holds $\frac{t}{a+t} < \frac{t}{a}$. □

Lemma 5.6. *Let φ be an N-function. Then*

$$\varphi_a'(t-\varepsilon) \le \varphi'(2t-\varepsilon)$$

for all a, t and ε such that $0 \le a \le t$ and $0 < \varepsilon < t$, and

$$\varphi_a'(t-\varepsilon) \le \frac{\varphi'(2a-\varepsilon)}{a}\,t \tag{5.23}$$

for all a, t and ε such that $0 < t \le a$ and $0 < \varepsilon < t$.

Proof. If $0 \le a \le t$ and $0 < \varepsilon < t$ then

$$\varphi_a'(t-\varepsilon) = \frac{\varphi'(a+t-\varepsilon)}{a+t-\varepsilon}\,(t-\varepsilon) \le \varphi'(2t-\varepsilon),$$

and, similarly,

$$\varphi_a'(t-\varepsilon) = \frac{\varphi'(a+t-\varepsilon)}{a+t-\varepsilon}\,(t-\varepsilon) \le \frac{\varphi'(a+t-\varepsilon)}{a+t}\,t \le \frac{\varphi'(2a-\varepsilon)}{a}\,t$$

if $0 < t \le a$ and $0 < \varepsilon < t$. □

Now we want to investigate the complementary function $(\varphi_a)^*$ of a shifted N-function φ_a and derive a relation to some shifted complementary N-function $(\varphi^*)_b$.

Lemma 5.7. *Let φ be an N-function such that φ and φ^* satisfy the Δ_2-condition. Then for all $a, t \ge 0$,*

$$(\varphi^*)'_{\varphi'(a)}(\varphi_a'(t)) \le 2K_*'K't.$$

Proof. For $t = 0$ the assertion is obvious. Thus we consider in the following only the case $t > 0$. For $a = 0$ and sufficiently small $\varepsilon > 0$, using (5.5), we have

$$(\varphi^*)'(\varphi'(t)) \le K_*'\,(\varphi^*)'\big(\tfrac{1}{2}\varphi'(t)\big) \le K_*'\,(\varphi^*)'(\varphi'(t)-\varepsilon) \le K_*'t.$$

For $t \geq a > 0$ we have $\varphi'_a(t) \leq \varphi'(2t)$. Taking sufficiently small $\varepsilon > 0$, Lemma 5.4 applied with $M = 1$ first for φ'_a and then once more for $(\varphi^*)'_{\varphi'(a)}$ and the first inequality in (5.5) with $2t$ in place of t yield

$$(\varphi^*)'_{\varphi'(a)}(\varphi'_a(t)) \leq (\varphi^*)'_{\varphi'(a)}(\varphi'(2t) - \varepsilon) \leq (\varphi^*)'(2(\varphi'(2t) - \varepsilon)) \leq 2K'_* t.$$

For $0 < t \leq a/K'$ we have $\varphi'_a(t) \leq \varphi'_a(a)$ and $\varphi'(2a)t/a \leq \varphi'(a)$. Lemma 5.5 applied with $M = 1$ for φ'_a, (5.23) and (5.5) yield

$$\begin{aligned}(\varphi^*)'_{\varphi'(a)}(\varphi'_a(t)) &\leq (\varphi^*)'_{\varphi'(a)}\Big(\varphi'(2a)\frac{t}{a} - \varepsilon\Big) \\ &\leq \frac{(\varphi^*)'(2\varphi'(a) - \varepsilon)}{\varphi'(a)}\,\varphi'(2a)\frac{t}{a} \\ &\leq K'_* K' t\end{aligned}$$

for sufficiently small $\varepsilon > 0$. For $a/K' \leq t \leq a$ we have $\varphi'_a(t) \leq \varphi'_a(a)$ and thus, using the above result for the case $t \geq a$ and (5.5), we obtain

$$(\varphi^*)'_{\varphi'(a)}(\varphi'_a(t)) \leq (\varphi^*)'_{\varphi'(a)}(\varphi'_a(a)) \leq 2K'_* a \leq 2K'_* K' t.$$

This proves the assertion. □

Lemma 5.8. *Let φ^* be an N-function satisfying the Δ_2-condition. Then for all $a, t \geq 0$,*

$$(\varphi^*)'_{\varphi'(a)}(\varphi'_a(t)) \geq \frac{1}{2K'_*}\, t.$$

Proof. For $t \geq a$ we have $\varphi'(t) \geq \varphi'(a)$. Applying Lemma 5.4 with $M = 1$ first for φ'_a and then once more for $(\varphi^*)'_{\varphi'(a)}$ and (5.5) we obtain

$$\begin{aligned}(\varphi^*)'_{\varphi'(a)}(\varphi'_a(t)) &\geq (\varphi^*)'_{\varphi'(a)}\Big(\frac{1}{2}\varphi'(t)\Big) \geq \frac{1}{K'_*}(\varphi^*)'_{\varphi'(a)}(\varphi'(t)) \\ &\geq \frac{1}{2K'_*}(\varphi^*)'(\varphi'(t)) \geq \frac{1}{2K'_*}\, t.\end{aligned}$$

For $t \leq a$ we have $\varphi'(a)t/a \leq \varphi'(a)$. Lemma 5.5 applied with $M = 1$ first for φ'_a and then again for $(\varphi^*)'_{\varphi'(a)}$ together with (5.5) yield

$$\begin{aligned}(\varphi^*)'_{\varphi'(a)}\big(\varphi'_a(t)\big) &\geq (\varphi^*)'_{\varphi'(a)}\Big(\frac{\varphi'(a)t}{2a}\Big) \geq \frac{1}{K'_*}(\varphi^*)'_{\varphi'(a)}\Big(\frac{\varphi'(a)t}{a}\Big) \\ &\geq \frac{1}{K'_*}\,\frac{(\varphi^*)'(\varphi'(a))}{2\varphi'(a)}\,\frac{\varphi'(a)t}{a} \geq \frac{1}{2K'_*}\, t.\end{aligned}$$

□

Lemma 5.9. *Let φ^* be an N-function such that φ and φ^* satisfy the Δ_2-condition. Then for all $a, u \geq 0$,*

$$\frac{1}{2K'_*K'}\,(\varphi^*)'_{\varphi'(a)}(u) \leq ((\varphi_a)^*)'(u) \leq 2K'_*(\varphi^*)'_{\varphi'(a)}(u), \tag{5.24}$$

$$\frac{1}{2K'_*K'}\,(\varphi^*)_{\varphi'(a)}(u) \leq (\varphi_a)^*(u) \leq 2K'_*(\varphi^*)_{\varphi'(a)}(u). \tag{5.25}$$

Proof. Setting $t = ((\varphi_a)^*)'(u) - \varepsilon$ in Lemma 5.8 and using (5.5) we have

$$((\varphi_a)^*)'(u) - \varepsilon \leq 2K'_*(\varphi^*)'_{\varphi'(a)}\big(\varphi'_a\big(((\varphi_a)^*)'(u) - \varepsilon\big)\big) \leq 2K'_*(\varphi^*)'_{\varphi'(a)}(u).$$

The limit $\varepsilon \to 0$ implies the second inequality in (5.24). By setting $t = ((\varphi_a)^*)'(u)$ in Lemma 5.7 and using (5.5) we obtain

$$(\varphi^*)'_{\varphi'(a)}(u) \leq (\varphi^*)'_{\varphi'(a)}\big(\varphi'_a\big(((\varphi_a)^*)'(u)\big)\big) \leq 2K'_*K'((\varphi_a)^*)'(u),$$

which yields the first inequality in (5.24). From (5.24) and (5.4) we obtain (5.25). □

Lemma 5.10. *Let φ^* be an N-function such that φ and φ^* satisfy the Δ_2-condition. Then*

$$(\varphi_a)^*(2t) \leq 16(K'_*)^2(K')^2(\varphi_a)^*(t) \tag{5.26}$$

for all $a, t \geq 0$, i.e., the Δ_2-constants of $(\varphi_a)^*$ are bounded uniformly for $a \geq 0$, depending only on K'_* and K'.

Proof. This follows immediately from Lemma 5.9 and (5.22). □

Now we want to investigate how shifted N-functions behave if the shift is changed. In view of the applications we derive tensor-valued versions.

Lemma 5.11. *Let φ be an N-function satisfying the Δ_2-condition. Then for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$,*

$$\frac{1}{2K'}\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) \leq \varphi'_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|) \leq 2K'\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|),$$

$$\frac{1}{8(K')^2}\,\varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) \leq \varphi_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|) \leq 8(K')^2\varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|).$$

Proof. The assertion is clear for $\mathbf{P} = \mathbf{Q}$. Thus we can assume $|\mathbf{P} - \mathbf{Q}| > 0$. From the obvious estimates $\frac{1}{2}(|\mathbf{Q}| + |\mathbf{P} - \mathbf{Q}|) \le |\mathbf{P}| + |\mathbf{P} - \mathbf{Q}| \le 2(|\mathbf{Q}| + |\mathbf{P} - \mathbf{Q}|)$ we get

$$\begin{aligned}\varphi'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|) &= \varphi'(|\mathbf{P}| + |\mathbf{P} - \mathbf{Q}|)\frac{|\mathbf{P} - \mathbf{Q}|}{|\mathbf{P}| + |\mathbf{P} - \mathbf{Q}|} \\ &\le K'\varphi'(|\mathbf{Q}| + |\mathbf{P} - \mathbf{Q}|)\frac{2|\mathbf{P} - \mathbf{Q}|}{|\mathbf{Q}| + |\mathbf{P} - \mathbf{Q}|} \\ &= 2K'\varphi'_{|\mathbf{Q}|}(|\mathbf{P} - \mathbf{Q}|).\end{aligned}$$

The assertion now follows by the symmetry, by the use of (5.16) for $\varphi_{|\mathbf{Q}|}$ and of (5.22). □

Remark 5.12. Applying Lemma 5.11 to $\mathbf{P} = |\mathbf{P}|\mathbf{G}$ and $\mathbf{Q} = |\mathbf{Q}|\mathbf{G}$ where $|\mathbf{G}| = 1$ we obtain that

$$\frac{1}{2K'}\varphi'_{|\mathbf{P}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big) \le \varphi'_{|\mathbf{Q}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big) \le 2K'\varphi'_{|\mathbf{P}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big),$$

$$\begin{aligned}\frac{1}{8(K')^2}\varphi_{|\mathbf{P}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big) &\le \varphi_{|\mathbf{Q}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big) \\ &\le 8(K')^2\varphi_{|\mathbf{P}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big).\end{aligned} \tag{5.27}$$

Lemma 5.13. *Let φ be an N-function satisfying the Δ_2-condition. Then for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ and all $t \ge 0$,*

$$\varphi'_{|\mathbf{P}|}(t) \le 2K'\varphi'_{|\mathbf{Q}|}(t) + \varphi'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|). \tag{5.28}$$

Proof. For $|\mathbf{P} - \mathbf{Q}| \le t$ we have $0 \le \frac{1}{2}(|\mathbf{Q}| + t) \le |\mathbf{P}| + t \le 2(|\mathbf{Q}| + t)$. Hence,

$$\varphi'_{|\mathbf{P}|}(t) = \frac{\varphi'(|\mathbf{P}| + t)}{|\mathbf{P}| + t}\,t \le \frac{\varphi'(2(|\mathbf{Q}| + t))}{\frac{1}{2}(|\mathbf{Q}| + t)}\,t \le 2K'\,\frac{\varphi'(|\mathbf{Q}| + t)}{|\mathbf{Q}| + t}\,t = 2K'\varphi'_{|\mathbf{Q}|}(t).$$

For $|\mathbf{P} - \mathbf{Q}| \ge t$ we have

$$\varphi'_{|\mathbf{P}|}(t) \le \varphi'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|)$$

since $\varphi'_{|\mathbf{P}|}$ is non-decreasing. Combining both cases we obtain (5.28). □

Remark 5.14. From Lemmas 5.11 and 5.13 it follows that for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ and all $t \geq 0$ we have

$$\varphi'_{|\mathbf{P}|}(t) \leq 2K'\varphi'_{|\mathbf{Q}|}(t) + 2K'\varphi'_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|).$$

Similarly as in Remark 5.12 we obtain that also

$$\varphi'_{|\mathbf{P}|}(t) \leq 2K'\varphi'_{|\mathbf{Q}|}(t) + \varphi'_{|\mathbf{P}|}\big(\big||\mathbf{P}|-|\mathbf{Q}|\big|\big), \tag{5.29}$$

$$\varphi'_{|\mathbf{P}|}(t) \leq 2K'\varphi'_{|\mathbf{Q}|}(t) + 2K'\varphi'_{|\mathbf{Q}|}\big(\big||\mathbf{P}|-|\mathbf{Q}|\big|\big). \tag{5.30}$$

Lemma 5.15 (Change of shift). *Let φ be an N-function such that φ and φ^* satisfy the Δ_2-condition. Then for all $\delta \in (0,1)$, $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ and $t \geq 0$, the inequalities*

$$\varphi_{|\mathbf{P}|}(t) \leq \big(1 + 2(4K')^M\big)\varphi_{|\mathbf{Q}|}(t) + \delta\varphi_{|\mathbf{Q}|}\big(\big||\mathbf{P}|-|\mathbf{Q}|\big|\big), \tag{5.31}$$

$$\varphi_{|\mathbf{P}|}(t) \leq \big(1 + 2(4K')^M\big)\varphi_{|\mathbf{Q}|}(t) + \delta\varphi_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|) \tag{5.32}$$

hold, where $M \in \mathbb{N}$ is such that $8(K')^2/\delta \leq 2^M$.

