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METRIC SOBOLEV SPACES

PEKKA KOSKELA

ABSTRACT. We describe an approach to establish a theory of metric Sobolev spaces based on Lipschitz functions and their pointwise Lipschitz constants and the Poincaré inequality.

1. INTRODUCTION

There have been recent attempts to do analysis on metric spaces that are equipped with a doubling measure. In these notes we try to explain an approach based on the pointwise Lipschitz constants.

In the Euclidean setting one can define the Sobolev spaces by many different ways. For example, one can begin with smooth functions defined in an open and connected set and then complete this class with respect to the norm

$$\|u\| := \|u\|_p + \| \, |\nabla u| \, \|_p,$$

where the norms on the right are usual L^p-norms and $|\nabla u|$ is the length of the gradient of u. Alternatively, one can consider Lipschitz functions instead of smooth functions. The point is that also Lipschitz functions have an (almost everywhere defined) gradient and usual density arguments show that smooth functions may well be replaced by Lipschitz functions.

In the setting of a metric space equipped with a measure, one cannot anymore talk about smooth functions. However, the concept of a Lipschitz function makes sense. What then about the gradient?

Above we did not really use the gradient but only the length of it, which is almost everywhere comparable to the pointwise Lipschitz constant. Recall that the pointwise Lipschitz constant of u is defined as

$$\operatorname{Lip} u(x) = \limsup_{r \to 0+} \sup_{\{y : d(x,y) \le r\}} \frac{|u(x) - u(y)|}{r}.$$

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This suggests to try building the theory of Sobolev functions based on Lipschitz functions and their pointwise Lipschitz constants.

We shall be somewhat sketchy in what follows and we refer the reader to [5], [4], [6], [15] for more details.

2. The Poincaré inequality and Sobolev spaces

The power of Sobolev space techniques lies in Sobolev inequalities: one wishes to control suitable integrals involving the function in terms of integrals of the length of the gradient. In order to obtain such estimates we shall take as a standing assumption a Poincaré inequality. For this we need a measure. The measures that we consider will be doubling, i.e. satisfying

$$\mu(B(x,2r)) \leq C_d\, \mu(B(x,r))$$

for every x in our metric space X and all $r > 0$. We shall also assume that $0 < \mu(B) < \infty$ for each ball B and that μ is a Borel measure. We then say that (X, d, μ) is a doubling space.

Let (X, d, μ) be a doubling space, and let $p \geq 1$. We say that X supports a p-Poincaré inequality if there exist $C_p > 0$ and $\lambda \geq 1$ such that

$$\fint_B |u - u_B|\, d\mu \leq C_p \operatorname{diam}(B) \left(\fint_{\lambda B} \big(\operatorname{Lip} u(x)\big)^p \, d\mu \right)^{1/p}, \tag{1}$$

for all balls B and for every Lipschitz function u.

We wish to warn the reader that this definition is somewhat different from the one in [6] and in the subsequent works. See, however, the discussion after Corollary 6.3 below.

Notice that our Poincaré inequality differs from the usual Euclidean one in two respects. First, the left-hand side is the averaged L^1-integral instead of the usual averaged L^p-integral; so our inequality is actually a $(1, p)$-inequality. Secondly, the integration on the right-hand side is over a larger ball than on the left-hand side.

Thus we should perhaps call our inequality a weak Poincaré inequality. However, it turns out that the Poincaré inequality always improves itself to a (p, p)-inequality, possibly with larger constants C_p and λ. Indeed, even a (q, p)-inequality follows with an optimal $q > p$. This will be explained below (see Section 3). Moreover, the constant λ can often be taken to be 1 by enlarging C_p. To be more precise, this holds if the metric d is a path metric (i.e. infimum of lengths of paths joining the points) and geodesic: in this case the geometry of balls can be controlled and one can iterate the

Poincaré inequality so as to decrease λ. We call such a metric a length metric and the corresponding space a length space. If we assume that X is *proper* (i.e. all closed balls are compact), then it follows from the Poincaré inequality that we can replace the metric d with a bi-Lipschitz equivalent length metric. Because bi-Lipschitz changes of the metric do not affect the validity of our Poincaré inequality, a p-Poincaré inequality also holds (with some new λ and C