Proof. By (5.11) and (5.30) we have

$$\varphi_{|\mathbf{P}|}(t) \leq \varphi'_{|\mathbf{P}|}(t)t \leq 2K'\varphi'_{|\mathbf{Q}|}(t)t + 2K'\varphi'_{|\mathbf{Q}|}\big(\big||\mathbf{P}|-|\mathbf{Q}|\big|\big)t =: I_1 + I_2.$$

For I_2 we deduce from (5.7), (5.18), (5.17) and (5.22) that

$$\begin{aligned} I_2 &\leq (\varphi_{|\mathbf{Q}|})^*\Big(\frac{\delta}{4K'}\varphi'_{|\mathbf{Q}|}\big(\big||\mathbf{P}|-|\mathbf{Q}|\big|\big)\Big) + \varphi_{|\mathbf{Q}|}\Big(\frac{8K'K'}{\delta}t\Big) \\ &\leq \delta\varphi_{|\mathbf{Q}|}\big(\big||\mathbf{P}|-|\mathbf{Q}|\big|\big) + (4K')^M\varphi_{|\mathbf{Q}|}(t), \end{aligned}$$

where $M \in \mathbb{N}$ is such that $8(K')^2/\delta \leq 2^M$. Further we have

$$I_1 \leq \varphi^*_{|\mathbf{Q}|}\Big(\frac{1}{4K'}\varphi'_{|\mathbf{Q}|}(t)\Big) + \varphi_{|\mathbf{Q}|}\big(8(K')^2t\big) \leq \varphi_{|\mathbf{Q}|}(t) + (4K')^M\varphi_{|\mathbf{Q}|}(t).$$

This proves (5.31). The inequality (5.32) follows immediately from (5.31). □

Remark 5.16. One can prove in the same way, using additionally (5.27) in the estimate of I_2, that if φ is an N-function such that φ and φ^* satisfy the Δ_2-condition, then for all $\delta \in (0,1)$, all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ and all $t \geq 0$ the inequalities

$$\varphi_{|\mathbf{P}|}(t) \leq \big(1 + 2(4K')^M\big)\varphi_{|\mathbf{Q}|}(t) + \delta\varphi_{|\mathbf{P}|}\big(\big||\mathbf{P}|-|\mathbf{Q}|\big|\big),$$

$$\varphi_{|\mathbf{P}|}(t) \leq \big(1 + 2(4K')^M\big)\varphi_{|\mathbf{Q}|}(t) + \delta\varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)$$

hold, where $M \in \mathbb{N}$ is now such that $64(K')^4/\delta \leq 2^M$.

In order to extend Lemma 5.13 and Lemma 5.15 and its versions in the remarks to complementary shifted N-functions we need an additional assumption, namely that there exists a constant γ_0 such that for all $s, t \geq 0$,

$$\big|\varphi'(s+t) - \varphi'(t)\big| \leq \gamma_0 \varphi_t'(s). \tag{5.33}$$

This assumption is closely related to the continuity properties of the operator possessing an N-potential φ (cf. (6.1) and Remark 6.9) and can be also written in a more conventional way as

$$\big|\varphi'(s+t) - \varphi'(t)\big| \leq \gamma_0 \, \frac{\varphi'\big(s+t\big)}{s+t}\, s.$$

In particular, these assumptions imply that φ' is continuous on $\mathbb{R}^{\geq 0}$ and locally Lipschitz-continuous on $(0,\infty)$.

Lemma 5.17. *Let φ^* be an N-function such that φ and φ^* satisfy the Δ_2-condition and let φ' satisfy* (5.33)*. Then the inequalities*

$$((\varphi_{|\mathbf{P}|})^*)'(t) \leq 8K'(K_*')^3((\varphi_{|\mathbf{Q}|})^*)'(t) + K'(2K_*')^{L+2}\big||\mathbf{P}| - |\mathbf{Q}|\big|, \tag{5.34}$$

$$((\varphi_{|\mathbf{P}|})^*)'(t) \leq 8K'(K_*')^3((\varphi_{|\mathbf{Q}|})^*)'(t) + K'(2K_*')^{L+2}|\mathbf{P} - \mathbf{Q}| \tag{5.35}$$

hold for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ and $t \geq 0$, where $L \in \mathbb{N}$ is such that $\gamma_0 < 2^L$.

Proof. Choose $L \in \mathbb{N}$ such that $\gamma_0 < 2^L$. Using Lemma 5.9, the estimate (5.29) for $(\varphi^*)'_{\varphi'(|\mathbf{P}|)}(t)$, (5.33), Lemma 5.9, the Δ_2-condition for $((\varphi_{|\mathbf{P}|})^*)'$, (5.22) and (5.5) we obtain

$$\begin{aligned}
((\varphi_{|\mathbf{P}|})^*)'(t) &\leq 2K_*'(\varphi^*)'_{\varphi'(|\mathbf{P}|)}(t)\\
&\leq 4(K_*')^2(\varphi^*)'_{\varphi'(|\mathbf{Q}|)}(t) + 2K_*'(\varphi^*)'_{\varphi'(|\mathbf{P}|)}\big(\big|\varphi'(|\mathbf{P}|) - \varphi'(|\mathbf{Q}|)\big|\big)\\
&\leq 4(K_*')^2(\varphi^*)'_{\varphi'(|\mathbf{Q}|)}(t) + 2K_*'(\varphi^*)'_{\varphi'(|\mathbf{P}|)}\big(\gamma_0\, \varphi'_{|\mathbf{P}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big)\big)\\
&\leq 8K'(K_*')^3((\varphi_{|\mathbf{Q}|})^*)'(t) + 4K'(K_*')^2((\varphi_{|\mathbf{P}|})^*)'\big(\gamma_0\, \varphi'_{|\mathbf{P}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big)\big)\\
&\leq 8K'(K_*')^3((\varphi_{|\mathbf{Q}|})^*)'(t) + K'(2K_*')^{L+2}\big||\mathbf{P}| - |\mathbf{Q}|\big|.
\end{aligned}$$

Inequality (5.35) follows immediately from (5.34). □

Lemma 5.18 (Change of shift)**.** *Let φ be an N-function such that φ and φ^* satisfy the Δ_2-condition and let φ' satisfy* (5.33)*. Then the inequalities*

$$(\varphi_{|\mathbf{P}|})^*(t) \leq \big(4K_*' + (4K_*')^I + (4K_*')^J\big)(\varphi_{|\mathbf{Q}|})^*(t) + \delta\varphi_{|\mathbf{Q}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big),$$

$$(\varphi_{|\mathbf{P}|})^*(t) \leq \big(4K_*' + (4K_*')^I + (4K_*')^J\big)(\varphi_{|\mathbf{Q}|})^*(t) + \delta\varphi_{|\mathbf{Q}|}(|\mathbf{P} - \mathbf{Q}|)$$

hold for all $\delta \in (0,1)$, $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ and $t \geq 0$, where $I, J \in \mathbb{N}$ are such that $8K'(K_*')^3 \leq 2^I$ and $\delta^{-1}K'(2K_*')^{L+2} \leq 2^J$ with L from Lemma 5.17.

Proof. We proceed as in the proof of Lemma 5.15. □

The following lemma quantifies the known fact that an N-function growths faster than linearly.

Lemma 5.19. *Let φ be an N-function such that φ and φ^* satisfy the Δ_2-condition. Then for all $\kappa \in \big(\frac{1}{2}(\log_{K_*}(\frac{3}{4}K_*)+1),1\big)$ there exists an N-function ρ^κ such that for all $t \geq 0$*

$$\rho^\kappa(t) \leq (\varphi(t))^\kappa \leq \frac{K_*^4}{4} K^{\kappa M} \rho^\kappa(t) \tag{5.36}$$

holds, where $M \in \mathbb{N}$ is such that $K_*^2/2 \leq 2^M$. Moreover, ρ^κ satisfies the Δ_2-condition, i.e.,

$$\rho^\kappa(2t) \leq \frac{K_*^4}{4} K^{\kappa(1+M)} \rho^\kappa(t)$$

holds for all $t \geq 0$. Let κ_0 be such that $(K_*/2)^{1-\kappa_0} = 2$. Then for all $\kappa \in (\kappa_0, 1)$ the complementary function $(\rho^\kappa)^*$ also satisfies the Δ_2-condition, i.e., for all $t \geq 0$,

$$(\rho^\kappa)^*(2t) \leq 2\left(\frac{K_*}{2}\right)^m (\rho^\kappa)^*(t),$$

where $m \in \mathbb{N}$ is such that $(K_*^4/2K)^{\kappa M}\big(2^{-1/2}(K_*/2)^{(1-\kappa)/2}\big)^m \leq 1$.

Proof. In [35, Theorem 1.4.2] it is shown that the Δ_2-condition for φ^*, i.e. $\varphi^*(2t) \leq K_*\varphi^*(t)$, is equivalent to the condition

$$\varphi(t) \leq \frac{1}{K_*}\,\varphi\left(\frac{K_*}{2}\,t\right). \tag{5.37}$$

The proofs of [33, Lemma 1.2.2, Lemma 1.2.3] imply that for all $0 < t_1 < t_2$ and all $\alpha \in \big(\log_{K_*}(\frac{3}{4}K_*),1\big)$

$$\frac{(\varphi(t_1))^\alpha}{t_1} \leq \frac{K_*^2}{2}\,\frac{\Big(\varphi\Big(\frac{K_*^2}{2}\,t_2\Big)\Big)^\alpha}{t_2}.$$

This in turn implies due to [33, Lemma 1.1.1] that there exists a convex function ρ^α such that

$$\rho^\alpha(t) \leq (\varphi(t))^\alpha \leq \frac{K_*^2}{2}\,\rho^\alpha\left(\frac{K_*^2}{2}\,t\right)$$

for all $t \geq 0$. Since φ satisfies the Δ_2-condition this yields

$$\rho^\alpha(t) \leq (\varphi(t))^\alpha \leq \frac{K_*^4}{4}\,K^{\alpha M}\rho^\alpha(t), \tag{5.38}$$

where $M \in \mathbb{N}$ is such that $K_*^2 \le 2^{M+1}$. If α is in the above interval then $\kappa := (1+\alpha)/2$ lies in the same interval. Using (5.38) once for κ and once for α, the convexity of ρ^α and that fact that φ is an N-function, we deduce for $t \to 0$

$$\begin{aligned}\frac{\rho^\kappa(t)}{t} &\le \frac{(\varphi(t))^{(1+\alpha)/2}}{t} \le \sqrt{\frac{\varphi(t)}{t}}\sqrt{\frac{(\varphi(t))^\alpha}{t}}\\ &\le \sqrt{\frac{\varphi(t)}{t}}\,\frac{K_*^2}{2}\sqrt{K^{\alpha M}}\sqrt{\rho^\alpha(1)} \to 0,\end{aligned}$$

and for $t \to \infty$

$$\frac{\rho^\kappa(t)}{t} \ge \frac{1}{K^{\alpha M}}\frac{4}{K_*^4}\sqrt{\frac{\varphi(t)}{t}}\sqrt{\frac{(\varphi(t))^\alpha}{t}} \ge \frac{1}{K^{\alpha M}}\frac{4}{K_*^4}\sqrt{\frac{\varphi(t)}{t}}\sqrt{\rho^\alpha(1)} \to \infty.$$

This proves that ρ^κ is an N-function. From (5.38) we also deduce

$$\rho^\kappa(2t) \le (\varphi(2t))^\kappa \le K^\kappa(\varphi(t))^\kappa \le \frac{K_*^4}{4}K^{\kappa(1+M)}\rho^\kappa(t),$$

which shows that ρ^κ satisfies the Δ_2-condition. In order to verify that $(\rho^\kappa)^*$ satisfies the Δ_2-condition we check the equivalent condition that there exists $a > 1$ such that $\rho^\kappa(t) \le \rho^\kappa(at)/(2a)$ for all $t \ge 0$. Since φ^* satisfies the Δ_2-condition, using the convexity of φ and (5.38), we get for $a := K_*/2$

$$\begin{aligned}\rho^\kappa(t) &\le \sqrt{\varphi(t)}\sqrt{(\varphi(t))^\alpha} \le \sqrt{\frac{\varphi(a^m t)}{(2a)^m}}\sqrt{\frac{(\varphi(a^m t))^\alpha}{a^{\alpha m}}}\\ &\le \frac{K_*^4}{2}K^{\kappa M}\left(\frac{a^{(1-\kappa)/2}}{\sqrt{2}}\right)^m \frac{\rho^\kappa(a^m t)}{2a^m} \le \frac{\rho^\kappa(a^m t)}{2a^m}.\end{aligned}$$

In order to justify the last step we proceed as follows: First we choose κ_0 such that $a^{1-\kappa_0} = 2$. For $\kappa \in (\kappa_0, 1)$ we then choose $m \in \mathbb{N}$ large enough such that $K_*^4\, K^{\kappa M}\left(a^{(1-\kappa)/2}/\sqrt{2}\right)^m/2 \le 1$. □

Complementarily to condition (5.37) we give an equivalent condition for $(\varphi^*)'$ to satisfy the Δ_2-condition in terms of φ' only.

Lemma 5.20. *Let φ be an N-function. Then $(\varphi^*)'$ satisfies the Δ_2-condition, i.e., $(\varphi^*)'(2t) \le K_*'(\varphi^*)'(t)$ for all $t \ge 0$ if and only if*

$$2\varphi'(t) \le \varphi'(K_*'t)$$

holds for all $t \ge 0$.

Proof. Assume that $(\varphi^*)'(2t) \le K_*'(\varphi^*)'(t)$ for all $t \ge 0$. Using $\varphi' = ((\varphi^*)')^{-1}$ we obtain for all $t \ge 0$

$$\begin{aligned}2\varphi'(t) &= 2\sup\{u \mid (\varphi^*)'(u) \le t\}\\ &= \sup\{2u \mid K_*'(\varphi^*)'(u) \le K_*'t\}\\ &\le \sup\{2u \mid (\varphi^*)'(2u) \le K_*'t\}\\ &= \sup\{u \mid (\varphi^*)'(u) \le K_*'t\}\\ &= \varphi'(K_*'t).\end{aligned}$$

Assume that $2\varphi'(t) \le \varphi'(K_*'t)$ for all $t \ge 0$. Then we obtain for all $t \ge 0$

$$\begin{aligned}(\varphi^*)'(2t) &= \sup\{u \mid \varphi'(u) \le 2t\}\\ &= \sup\{K_*'u \mid \varphi'(K_*'u) \le 2t\}\\ &\le \sup\{K_*'u \mid 2\varphi'(u) \le 2t\}\\ &= K_*'(\varphi^*)'(t).\end{aligned}$$

This proves the lemma. □

Let us finish this section with some improvements of (5.18).

Lemma 5.21. *Let φ be an N-function satisfying the Δ_2-condition. Then*

$$\varphi_a(\delta a) \le \delta^2 KK'\varphi(a) \tag{5.39}$$

holds for all $\delta \in [0,1]$ and $a \ge 0$.

Proof. We have

$$\varphi_a(\delta a) \le \delta a\varphi_a'(\delta a) = \varphi'(a+\delta a)\frac{\delta^2 a^2}{a+\delta a} \le \delta^2 K'\varphi'(a)a \le \delta^2 KK'\varphi(a).$$

□

Lemma 5.22. *Let φ be an N-function such that φ and φ^* satisfy the Δ_2-condition. Then*

$$(\varphi_a)^*(\delta\varphi'(a)) \le \delta^2 2K'KK_*'K_*\varphi(a).$$

holds for all $\delta \in [0,1]$ and $a \ge 0$.

Proof. Using (5.39) and (5.17), we have

$$\begin{aligned}(\varphi_a)^*(\delta\varphi'(a)) &\le 2K'(\varphi)^*_{\varphi'(a)}(\delta\varphi'(a))\\ &\le \delta^2 2K'K_*'K_*\varphi^*(\varphi'(a))\\ &\le \delta^2 2K'KK_*'K_*\varphi(a).\end{aligned}$$

□

Lemma 5.23. *Let φ be an N-function such that φ and φ^* satisfy the Δ_2-condition. Then*

$$\varphi(\delta t) \le \left(\frac{K_*^4}{4}\right)^{1/\kappa} K^M \delta^{1/\kappa} \varphi(t)$$

for all $\kappa \in \big(2^{-1}(\log_{K_*}(3\,K_*/4)+1),1\big)$, $t \ge 0$ and $\delta \in [0,1]$, where $M \in \mathbb{N}$ is such that $K_*^2/2 \le 2^M$. In particular, for all $\kappa \in \big(2^{-1}(\log_{\tilde{K}_*}(3\,\tilde{K}_*/4)+1),1\big)$, $t, a \ge 0$ and $\delta \in [0,1]$, we have

$$\varphi_a(\delta t) \le \left(\frac{\tilde{K}_*^4}{4}\right)^{1/\kappa} 2^M K^{2M} \delta^{1/\kappa} \varphi_a(t), \tag{5.40}$$

where $\tilde{K}_* = 16(K_*')^2(K')^2$ (cf. (5.26)).

Proof. From (5.36) it follows for all $t \ge 0$ and $\delta \in [0,1]$ that

$$\begin{aligned}\varphi(\delta t) &\le \left(\frac{K_*^4}{4}\right)^{1/\kappa} K^M \big(\rho^\kappa(\delta t)\big)^{1/\kappa} \\ &\le \left(\frac{K_*^4}{4}\right)^{1/\kappa} K^M \big(\delta \rho^\kappa(t)\big)^{1/\kappa} \\ &\le \left(\frac{K_*^4}{4}\right)^{1/\kappa} K^M\, \delta^{1/\kappa}\, \varphi(t),\end{aligned}$$

where we have used the convexity of ρ^κ and $\rho^\kappa(0) = 0$. Now, (5.40) follows from Lemma 5.10 and (5.22). □

6. Problems with N-potential

In this section we want to derive with the help of shifted N-functions some useful results for problems with the N-potential φ. We say that the operator $\mathbf{A}$ possesses an *N-potential* φ, where φ is some N-function, if $\mathbf{A}(\mathbf{0}) = \mathbf{0}$ and for all $\mathbf{P} \in \mathbb{R}^{N\times n} \setminus \{\mathbf{0}\}$ holds

$$\mathbf{A}(\mathbf{P}) = \mathbf{A}_\varphi(\mathbf{P}) := \frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|}\,\mathbf{P}. \tag{6.1}$$

We also consider the more general situation, where $\mathbf{A}$ has no N-potential. We say that the operator $\mathbf{A}$ has a *φ-structure*, where φ is an N-function, if $\mathbf{A}(\mathbf{0}) = \mathbf{0}$ and there exist constants $\gamma_1, \gamma_2 > 0$ such that for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$,

$$(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) \ge \gamma_1\, \varphi'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|)|\mathbf{P} - \mathbf{Q}|, \tag{6.2}$$

$$|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})| \le \gamma_2\, \varphi'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|). \tag{6.3}$$

From (6.2) and (6.3) one deduces by the Cauchy–Schwarz inequality that

$$\gamma_1\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)\le|\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q})|\le\gamma_2\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|),\tag{6.4}$$

$$\begin{aligned}\gamma_1\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}|&\le(\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q})\\&\le\gamma_2\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}|.\end{aligned}\tag{6.5}$$

If $\mathbf{A}$ has an N-potential φ or a φ-structure, we say that the associated elliptic problem

$$\begin{aligned}-\operatorname{div}(\mathbf{A}(\nabla\mathbf{v}))&=\mathbf{f}\quad\text{in }\Omega,\\\mathbf{v}&=\mathbf{0}\quad\text{on }\partial\Omega,\end{aligned}$$

has a *φ-structure.*

There are not many investigations of problems with a general φ-structure. E.g., the contributions in [8], [22], [29], [37], [57] are devoted to the study of PDEs in Orlicz–Sobolev spaces. However, the usage of shifted N-functions in this context seems to be new.

Let $\mathbf{A}$ be an operator with an N-potential φ. We want to investigate under which conditions on φ the operator $\mathbf{A}$ has a φ-structure. Let us start with the following assumption:

Assumption 6.1. Let φ be an N-function such that φ and its complementary function φ^* satisfy the Δ_2-condition. Further, assume that φ is C^1 on $[0,\infty)$ and C^2 on $(0,\infty)$ and that there exist constants $0<\gamma_3\le 1$, $\gamma_4>0$ such that for all $t>0$

$$\gamma_3\varphi'(t)\le t\varphi''(t)\le\gamma_4\varphi'(t).\tag{6.6}$$

Remark 6.2. Note that under Assumption 6.1 we have that $\varphi''(t)>0$ for all $t>0$. Thus, $\varphi'(t)$ is strictly increasing and $(\varphi^*)'(t)=(\varphi')^{-1}(t)$ is the inverse function of $\varphi'(t)$.

Lemma 6.3. *Let φ satisfy Assumption 6.1. Then for all $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{N\times n}$ with $|\mathbf{P}|+|\mathbf{Q}|>0$,*

$$\frac{\gamma_3}{2K'}\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)\le\varphi''(|\mathbf{P}|+|\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}|\le 2K'\gamma_4\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|).$$

Proof. Using $\frac12(|\mathbf{P}|+|\mathbf{Q}|)\le|\mathbf{P}|+|\mathbf{P}-\mathbf{Q}|\le 2(|\mathbf{P}|+|\mathbf{Q}|)$ and (6.6) we obtain

$$\begin{aligned}\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)&=\frac{\varphi'(|\mathbf{P}|+|\mathbf{P}-\mathbf{Q}|)}{|\mathbf{P}|+|\mathbf{P}-\mathbf{Q}|}\,|\mathbf{P}-\mathbf{Q}|\\&\le 2K'\,\frac{\varphi'(|\mathbf{P}|+|\mathbf{Q}|)}{|\mathbf{P}|+|\mathbf{Q}|}\,|\mathbf{P}-\mathbf{Q}|\\&\le\frac{2K'}{\gamma_3}\,\varphi''(|\mathbf{P}|+|\mathbf{Q}|)\,|\mathbf{P}-\mathbf{Q}|.\end{aligned}$$

The second inequality follows analogously. □

Thus, if φ satisfies Assumption 6.1, conditions (6.2) and (6.3) can be also formulated with $\varphi''(|\mathbf{P}|+|\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}|$ instead of $\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)$.

Lemma 6.4. *Let φ satisfy Assumption 6.1. Then also φ^* statisfies Assumption 6.1. In particular, for all $t>0$*

$$\frac{1}{\gamma_4}\,(\varphi^*)'(t) \le t(\varphi^*)''(t) \le \frac{1}{\gamma_3}\,(\varphi^*)'(t).$$

Proof. Since φ' is continuous on $[0,\infty)$ and increasing, we deduce from the definition of $(\varphi^*)'$ that it also has these properties. From $(\varphi^*)'(t)=(\varphi')^{-1}(t)$ and the theorem on the derivative of inverse functions it follows that

$$(\varphi^*)''(t) = \frac{1}{\varphi''((\varphi^*)'(t))}, \tag{6.7}$$

i.e., φ^* is C^2 on $(0,\infty)$. Moreover, from (6.7), (6.6) and $(\varphi^*)'(t)=(\varphi')^{-1}(t)$, we deduce for $t>0$

$$(\varphi^*)''(t) \le \frac{1}{\gamma_3}\,\frac{(\varphi^*)'(t)}{\varphi'((\varphi^*)'(t))} = \frac{1}{\gamma_3}\,\frac{(\varphi^*)'(t)}{t}.$$

The second inequality in (6.6) follows analogously. □

We need some auxiliary results.

Lemma 6.5. *Let $\alpha>-1$ and $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{N\times n}$ with $|\mathbf{P}|+|\mathbf{Q}|>0$. Then*

$$c_1(\alpha)(|\mathbf{P}|+|\mathbf{Q}|)^\alpha \le \int_0^1 |\theta\mathbf{P}+(1-\theta)\mathbf{Q}|^\alpha\,d\theta \le c_2(\alpha)(|\mathbf{P}|+|\mathbf{Q}|)^\alpha$$

with

$$c_1(\alpha) := \min\Big\{\frac{1}{\alpha+1},\frac{2^{-\alpha}}{\alpha+1},2^{-\alpha}\Big\},\quad c_2(\alpha) := \max\Big\{\frac{1}{\alpha+1},\frac{2^{-\alpha}}{\alpha+1},2^{-\alpha}\Big\}.$$

The constants c_1 and c_2 are optimal.

Proof. This results is essentially contained in [28] and proved with optimal constants in [14]. □

Lemma 6.6. *Let φ be an N-function such that φ and φ^* satisfy the Δ_2-condition. Then*

$$\frac{1}{K^3}\,\frac{\varphi'(|\mathbf{P}|+|\mathbf{Q}|)}{|\mathbf{P}|+|\mathbf{Q}|} \le \int_0^1 \frac{\varphi'(|\theta\mathbf{P}+(1-\theta)\mathbf{Q}|)}{|\theta\mathbf{P}+(1-\theta)\mathbf{Q}|}\,d\theta \le \Big(\frac{K_*^4}{4}\Big)^{1/\kappa} K^{M+2} c_2\,\frac{\varphi'(|\mathbf{P}|+|\mathbf{Q}|)}{|\mathbf{P}|+|\mathbf{Q}|}$$

holds for all $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{N\times n}$ with $|\mathbf{P}|+|\mathbf{Q}|>0$, where κ and M are taken from Lemma 5.19, and $C-2=c_2(\kappa)$.

Proof. From (5.11), the convexity of φ, Lemma 6.5 with $\alpha=1$, and again (5.11) we derive

$$\begin{aligned}\int_0^1 \frac{\varphi'(|\theta\mathbf{P}+(1-\theta)\mathbf{Q}|)}{|\theta\mathbf{P}+(1-\theta)\mathbf{Q}|}\,d\theta &\ge \int_0^1 \frac{\varphi(|\theta\mathbf{P}+(1-\theta)\mathbf{Q}|)}{(|\mathbf{P}|+|\mathbf{Q}|)^2}\,d\theta\\ &\ge \frac{1}{(|\mathbf{P}|+|\mathbf{Q}|)^2}\,\varphi\big(\textstyle\int_0^1|\theta\mathbf{P}+(1-\theta)\mathbf{Q}|\,d\theta\big)\\ &\ge \frac{\varphi\big(\frac14(|\mathbf{P}|+|\mathbf{Q}|)\big)}{(|\mathbf{P}|+|\mathbf{Q}|)^2}\\ &\ge \frac{\varphi\big(2(|\mathbf{P}|+|\mathbf{Q}|)\big)}{K^3(|\mathbf{P}|+|\mathbf{Q}|)^2}\\ &\ge \frac{\varphi'(|\mathbf{P}|+|\mathbf{Q}|)}{K^3(|\mathbf{P}|+|\mathbf{Q}|)}.\end{aligned}$$

This proves the first part. Due to Lemma 5.19 and (5.16) we have

$$\begin{aligned}\varphi'(t) &\le K\,\frac{\varphi(t)}{t}\\ &\le \Big(\frac{K_*^4}{4}\Big)^{1/\kappa} K^{M+1}\,\frac{(\rho^\kappa(t))^{1/\kappa}}{t}\\ &\le \Big(\frac{K_*^4}{4}\Big)^{1/\kappa} K^{M+1} t^{1/\kappa-1}\big((\rho^\kappa)'(t)\big)^{1/\kappa},\end{aligned}$$

from which, by the monotonicity of $(\rho^\kappa)'$, Lemma 6.5 with $\alpha := 1/\kappa-2$, (5.11) and Lemma 5.19, it follows

$$\begin{aligned}
&\int_0^1 \frac{\varphi'(|\theta\mathbf{P}+(1-\theta)\mathbf{Q}|)}{|\theta\mathbf{P}+(1-\theta)\mathbf{Q}|}\,d\theta \\
&\quad\le \Big(\frac{K_*^4}{4}\Big)^{1/\kappa} K^{M+1}\int_0^1 \big((\rho^\kappa)'(|\mathbf{P}|+|\mathbf{Q}|)\big)^{1/\kappa}\big|\theta\mathbf{P}+(1-\theta)\mathbf{Q}\big|^{1/\kappa-2}\,d\theta \\
&\quad\le \Big(\frac{K_*^4}{4}\Big)^{1/\kappa} K^{M+1}c_2\big((\rho^\kappa)'(|\mathbf{P}|+|\mathbf{Q}|)\big)^{1/\kappa}\big(|\mathbf{P}|+|\mathbf{Q}|\big)^{1/\kappa-2} \\
&\quad\le \Big(\frac{K_*^4}{4}\Big)^{1/\kappa} K^{M+2}c_2\frac{\varphi'(|\mathbf{P}|+|\mathbf{Q}|)}{|\mathbf{P}|+|\mathbf{Q}|}.
\end{aligned}$$

This proves the lemma. □

Lemma 6.7. *Let* $\mathbf{A}$ *have the N-potential* φ *and let* φ *satisfy Assumption 6.1. Then* $\mathbf{A}$ *has the* φ*-structure. In particular,* $\mathbf{A}$ *satisfies the inequalities* (6.2) *and* (6.3) *with* $\gamma_1 = 2\gamma_3/(K^3K')$, $\gamma_2 = 2(2+\gamma_4)(n+N)(K_*^4/4)^{1/\kappa}K^{M+2}c_2K'$, *where* $\kappa \in \big(2^{-1}(\log_{K_*}(3\,K_*/4)+1),1\big)$ *and* $M\in\mathbb{N}$ *is such that* $K_*^2 \le 2^{M+1}$.

Proof. Note that

$$\frac{A_{jk}(\mathbf{P})}{\partial P_{lm}} = \frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|}\Big(\delta_{jk}\delta_{lm} - \frac{P_{jk}P_{lm}}{|\mathbf{P}|^2}\Big) + \varphi''(|\mathbf{P}|)\frac{P_{jk}}{|\mathbf{P}|}\frac{P_{lm}}{|\mathbf{P}|} \tag{6.8}$$

holds for all $\mathbf{P}\in\mathbb{R}^{N\times n}\setminus\{\mathbf{0}\}$ and all j, k, l, m. Especially, with (6.6) we obtain for all j, k, l, m

$$\Big|\frac{A_{jk}(\mathbf{P})}{\partial P_{lm}}\Big| \le 2\frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|} + \varphi''(|\mathbf{P}|) \le (2+\gamma_4)\frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|}. \tag{6.9}$$

Moreover, for all j,k we have

$$A_{jk}(\mathbf{P}) - A_{jk}(\mathbf{Q}) = \sum_{l,m}\int_0^1 \frac{A_{jk}\left([\mathbf{Q},\mathbf{P}]_\theta\right)}{\partial P_{lm}}(P_{lm}-Q_{lm})\,d\theta, \tag{6.10}$$

where $[\mathbf{Q},\mathbf{P}]_\theta := (1-\theta)\mathbf{Q}+\theta\mathbf{P}$. So by (6.9), Lemma 6.6, (6.6) and the inequality $\frac12(|\mathbf{P}|+|\mathbf{Q}|)\le|\mathbf{P}|+|\mathbf{P}-\mathbf{Q}|\le 2(|\mathbf{P}|+|\mathbf{Q}|)$ we deduce

$$\begin{aligned}
|\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q})| &\le (2+\gamma_4)(n+N)\int_0^1 \frac{\varphi'(|[\mathbf{Q},\mathbf{P}]_\theta|)}{|[\mathbf{Q},\mathbf{P}]_\theta|}\,d\theta|\mathbf{P}-\mathbf{Q}| \\
&\le (2+\gamma_4)(n+N)c_3\frac{\varphi'(|\mathbf{P}|+|\mathbf{Q}|)}{|\mathbf{P}|+|\mathbf{Q}|}\,|\mathbf{P}-\mathbf{Q}| \\
&\le 2(2+\gamma_4)(n+N)c_3K'\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|),
\end{aligned}$$

where $c_3 := (K_*^4/4)^{1/\kappa} K^{M+2} c_2$. From (6.6), (6.10) and $\gamma_3 \le 1$ we obtain for $\mathbf{G}, \mathbf{B} \in \mathbb{R}^{N\times n}$ with $\mathbf{G} \neq \mathbf{0}$,

$$\begin{aligned}\sum_{l,m,j,k} B_{jk} \frac{A_{jk}(\mathbf{G})}{\partial P_{lm}} B_{lm} &= \frac{\varphi'(|\mathbf{G}|)}{|\mathbf{G}|}\left(|\mathbf{B}|^2 - \frac{|\mathbf{BG}|^2}{|\mathbf{G}|^2}\right) + \varphi''(|\mathbf{G}|)\frac{|\mathbf{BG}|^2}{|\mathbf{G}|^2} \\ &\ge \gamma_3 \frac{\varphi'(|\mathbf{G}|)}{|\mathbf{G}|}\left(|\mathbf{B}|^2 - \frac{|\mathbf{BG}|^2}{|\mathbf{G}|^2}\right) + \gamma_3 \frac{\varphi'(|\mathbf{G}|)}{|\mathbf{G}|}\frac{|\mathbf{BG}|^2}{|\mathbf{G}|^2} \\ &\ge \gamma_3 \frac{\varphi'(|\mathbf{G}|)}{|\mathbf{G}|}|\mathbf{B}|^2.\end{aligned}$$

This, (6.10), Lemma 6.6, (5.11) and (5.16) imply

$$\begin{aligned}\big(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big)\cdot(\mathbf{P}-\mathbf{Q}) &\ge \gamma_3 \int_0^1 \frac{\varphi'([\mathbf{Q},\mathbf{P}]_\theta)}{|[\mathbf{P},\mathbf{Q}]_\theta|}|\mathbf{P}-\mathbf{Q}|^2\,d\theta \\ &\ge \frac{\gamma_3}{K^3}\,\frac{\varphi'(|\mathbf{P}|+|\mathbf{Q}|)}{|\mathbf{P}|+|\mathbf{Q}|}\,|\mathbf{P}-\mathbf{Q}|^2 \\ &\ge \frac{2\gamma_3}{K^3K'}\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)\,|\mathbf{P}-\mathbf{Q}|.\end{aligned}$$

This proves the lemma. □

We now want to establish the assertions of Lemma 6.7 under weaker requirements on φ than Assumption 6.1. For that we define another *shifted N-function* $\varphi_{[a]}$ by setting

$$\varphi'_{[a]}(t) := \varphi'(a+t) - \varphi'(a)$$

for all $a, t \ge 0$.

Lemma 6.8. *Let the operator* $\mathbf{A}$ *have the N-potential* φ *and the* φ*-structure. Then*

$$\gamma_1\varphi'_a(t) \le \varphi'_{[a]}(t) \le \gamma_2\varphi'_a(t) \tag{6.11}$$

holds for all $a, t \ge 0$.

Proof. Since $\mathbf{A}$ has N-potential φ we obtain for $\mathbf{Q} = (t+a)\mathbf{R}$, $\mathbf{P} = a\mathbf{R}$ with $|\mathbf{R}| = 1$ and the definition of $\varphi'_{[a]}$ that

$$\begin{aligned}\varphi'_{[a]}(t) &= |\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})|, \\ \varphi'_a(t) &= \varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|).\end{aligned}$$

The assertion now follows from (6.4). □

Remark 6.9. Note, that if the operator $\mathbf{A}$ has an N-potential φ and a φ-structure, then the constants γ_1, γ_2 have to satisfy $\gamma_1 \le 1 \le \gamma_2$. This follows from (6.11) by letting $a \to 0$.

If $\mathbf{A}$ possesses an N-potential φ, then the tensor-valued inequality (6.3) follows already from the scalar inequality (5.33). Indeed, if φ satisfies (5.33), we have for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$ with $|\mathbf{P}| \ge |\mathbf{Q}|$

$$\begin{aligned}
\big|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big| &= \left|\frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|}\mathbf{P} - \frac{\varphi'(|\mathbf{Q}|)}{|\mathbf{Q}|}\mathbf{Q}\right| \\
&\le \big|\varphi'(|\mathbf{P}|) - \varphi'(|\mathbf{Q}|)\big|\left|\frac{\mathbf{Q}}{|\mathbf{Q}|}\right| + \varphi'(|\mathbf{P}|)\left|\frac{\mathbf{P}}{|\mathbf{P}|} - \frac{\mathbf{Q}}{|\mathbf{Q}|}\right| \\
&\le \gamma_0\, \varphi'_{|\mathbf{Q}|}\big(\big||\mathbf{P}| - |\mathbf{Q}|\big|\big) + 4\varphi'(|\mathbf{P}|)\,\frac{|\mathbf{P}-\mathbf{Q}|}{|\mathbf{P}|+|\mathbf{Q}|} \\
&\le (8+\gamma_0)\,\varphi'_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|) \\
&\le 2K'(8+\gamma_0)\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|),
\end{aligned}$$

where we used Lemma 5.11. In the case $|\mathbf{P}| \le |\mathbf{Q}|$ one analogously shows that

$$\big|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big| \le (8+\gamma_0)\,\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|).$$

Note, that (5.33) can be written as:

$$\varphi'_{[a]}(t) \le \gamma_0 \varphi'_a(t), \quad \text{for all } t, a \ge 0.$$

Lemma 6.10. *Let φ be an N-function which satisfies* (6.11)*. Then φ is C^1 on $[0,\infty)$ and φ' is locally Lipschitz-continuous on $(0,\infty)$. Moreover, φ and φ^* satisfy the Δ_2-condition. If additionally φ is C^2 on $(0,\infty)$, then*

$$\gamma_1\varphi'(t) \le t\varphi''(t) \le \gamma_2\varphi'(t), \tag{6.12}$$

holds for all $t > 0$, i.e., (6.6) *is satisfied with $\gamma_3 = \gamma_1$ and $\gamma_4 = \gamma_2$. In particular, φ satisfies Assumption 6.1.*

Proof. From (6.11) follows for $0 < t \le a$,

$$\big|\varphi'(a+t) - \varphi'(a)\big| \le \gamma_2\,\frac{\varphi'(a+t)}{a+t}\,t \le \gamma_2\,\frac{\varphi'(2a)}{a}\,t,$$

which shows that φ' is locally Lipschitz-continuous on $(0,\infty)$. In particular, φ' is continuous on $(0,\infty)$. Since φ is an N-function we also know that

$\varphi'(t)$ is right-continuous at $t = 0$ and thus φ is C^1 on $\mathbb{R}^{\geq 0}$. Setting $\lambda_2 := (2\gamma_2 + 1)^{-1} \in (0, 1)$, then we have for all $t \geq 0$

$$\varphi'((1+\lambda_2)t) - \varphi'(t) = \varphi'_{[t]}(\lambda_2 t) \leq \gamma_2 \varphi'_t(\lambda_2 t) = \frac{1}{2}\frac{\gamma_2}{1+\gamma_2}\,\varphi'((1+\lambda_2)t).$$

This implies

$$\varphi'((1+\lambda_2)t) \leq \frac{2(1+\gamma_2)}{2+\gamma_2}\,\varphi'(t).$$

Since $\lambda_2 \in (0,1)$ there exists $m \in \mathbb{N}$ such that $(1+\lambda_2)^m \geq 2$ and we get

$$\varphi'(2t) \leq \varphi'((1+\lambda_2)^m t) \leq \left(\frac{2(1+\gamma_2)}{2+\gamma_2}\right)^m \varphi'(t).$$

Thus φ' satisfies the Δ_2-condition and consequently also φ. Analogously, we obtain for all $t \geq 0$,

$$\varphi'(2t) - \varphi'(t) = \varphi'_{[t]}(t) \geq \gamma_1 \varphi'_t(t) = \frac{\gamma_1}{2}\,\varphi'(2t),$$

which implies

$$\varphi'(t) \leq \Big(1 - \frac{\gamma_1}{2}\Big)\varphi'(2t),$$

because $0 < \gamma_1 \leq 1$. We can chose $m \in \mathbb{N}$ such that $\big(1 - \gamma_1/2\big)^m \leq 1/2$. Using (5.3), we thus obtain

$$\varphi(t) \leq \frac{\big(1 - \frac{\gamma_1}{2}\big)^m}{2^m}\,\varphi(2^m t) \leq \frac{1}{2 \cdot 2^m}\,\varphi(2^m t), \tag{6.13}$$

which, due to (5.37), implies that φ^* satisfies the Δ_2-condition.

Assume now that φ is C^2 on $(0,\infty)$. For $t > 0$ we then have

$$t \lim_{s\to 0} \frac{\varphi'_{[t]}(s)}{s} = t \lim_{s\to 0} \frac{\varphi'(t+s) - \varphi'(t)}{s} = t\varphi''(t),$$
$$t \lim_{s\to 0} \frac{\varphi'_t(s)}{s} = t \lim_{s\to 0} \frac{\varphi'(t+s)}{t+s} = \varphi'(t),$$

which together with (6.11) implies (6.12). □

Also the reverse implication of Lemma 6.10 is true.

Lemma 6.11. *Let φ be an N-function which belongs to $C^1(\mathbb{R}^{\geq 0}) \cap C^2(0,\infty)$ and satisfies the Δ_2-condition and* (6.12)*. Then for all $a, t \geq 0$,*

$$\frac{\gamma_1}{4(K')^2}\,\varphi_a'(t) \leq \varphi_{[a]}'(t) \leq 2\max\{\gamma_2, K'\}\varphi_a'(t). \tag{6.14}$$

Proof. The assertion is obvious if either $a = 0$ or $t = 0$. For all $a, t > 0$ we have

$$\varphi_{[a]}'(t) = \varphi'(a+t) - \varphi'(a) = \int_0^1 \varphi''(a+\lambda t)t\,d\lambda. \tag{6.15}$$

If $0 < t \leq a$, we use the second equality in (6.15) and (6.12), the fact that φ' is non-decreasing and the inequality $a \geq (a+t)/2$ to obtain

$$\varphi_{[a]}'(t) \leq \gamma_2 \int_0^1 \frac{\varphi'(a+\lambda t)}{a+\lambda t}\,t\,d\lambda \leq \gamma_2\,\frac{\varphi'(a+t)}{a}\,t \leq 2\gamma_2\varphi_a'(t)$$

and

$$\varphi_{[a]}'(t) \geq \gamma_1\,\frac{\varphi'(a)}{a+t}\,t \geq \gamma_1\,\frac{\varphi'\big(\frac{1}{2}(a+t)\big)}{a+t}\,t \geq \frac{\gamma_1}{K'}\,\frac{\varphi'(a+t)}{a+t}\,t = \frac{\gamma_1}{K'}\,\varphi_a'(t).$$

If $0 < a \leq t$, we use the first equality in (6.15), the inequality $a+t \leq 2t$ and Lemma 5.4 with $M = 1$ to obtain

$$\varphi_{[a]}'(t) \leq \varphi'(2t) \leq K'\varphi'(t) \leq 2K'\varphi_a'(t),$$

and using the second equality in (6.15), (6.12), that φ' is positive, $a+\lambda t \leq 2t$, and Lemma 5.5 with $M = 1$

$$\begin{aligned}\varphi_{[a]}(t) &\geq \gamma_1 \int_{\frac{1}{2}}^1 \frac{\varphi'(a+\lambda t)}{a+\lambda t}\,t\,d\lambda \geq \frac{\gamma_1}{4}\,\varphi'\Big(\frac{1}{2}t\Big)\\ &\geq \frac{\gamma_1}{4(K')^2}\,\varphi'(2t) \geq \frac{\gamma_1}{4(K')^2}\,\varphi_a'(t).\end{aligned}$$

Thus, the assertion of the lemma is proved. □

In general, an N-function φ does not have to be $C^2(0,\infty)$ even if it satisfies (5.33). However, we can mollify it. For that let $\varepsilon \in (0, 1/2)$ and let $\eta_\varepsilon \in C^\infty(0,\infty)$ be such that $\eta_\varepsilon \geq 0$, $\int_0^\infty \eta_\varepsilon\,dx = 1$, and $\operatorname{supp}(\eta_\varepsilon) \subseteq\subseteq (1-\varepsilon, 1+\varepsilon)$. We define for $t \geq 0$

$$(\omega^\varepsilon)'(t) := \int_{1-\varepsilon}^{1+\varepsilon} \varphi'(st)\eta_\varepsilon(s)\,ds, \tag{6.16}$$

and set for $t \geq 0$

$$\omega^\varepsilon(t) := \int_0^t (\omega^\varepsilon)'(s)\,ds. \tag{6.17}$$

Lemma 6.12. *Let φ be an N-function such that φ and φ^* satisfy the Δ_2-condition. Let $(\omega^\varepsilon)'$ be defined by (6.16). Then ω^ε and $(\omega^\varepsilon)^*$ are N-functions satisfying for all $t \geq 0$*

$$\frac{1}{K'}\,\varphi(t) \leq \omega^\varepsilon(t) \leq K'\varphi(t), \tag{6.18}$$

$$\omega^\varepsilon(2t) \leq K\omega^\varepsilon(t), \tag{6.19}$$

$$(\omega^\varepsilon)'(2t) \leq K'(\omega^\varepsilon)'(t), \tag{6.20}$$

$$(\omega^\varepsilon)^*(2t) \leq K_*(\omega^\varepsilon)^*(t), \tag{6.21}$$

$$\big((\omega^\varepsilon)^*\big)'(2t) \leq K_*'\big((\omega^\varepsilon)^*\big)'(t). \tag{6.22}$$

Proof. Using the properties of φ' and (6.16) one immediately sees that $(\omega^\varepsilon)'$ has the same properties and thus ω^ε is an N-function. Since φ' satisfies the Δ_2-condition and φ' is non-decreasing we obtain for all $t \geq 0$ and all $0 < \varepsilon < 1/2$

$$\frac{1}{K'}\,\varphi'(t) \leq (\omega^\varepsilon)'(t) \leq K'\varphi'(t).$$

This, (6.17) and (5.3) immediately yield (6.18). From (6.16) we get for all $t \geq 0$ and all $0 < \varepsilon < 1/2$

$$(\omega^\varepsilon)'(2t) = \int_{1-\varepsilon}^{1+\varepsilon} \varphi'(s2t)\,\eta_\varepsilon(s)\,ds \leq K' \int_{1-\varepsilon}^{1+\varepsilon} \varphi'(st)\,\eta_\varepsilon(s)\,ds = K'(\omega^\varepsilon)'(t),$$

which proves (6.20). From (6.16), (6.17), and Fubini's theorem we obtain

$$\begin{aligned}\omega^\varepsilon(t) &= \int_0^t \int_{1-\varepsilon}^{1+\varepsilon} \varphi'(s\tau)\,\eta_\varepsilon(s)\,ds\,d\tau \\ &= \int_{1-\varepsilon}^{1+\varepsilon} \int_0^t \varphi'(s\tau)\,d\tau\,\eta_\varepsilon(s)\,ds \\ &= \int_{1-\varepsilon}^{1+\varepsilon} \frac{\varphi(st)}{s}\,\eta_\varepsilon(s)\,ds,\end{aligned}$$

from which (6.19) follows. Moreover, using this representation and the equivalent condition for φ^* satisfying the Δ_2-condition, we obtain (cf. (5.37))

$$\omega^\varepsilon(t) \leq \int_{1-\varepsilon}^{1+\varepsilon} \frac{\varphi\big(s\frac{tK_*}{2}\big)}{K_* s}\,\eta_\varepsilon(s)\,ds = \frac{1}{K_*}\,\omega^\varepsilon\Big(\frac{tK_*}{2}\Big),$$

which implies (6.21). Since $(\varphi^*)'$ satisfies the Δ_2-condition we have for all $t \geq 0$ using Lemma 5.20

$$2(\omega^\varepsilon)'(t) = \int_{1-\varepsilon}^{1+\varepsilon} 2\varphi'(st)\,\eta_\varepsilon(s)\,ds \leq \int_{1-\varepsilon}^{1+\varepsilon} \varphi'(sK_*'t)\,\eta_\varepsilon(s)\,ds = (\omega^\varepsilon)'(K_*'t),$$

which implies (6.22) due to Lemma 5.20. This finishes the proof. $\square$

Lemma 6.13. *Let φ satisfy* (6.11) *and let $(\omega^\varepsilon)'$ be defined by* (6.16). *Then $(\omega^\varepsilon)'$ satisfies* (6.11) *and ω^ε satisfies Assumption 6.1. In particular, for all $t \geq 0$,*

$$\gamma_1\,(\omega^\varepsilon)'(t) \leq t\,(\omega^\varepsilon)''(t) \leq \gamma_2\,(\omega^\varepsilon)'(t). \tag{6.23}$$

Proof. From (6.11) and (6.16) we deduce

$$(\omega^\varepsilon)'_{[a]}(t) = \int_{1-\varepsilon}^{1+\varepsilon} \varphi'_{[sa]}(st)\,\eta_\varepsilon(s)\,ds \leq \gamma_2 \int_{1-\varepsilon}^{1+\varepsilon} \varphi'_{sa}(st)\,\eta_\varepsilon(s)\,ds = \gamma_2(\omega^\varepsilon)'_a(t),$$

which proves the second inequality in (6.23). The first one follows analogously. For $t > 0$ the expression (6.16) can be written as

$$(\omega^\varepsilon)'(t) = \frac{1}{t}\int_{t(1-\varepsilon)}^{t(1+\varepsilon)} \varphi'(\tau)\,\eta_\varepsilon\Big(\frac{\tau}{t}\Big)\,d\tau,$$

which yields $\omega^\varepsilon \in C^2(0,\infty)$. Thus in view of Lemma 6.10 we obtain that ω^ε satisfies Assumption 6.1. □

Now we are ready to prove the assertions of Lemma 6.7 under weaker assumptions on φ.

Lemma 6.14. *Let the operator $\mathbf{A}$ have an N-potential φ and let φ satisfy for all $t \geq 0$,*

$$\gamma_5\varphi'_a(t) \leq \varphi'_{[a]}(t) \leq \gamma_6\varphi'_a(t).$$

Then $\mathbf{A}$ has the φ-structure. In particular, $\mathbf{A}$ satisfies(6.2) *and* (6.3) *with $\gamma_1 = 2\gamma_5/(K^3K')$ and $\gamma_2 = 2c_2(2+\gamma_6)(n+N)(K_*^4/4)^{1/\kappa}K^{M+2}K'$, where $\kappa \in \big(2^{-1}(\log_{K_*}(3\,K_*/4)+1),1\big)$ and $M \in \mathbb{N}$ is such that $K_*^2 \leq 2^{M+1}$.*

Proof. Let us denote by $\mathbf{A}^\varepsilon$ the operator with the N-potential ω^ε, i.e., $\mathbf{A}^\varepsilon(\mathbf{0}) = \mathbf{0}$ and

$$\mathbf{A}^\varepsilon(\mathbf{P}) := \frac{(\omega^\varepsilon)'(|\mathbf{P}|)}{|\mathbf{P}|}\,\mathbf{P}$$

for all $\mathbf{P} \in \mathbb{R}^{N\times n}$. From Lemma 6.13 it follows that ω^ε satisfies Assumption 6.1 with $\gamma_3 = \gamma_5$ and $\gamma_4 = \gamma_6$. Lemma 6.7 thus yields that the operator $\mathbf{A}^\varepsilon$ has the ω^ε-structure, i.e.,

$$(\mathbf{A}^\varepsilon(\mathbf{P}) - \mathbf{A}^\varepsilon(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q}) \geq c_4(\omega^\varepsilon)'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}|, \tag{6.24}$$

$$\big|\mathbf{A}^\varepsilon(\mathbf{P}) - \mathbf{A}^\varepsilon(\mathbf{Q})\big| \leq c_5(\omega^\varepsilon)'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) \tag{6.25}$$

holds for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N \times n}$, where $c_4 := 2\gamma_5/(K^3 K')$ and $c_5 := 2c_2(2+\gamma_6)(n+N)(K_*^4/4)^{1/\kappa} K^{M+2} K'$ with κ and M as above. According to Lemma 6.10 φ' is continuous and so we easily deduce from the definition of $(\omega^\varepsilon)'$ that for all $t \geq 0$,

$$\lim_{\varepsilon \to 0} (\omega^\varepsilon)'(t) = \varphi'(t).$$

Thus also $\mathbf{A}^\varepsilon(\mathbf{P})$ and $(\omega^\varepsilon)'_{|\mathbf{P}|}$ converge to $\mathbf{A}(\mathbf{P})$ and $\varphi'_{|\mathbf{P}|}$, respectively, as $\varepsilon \to 0$. The assertions of the lemma now follow immediately from (6.24) and (6.25) letting $\varepsilon \to 0$. □

Now, we want to derive another very useful property of an operator $\mathbf{A}$ with an N-potential φ. For that let φ be a given N-function. We set for $t \geq 0$

$$\psi'(t) := \sqrt{\varphi'(t) t} \tag{6.26}$$

and define the *associated N-function* ψ for $t \geq 0$ by

$$\psi(t) := \int_0^t \psi'(s)\, ds. \tag{6.27}$$

Lemma 6.15. *Let φ be an N-function satisfying* (6.11). *Then ψ defined in* (6.27) *is an N-function which satisfies for all $t \geq 0$*

$$\frac{\gamma_1}{2}\, \psi_a'(t) \leq \psi_{[a]}'(t) \leq (\gamma_2 + 1) \psi_a'(t). \tag{6.28}$$

Moreover, ψ is C^1 on $[0, \infty)$, ψ' is locally Lipschitz-continuous on $(0, \infty)$, and ψ and ψ^* satisfy the Δ_2-condition. In particular, we have for all $t \geq 0$

$$\psi'(2t) \leq \sqrt{2K'}\, \psi(t), \tag{6.29}$$

$$\psi(2t) \leq 2\sqrt{2K'}\, \psi(t), \tag{6.30}$$

$$(\psi^*)'(2t) \leq \max\{2, K_*'\} (\psi^*)'(t), \tag{6.31}$$

$$\psi^*(2t) \leq 2 \max\{2, K_*'\} \psi^*(t). \tag{6.32}$$

Proof. Using the properties of φ' and (6.26) one immediately sees that ψ' has the same properties and thus ψ is an N-function. From (6.26) and the definition of the shifted N-function it follows that for all $a, t \geq 0$,

$$\psi_a'(t) = \sqrt{\varphi'(a+t)(a+t)}\, \frac{t}{a+t} = \sqrt{\varphi_a'(t) t}.$$

Furthermore, we have for all $a, t \geq 0$

$$\begin{aligned}\psi'_{[a]}(t) &= \frac{\varphi'(a+t)(a+t) - \varphi'(a)a}{\sqrt{\varphi'(a+t)(a+t)} + \sqrt{\varphi'(a)a}} \\ &= \frac{\big(\varphi'(a+t) - \varphi'(a)\big)(a+t) + \varphi'(a)t}{\sqrt{\varphi'(a+t)(a+t)} + \sqrt{\varphi'(a)a}} =: I_1 + I_2.\end{aligned} \tag{6.33}$$

For I_1 we obtain

$$I_1 \leq \frac{\varphi'_{[a]}(t)(a+t)}{\sqrt{\varphi'(a+t)(a+t)}} \leq \gamma_2 \frac{\varphi'_a(t)(a+t)}{\sqrt{\varphi'(a+t)(a+t)}} = \gamma_2\, \psi'_a(t),$$

and

$$I_1 \geq \frac{\varphi'_{[a]}(t)(a+t)}{2\sqrt{\varphi'(a+t)(a+t)}} \geq \frac{\gamma_1}{2} \frac{\varphi'_a(t)(a+t)}{\sqrt{\varphi'(a+t)(a+t)}} = \frac{\gamma_1}{2}\, \psi'_a(t).$$

For I_2 we get

$$0 \leq I_2 \leq \frac{\varphi'(a+t)t}{\sqrt{\varphi'(a+t)(a+t)}} = \psi'_a(t).$$

These inequalities and (6.33) prove (6.28). The assertions concerning the smoothness of ψ and the Δ_2-conditions for ψ and ψ^* follow from Lemma 6.10. We deduce from (5.11) and (6.26) that for all $t \geq 0$,

$$\psi'(2t) = \sqrt{\varphi'(2t)2t} \leq \sqrt{2K'}\sqrt{\varphi'(t)t} = \sqrt{2K'}\psi(t),$$

which proves (6.29). This and Lemma 5.2 imply (6.30). In order to show (6.31) we use Lemma 5.20. For $t \geq 0$ we have

$$\begin{aligned}2\psi'(t) &= \sqrt{4t\varphi'(t)} \leq \sqrt{2t\varphi(K'_* t)} \\ &\leq \sqrt{\max\{2, K'_*\}\, t\varphi(\max\{2, K'_*\}t)} = \psi'(\max\{2, K'_*\}t).\end{aligned}$$

This and Lemma 5.10 imply (6.32). □

For a given N-function φ we denote by $\mathbf{F}$ the operator with the N-potential ψ, where ψ is the associated N-function defined in (6.27), i.e., $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and for all $\mathbf{P} \in \mathbb{R}^{N\times n} \setminus \{\mathbf{0}\}$,

$$\mathbf{F}(\mathbf{P}) := \mathbf{A}_\psi(\mathbf{P}) = \frac{\psi'(|\mathbf{P}|)}{|\mathbf{P}|}\, \mathbf{P}. \tag{6.34}$$

Using this operator we can show that $(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q})$ and $|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2$ are equivalent. More precisely we have:

Lemma 6.16. *Let the operator* $\mathbf{A}$ *have an N-potential* φ *and* φ*-structure. Let* $\mathbf{F}$ *be defined by* (6.34) *with* ψ *from* (6.27)*. Then* $\mathbf{F} = \mathbf{A}_\psi$ *has* ψ*-structure. Moreover,*

$$\gamma_1 \varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) \le (\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P}-\mathbf{Q}) \le \gamma_2 4K' \varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|), \tag{6.35}$$

$$c_5 |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 \le (\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P}-\mathbf{Q}) \le c_6 |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 \tag{6.36}$$

holds for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$*, where* c_5 *and* c_6 *are constants depending only on* γ_1*,* γ_2*,* K*,* K'*,* K_**,* n *and* N*. In particular, we have*

$$\gamma_1 \varphi(|\mathbf{P}|) \le \mathbf{A}(\mathbf{P}) \cdot \mathbf{P} \le \gamma_2 4K' \varphi(|\mathbf{P}|), \tag{6.37}$$

$$c_5 |\mathbf{F}(\mathbf{P})|^2 \le \mathbf{A}(\mathbf{P}) \cdot \mathbf{P} \le c_6 |\mathbf{F}(\mathbf{P})|^2. \tag{6.38}$$

Proof. In view of Lemma 6.8 and Lemma 6.10 the N-functions φ and φ^* satisfy the Δ_2-condition. The assertion (6.35) follows from (6.5), (5.16) and (5.22). Lemma 6.8 also implies that φ satisfies (6.11) and so Lemma 6.15 yields

$$\frac{\gamma_1}{2}\, \psi_a'(t) \le \psi_{[a]}'(t) \le (\gamma_2 + 1)\psi_a'(t)$$

for all $a, t \ge 0$. Thus Lemma 6.14 proves that $\mathbf{F}$ has the ψ-structure, i.e.,

$$(\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})) \cdot (\mathbf{P}-\mathbf{Q}) \ge c_7 \psi_{|\mathbf{P}|}'(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}|, \tag{6.39}$$

$$|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})| \le c_8 \psi_{|\mathbf{P}|}'(|\mathbf{P}-\mathbf{Q}|) \tag{6.40}$$

holds for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$, with constants c_7 and c_8 depending only on γ_1, γ_2, K, K', K_*, n and N (cf. Lemma 6.14, Lemma 6.15). Moreover, we have (cf. (6.4))

$$c_7 \psi_{|\mathbf{P}|}'(|\mathbf{P}-\mathbf{Q}|) \le |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})| \le c_8 \psi_{|\mathbf{P}|}'(|\mathbf{P}-\mathbf{Q}|), \tag{6.41}$$

which yields

$$\begin{aligned} c_7^2 \varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) &\le c_7^2 \varphi_{|\mathbf{P}|}'(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}| \\ &= \big(c_7 \psi_{|\mathbf{P}|}'(|\mathbf{P}-\mathbf{Q}|)\big)^2 \\ &\le |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 \end{aligned}$$

and

$$\begin{aligned} \big|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})\big|^2 &\le \big(c_8 \psi_{|\mathbf{P}|}'(|\mathbf{P}-\mathbf{Q}|)\big)^2 \\ &= c_8^2 \varphi_{|\mathbf{P}|}'(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}| \\ &\le c_8^2 4K' \varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|). \end{aligned}$$

These two inequalities together with (6.35) immediately imply (6.36). The inequalities (6.37) and (6.38) follow from (6.35), (6.36), and $\varphi_0(t) = \varphi(t)$. □

Remark 6.17. In view of Lemma 6.14 one can replace in Lemma 6.16 the assumption that $\mathbf{A}$ has φ-structure by the assumption that φ satisfies (6.11).

In concrete examples it might be complicated to verify condition (6.11) if φ is only an N-function. If φ possesses a second derivative, then the situation becomes easier, since it is sufficient to verify (6.6) (cf. Lemma 6.10, Lemma 6.11).

Concerning applications, also the following situation is of interest, for which we derive the same assertions as in Lemma 6.16. Let $\mathbf{A}$ have an N-potential φ, i.e. (6.1) holds, where the N-function φ belongs to $C^1(\mathbb{R}^{\geq 0}) \cap C^2(0,\infty)$.[2] Furthermore we assume that there exists another N-function ζ, which satisfies Assumption 6.1, and constants $\gamma_7, \gamma_8 > 0$ such that for all $t > 0$,

$$\gamma_7 \zeta''(t) \leq \varphi''(t) \leq \gamma_8 \zeta''(t). \tag{6.42}$$

Since both ζ and φ are N-functions, this implies

$$\gamma_7 \zeta'(t) \leq \varphi'(t) \leq \gamma_8 \zeta'(t), \tag{6.43}$$

$$\gamma_7 \zeta(t) \leq \varphi(t) \leq \gamma_8 \zeta(t). \tag{6.44}$$

Moreover, φ also satisfies (6.6), i.e. for all $t > 0$,

$$\frac{\gamma_7 \gamma_3}{\gamma_8}\, \varphi'(t) \leq t\varphi''(t) \leq \frac{\gamma_4 \gamma_8}{\gamma_7} \varphi'(t),$$

where γ_3, γ_4 are the constants from (6.6) with φ replaced by ζ. Lemma 6.7 now implies that $\mathbf{A}$ has the φ-structure and due to (6.43) also the ζ-structure. In particular, for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$,

$$\begin{aligned}
(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) &\geq c_9 \varphi'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|)|\mathbf{P} - \mathbf{Q}|,\\
(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) &\geq c_9 \gamma_7 \zeta'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|)|\mathbf{P} - \mathbf{Q}|,\\
|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})| &\leq c_{10} \varphi'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|),\\
|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})| &\leq c_{10} \gamma_8 \zeta'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|),
\end{aligned}$$

with constants c_9 and c_{10} depending only on γ_3, γ_4, γ_7, γ_8, n, N and on the Δ_2-constants of ζ, ζ', ζ^*, which are denoted by $\hat{K}$, $\hat{K}'$, $\hat{K}_*$. We also used

[2]The assumptions on φ can easily be weakened.

that

$$\varphi'(2t) \le \frac{\gamma_8}{\gamma_7}\,\hat{K}'\varphi'(t),$$
$$\varphi(2t) \le \frac{\gamma_8}{\gamma_7}\,\hat{K}\varphi(t),$$
$$\varphi^*(2t) \le \frac{\gamma_8}{\gamma_7}\,\hat{K}_*\varphi(t),$$

which follow from (6.43), (6.44), and (5.37). From the proof of Lemma 6.7 we obtain that

$$\sum_{l,m,j,k} B_{jk}\frac{A_{jk}(\mathbf{P})}{\partial P_{lm}}B_{lm} \ge \frac{\gamma_3\gamma_7}{\gamma_8}\,\frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|}\,|\mathbf{B}|^2 \ge \gamma_7\frac{\gamma_3\gamma_7}{\gamma_8}\,\frac{\zeta'(|\mathbf{P}|)}{|\mathbf{P}|}\,|\mathbf{B}|^2,$$
$$\Big|\frac{A_{jk}(\mathbf{P})}{\partial P_{lm}}\Big| \le \Big(2+\frac{\gamma_4\gamma_8}{\gamma_7}\Big)\frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|} \le \gamma_8\Big(2+\frac{\gamma_4\gamma_8}{\gamma_7}\Big)\frac{\zeta'(|\mathbf{P}|)}{|\mathbf{P}|}$$

holds for all $\mathbf{P},\mathbf{B} \in \mathbb{R}^{N\times n}$ with $\mathbf{P} \ne \mathbf{0}$. Moreover, from Lemma 6.16 it follows that

$$c_9\gamma_7\zeta_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) \le (\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q}) \le c_{10}\gamma_8 4\hat{K}'\zeta_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|),$$

$$\begin{aligned} c_{11}|\mathbf{F}(\mathbf{P})-\mathbf{F}(\mathbf{Q})|^2 &\le (\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q}) \\ &\le c_{12}|\mathbf{F}(\mathbf{P})-\mathbf{F}(\mathbf{Q})|^2 \end{aligned} \tag{6.45}$$

holds for all $\mathbf{P},\mathbf{Q} \in \mathbb{R}^{N\times n}$, with constants c_{11} and c_{12} depending only on γ_3, γ_4, γ_7, γ_8, n, N and $\hat{K}$, $\hat{K}'$, $\hat{K}_*$. Here $\mathbf{F} = \mathbf{A}_\psi$ denotes the operator with the N-potential ψ, where ψ is the associated N-function for φ (cf. (6.27)). Let us denote by $\hat{\psi}$ the associated N-function for ζ, i.e. for all $t \ge 0$,

$$\hat{\psi}'(t) := \sqrt{\zeta'(t)t},$$

and by $\hat{\mathbf{F}}$ the operator with the N-potential $\hat{\psi}$. Since

$$\sqrt{\gamma_7}\,\hat{\psi}'(t) \le \psi'(t) \le \sqrt{\gamma_8}\,\hat{\psi}'(t)$$

we obtain, using (6.41) once for $\mathbf{F}$ and ψ and once for $\hat{\mathbf{F}}$ and $\hat{\psi}$, that

$$c_7c_8\sqrt{\gamma_7}\,|\hat{\mathbf{F}}(\mathbf{P})-\hat{\mathbf{F}}(\mathbf{Q})| \le |\mathbf{F}(\mathbf{P})-\mathbf{F}(\mathbf{Q})| \le c_7c_8\sqrt{\gamma_8}\,|\hat{\mathbf{F}}(\mathbf{P})-\hat{\mathbf{F}}(\mathbf{Q})|.$$

From this and (6.45) we obtain

$$c_{13}|\hat{\mathbf{F}}(\mathbf{P}) - \hat{\mathbf{F}}(\mathbf{Q})|^2 \leq (\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) \leq c_{14}|\hat{\mathbf{F}}(\mathbf{P}) - \hat{\mathbf{F}}(\mathbf{Q})|^2,$$

with constants c_{13} and c_{14} depending only on γ_3, γ_4, γ_7, γ_8, n, N and $\hat{K}$, $\hat{K}'$, $\hat{K}_*$. In particular, we have

$$\begin{aligned} c_9\gamma_7\zeta(|\mathbf{P}|) &\leq \mathbf{A}(\mathbf{P}) \cdot \mathbf{P} \leq c_{10}\gamma_8 4\hat{K}'\zeta(|\mathbf{P}|), \\ c_{13}|\mathbf{F}(\mathbf{P})|^2 &\leq \mathbf{A}(\mathbf{P}) \cdot \mathbf{P} \leq c_{14}|\mathbf{F}(\mathbf{P})|^2. \end{aligned} \tag{6.46}$$

Remark 6.18. A typical example for the above situation is the following: Let the operator $\mathbf{A}$ have an N-potential φ, where the N-function φ belongs to $C^1(\mathbb{R}^{\geq 0}) \cap C^2(0,\infty)$. Assume that there exists $p \in (1,\infty)$, $\kappa \in [0,\infty)$, and constants $\gamma_9, \gamma_{10} > 0$ such that for all $t > 0$,

$$\gamma_9(\kappa + t)^{p-2} \leq \varphi''(t) \leq \gamma_{10}(\kappa + t)^{p-2}. \tag{6.47}$$

Then all assertions from (6.42) to (6.46) hold with

$$\zeta'(t) := \frac{1}{p-1}\big((\kappa + t)^{p-1} - \kappa^{p-1}\big).$$

Note that there exist constants c_{15}, c_{16} depending only on p, such that for all $\kappa, t \geq 0$,

$$c_{15}\zeta'(t) \leq (\kappa + t)^{p-2}t \leq c_{16}\zeta'(t).$$

If (6.47) holds, one says that φ has a *p-structure.*

7. Problems with a φ-structure

In this section we want to generalize the results of the previous section to operators $\mathbf{A}$ with a φ-structure, cf. (6.2) and (6.3), but possessing no potential. We first investigate the situation when φ satisfies Assumption 6.1.

Lemma 7.1. *Let φ satisfy Assumption 6.1. Then ψ defined in* (6.27) *satisfies Assumption 6.1. In particular,* (6.29)–(6.32) *are satisfied and*

$$\gamma_3\psi'(t) \leq t\psi''(t) \leq \frac{\gamma_4 + 1}{2}\,\psi'(t) \tag{7.1}$$

holds for all $t > 0$.

Proof. From Assumption 6.1 and the definition of ψ it follows that ψ has the same regularity properties as φ. In the proof of (6.29)–(6.32) we used only the definition of ψ and the fact that φ and φ^* satisfy the Δ_2-condition. Thus it follows also in our situation that ψ and ψ^* satisfy the Δ_2-condition. The definition of ψ', (6.6) and $0 < \gamma_3 \le 1$ yield

$$\psi''(t) = \frac{1}{2}\frac{t\varphi''(t) + \varphi'(t)}{\sqrt{t\varphi'(t)}} \le \frac{\gamma_4 + 1}{2}\frac{\varphi'(t)}{\sqrt{t\varphi'(t)}} = \frac{\gamma_4 + 1}{2}\frac{\psi'(t)}{t}$$

and

$$\frac{\psi'(t)}{t} = \frac{\varphi'(t)}{\sqrt{t\varphi'(t)}} \le \frac{1}{2}\frac{\gamma_3^{-1}t\varphi''(t) + \varphi'(t)}{\sqrt{t\varphi'(t)}} \le \frac{1}{\gamma_3}\psi''(t),$$

for $t > 0$. This proves (7.1). □

Lemma 7.2. *Let the operator* $\mathbf{A}$ *have* φ*-structure and let* φ *satisfy Assumption 6.1. Let* $\mathbf{F}$ *be defined by* (6.34) *with* ψ *from* (6.27). *Then* $\mathbf{F} = \mathbf{A}_\psi$ *has* ψ*-structure. Moreover,*

$$\gamma_1\varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) \le (\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q}) \le \gamma_2 4K'\varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|), \quad (7.2)$$
$$c_{17}|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 \le (\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q}) \le c_{18}|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2$$

holds for all $\mathbf{P},\mathbf{Q} \in \mathbb{R}^{N\times n}$, *where* c_{17} *and* c_{18} *are constants depending only on* γ_1, γ_2, K, K', K_*, n *and* N. *In particular, we have*

$$\gamma_1\varphi(|\mathbf{P}|) \le \mathbf{A}(\mathbf{P})\cdot(\mathbf{P}) \le \gamma_2 4K'\varphi(|\mathbf{P}|),$$
$$c_{17}|\mathbf{F}(\mathbf{P})|^2 \le \mathbf{A}(\mathbf{P})\cdot(\mathbf{P}) \le c_{18}|\mathbf{F}(\mathbf{P})|^2.$$

Proof. The assertion (7.2) follows from (6.5), (5.16) and (5.22). Lemma 7.1 yields that ψ satisfies Assumption 6.1 and so Lemma 6.7 implies that $\mathbf{F}$ has the ψ-structure. In particular, (6.39) and (6.40) are satisfied and we can finish the proof of this lemma exactly as in the proof of Lemma 6.16. □

In order to relax the assumption on φ we introduce another associated N-function $\bar\psi$. For a given N-function φ we set for $t \ge 0$

$$\bar\psi'(t) := \sqrt{\varphi(t)}, \quad (7.3)$$

and define the *associated N-function* $\bar\psi$ for $t \ge 0$ by

$$\bar\psi(t) := \int_0^t \bar\psi'(s)\,ds. \quad (7.4)$$

From the properties of the N-function φ it follows that $\bar\psi'(t)$ is continuous and always possesses a right continuous right derivative which we denote by $\bar\psi''(t)$.

Lemma 7.3. *Let φ be an N-function satisfying the Δ_2-condition. Then $\bar\psi$ defined in (7.4) is an N-function, which satisfies for all $t \geq 0$,*

$$\frac{1}{2}\,\bar\psi'(t) \leq t\bar\psi''(t) \leq \frac{K}{2}\,\bar\psi'(t), \tag{7.5}$$

$$\frac{1}{8K}\,\bar\psi'_a(t) \leq \bar\psi'_{[a]}(t) \leq 2\max\left\{\frac{K}{2}, \sqrt{K}\right\}\bar\psi'_a(t). \tag{7.6}$$

Moreover, $\bar\psi$ is C^1 on $[0,\infty)$, $\bar\psi'$ is locally Lipschitz-continuous on $(0,\infty)$ and $\bar\psi$ and $\bar\psi^*$ satisfy the Δ_2-condition. In particular, we have for all $t \geq 0$,

$$\bar\psi'(2t) \leq \sqrt{K}\,\bar\psi'(t), \tag{7.7}$$

$$\bar\psi(2t) \leq 2\sqrt{K}\,\bar\psi(t), \tag{7.8}$$

$$\bar\psi^*(2t) \leq 2^{m+1}\bar\psi^*(t), \tag{7.9}$$

where $m \in \mathbb{N}$ is such that $\big(1 - 1/(16K)\big)^m \leq \frac{1}{2}$. Finally, let ψ be defined in (6.27). Then we have for all $t \geq 0$,

$$\bar\psi'(t) \leq \psi'(t) \leq \sqrt{K}\,\bar\psi'(t). \tag{7.10}$$

Proof. The properties of the N-function φ, (7.3) and (7.4) imply that $\bar\psi$ is an N-function. Using the Δ_2-condition for φ and (7.3), we obtain (7.7). The inequality (7.8) then follows from Lemma 5.2. Inequalities in (7.5) follow from (5.16) and (7.3). The proof of inequality (6.14) in Lemma 6.11 works also under the assumptions on $\bar\psi$ here. Thus (7.6) is a consequence of (7.7) and (6.14). The regularity properties of $\bar\psi$ and the fact that $\bar\psi^*$ satisfies the Δ_2-condition follow from (7.6) and Lemma 6.10. Inequality (7.9) follows from (6.13) and (5.37). Inequality (7.10) follows from the definitions of $\bar\psi$ and ψ. ☐

For a given N-function φ we denote by $\bar{\mathbf{F}}$ the operator with the N-potential $\bar\psi$, where $\bar\psi$ is defined in (7.4),

$$\bar{\mathbf{F}}(\mathbf{P}) := \mathbf{A}_{\bar\psi}(\mathbf{P}) = \frac{\bar\psi'(|\mathbf{P}|)}{|\mathbf{P}|}\,\mathbf{P} \tag{7.11}$$

holds, i.e., $\bar{\mathbf{F}}(\mathbf{0}) = \mathbf{0}$ and for all $\mathbf{P} \in \mathbb{R}^{N\times n} \setminus \{\mathbf{0}\}$.

Lemma 7.4. *Let the operator* $\mathbf{A}$ *have a* φ*-structure, where* φ *is an N-function satisfying the* Δ_2*-condition. Let* $\bar{\mathbf{F}}$ *be defined by* (7.11) *with* $\bar\psi$ *from* (7.4)*. Then* $\bar{\mathbf{F}} = \mathbf{A}_{\bar\psi}$ *has the* $\bar\psi$*-structure. Moreover,*

$$\begin{aligned}\gamma_1\varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) &\le (\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q})\\ &\le \gamma_2 4K'\varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|),\end{aligned} \tag{7.12}$$

$$\begin{aligned}c_{19}|\bar{\mathbf{F}}(\mathbf{P})-\bar{\mathbf{F}}(\mathbf{Q})|^2 &\le (\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q})\\ &\le c_{20}|\bar{\mathbf{F}}(\mathbf{P})-\bar{\mathbf{F}}(\mathbf{Q})|^2\end{aligned} \tag{7.13}$$

holds for all $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{N\times n}$*, where* c_{19} *and* c_{20} *are constants depending only on* γ_1*,* γ_2*,* K*,* K'*,* K_**,* n *and* N*. In particular, we have*

$$\gamma_1\varphi(|\mathbf{P}|) \le \mathbf{A}(\mathbf{P})\cdot\mathbf{P} \le \gamma_2 4K'\varphi(|\mathbf{P}|), \tag{7.14}$$

$$c_{19}|\bar{\mathbf{F}}(\mathbf{P})|^2 \le \mathbf{A}(\mathbf{P})\cdot\mathbf{P} \le c_{20}|\bar{\mathbf{F}}(\mathbf{P})|^2. \tag{7.15}$$

Proof. Since the operator $\mathbf{A}$ has φ-structure, we obtain (cf. (6.5))

$$\begin{aligned}\gamma_1\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}| &\le (\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q})\\ &\le \gamma_2\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}|.\end{aligned} \tag{7.16}$$

From the definition of ψ (cf. (6.27)) we get

$$\varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}| = \big(\psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)\big)^2.$$

This and inequalities (7.10) yield

$$\big(\bar\psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)\big)^2 \le \varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}| \le K\big(\bar\psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)\big)^2. \tag{7.17}$$

Due to (7.6) Lemma 6.14 implies that $\bar{\mathbf{F}}$ has the $\bar\psi$-structure, i.e.,

$$\begin{aligned}(\bar{\mathbf{F}}(\mathbf{P})-\bar{\mathbf{F}}(\mathbf{Q}))\cdot(\mathbf{P}-\mathbf{Q}) &\ge c_{21}\bar\psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)|\mathbf{P}-\mathbf{Q}|,\\ |\bar{\mathbf{F}}(\mathbf{P})-\bar{\mathbf{F}}(\mathbf{Q})| &\le c_{22}\bar\psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)\end{aligned}$$

holds for all $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{N\times n}$, with constants c_{21} and c_{22} depending only on K, K', K_*, n and N (cf. Lemma 6.14, Lemma 7.3). Moreover, we have (cf. (6.4))

$$c_{19}\bar\psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|) \le |\bar{\mathbf{F}}(\mathbf{P})-\bar{\mathbf{F}}(\mathbf{Q})| \le c_{20}\bar\psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|). \tag{7.18}$$

Inequalities (7.13) now follow from (7.16), (7.17) and (7.18). The inequalities (7.14) and (7.15) follow immediately from (7.12), (7.13) and $\varphi_0(t) = \varphi(t)$. □

8. Applications to fluid dynamics

In this section we want to show how the results of the previous sections can be modified to fit to the setting of fluid dynamics. We study a similar system as in Section 7. However, due to the principle of objectivity the extra stress tensor $\mathbf{S}$ depends on the velocity gradient $\nabla\mathbf{v}$ only through its symmetric part $\mathbf{D}\mathbf{v} := \frac{1}{2}\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\big)$, i.e. $\mathbf{S}(\nabla\mathbf{v}) = \mathbf{S}(\mathbf{D}\mathbf{v})$. For instance, in the case of viscous, incompressible fluids we study the system

$$\begin{aligned} -\operatorname{div}\big(\mathbf{S}(\mathbf{D}\mathbf{v})\big) + \nabla\pi &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{v} &= \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{v} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$, $\mathbf{v} : \Omega \to \mathbb{R}^n$. Let $\mathbb{R}^{n\times n}_{\text{sym}}$ denote the set of symmetric $(n\times n)$-tensors. The extra stress $\mathbf{S} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\text{sym}}$ is defined for all $\mathbf{P} \in \mathbb{R}^{n\times n}$ by

$$\mathbf{S}(\mathbf{P}) = \mathbf{S}_\varphi(\mathbf{P}) := \frac{\varphi'(|\mathbf{P}^{\text{sym}}|)}{|\mathbf{P}^{\text{sym}}|}\,\mathbf{P}^{\text{sym}}, \tag{8.1}$$

where φ is an N-function and

$$\mathbf{P}^{\text{sym}} := \tfrac{1}{2}\big(\mathbf{P} + \mathbf{P}^\top\big).$$

In analogy with Section 6, we say that $\mathbf{S}_\varphi$ possesses an *N-potential* φ. By definition of $\mathbf{S}$ we have $\mathbf{S}(\mathbf{P}) = \mathbf{S}(\mathbf{P}^{\text{sym}})$ for all $\mathbf{P} \in \mathbb{R}^{n\times n}$. Then for all $\mathbf{D} \in \mathbb{R}^{n\times n}_{\text{sym}}$ we have $\mathbf{S}_\varphi(\mathbf{D}) = \mathbf{A}_\varphi(\mathbf{D})$ with $\mathbf{A}_\varphi$ defined by (6.1). Therefore, most of the results for $\mathbf{A}$ are extended to $\mathbf{S}$. It is the purpose of this section to present the necessary changes.

We also consider the more general situation when $\mathbf{S}$ has no N-potential. We say that the operator $\mathbf{S}$ has a *φ-structure*, where φ is an N-function, if $\mathbf{S}(\mathbf{0}) = \mathbf{0}$ and there exist constants $\gamma_1, \gamma_2 > 0$ such that for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times d}$,

$$(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q}))\cdot(\mathbf{P} - \mathbf{Q}) \geq \gamma_1\,\varphi'_{|\mathbf{P}^{\text{sym}}|}(|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|)|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|, \tag{8.2}$$

$$|\mathbf{S}(\mathbf{P}^{\text{sym}}) - \mathbf{S}(\mathbf{Q}^{\text{sym}})| \leq \gamma_2\,\varphi'_{|\mathbf{P}^{\text{sym}}|}(|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|). \tag{8.3}$$

With this new notation all the results from Section 6 and 7 for the operator $\mathbf{A}$ and related quantities carry over to the operator $\mathbf{S}$ and the corresponding quantities. As an example we formulate Lemma 6.16 for $\mathbf{S}$. Certainly, the definition of $\mathbf{F}$ has to be adapted accordingly; we set

$$\mathbf{F}(\mathbf{P}) := \mathbf{S}_\psi(\mathbf{P}) = \frac{\psi'(|\mathbf{P}^{\text{sym}}|)}{|\mathbf{P}^{\text{sym}}|}\,\mathbf{P}^{\text{sym}}. \tag{8.4}$$

Then Lemma 6.16 reads as follows.

Lemma 8.1. *Let the operator* $\mathbf{S}$ *have an N-potential* φ, *i.e.* (8.1) *holds, and the* φ*-structure, i.e.* (8.2) *and* (8.3) *are satisfied. Let* $\mathbf{F}$ *be defined by* (8.4) *with* $\psi'(t) = \sqrt{\varphi'(t)t}$. *Then* $\mathbf{F} = \mathbf{S}_\psi$ *has the* ψ*-structure. Moreover,*

$$\begin{aligned}
\gamma_1\, \varphi_{|\mathbf{P}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|) &\le (\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}),\\
\gamma_1\, \varphi_{|\mathbf{P}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|) &\le \gamma_2\, 4K' \varphi_{|\mathbf{P}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|),\\
c_5 |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 &\le (\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}),\\
c_5 |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 &\le c_6 |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2
\end{aligned}$$

holds for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times d}$, *where* c_5 *and* c_6 *are constants depending only on* γ_1, γ_2, K, K', K_* *and* n. *In particular, we have*

$$\begin{aligned}
\gamma_1\, \varphi(|\mathbf{P}|) \le \mathbf{S}(\mathbf{P}) \cdot \mathbf{P} &\le \gamma_2\, 4K' \varphi(|\mathbf{P}|),\\
c_5 |\mathbf{F}(\mathbf{P})|^2 \le \mathbf{S}(\mathbf{P}) \cdot \mathbf{P} &\le c_6 |\mathbf{F}(\mathbf{P})|^2.
\end{aligned}$$

Thus, the results of Section 6 and 7 carry over, if we define $\mathbf{F}$ and $\bar{\mathbf{F}}$ accordingly and replace $\mathbf{P}$ and $\mathbf{Q}$ in all the estimates involving φ, ψ, $\bar{\psi}$ and ω^ε by $\mathbf{P}^{\mathrm{sym}}$ and $\mathbf{Q}^{\mathrm{sym}}$, respectively.

Only the proof of Lemma 6.7 needs a little more attention. There (6.8) has to be replaced by

$$\frac{S_{jk}(\mathbf{P})}{\partial P_{lm}} = \frac{\varphi'(|\mathbf{P}^{\mathrm{sym}}|)}{|\mathbf{P}^{\mathrm{sym}}|}\left(\delta^{\mathrm{sym}}_{jk,lm} - \frac{P^{\mathrm{sym}}_{jk} P^{\mathrm{sym}}_{lm}}{|\mathbf{P}^{\mathrm{sym}}|^2}\right) + \varphi''(|\mathbf{P}^{\mathrm{sym}}|)\frac{P^{\mathrm{sym}}_{jk}}{|\mathbf{P}^{\mathrm{sym}}|}\frac{P^{\mathrm{sym}}_{lm}}{|\mathbf{P}^{\mathrm{sym}}|},$$

where

$$\delta^{\mathrm{sym}}_{ij,kl} := \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

The rest of the proof remains the same.

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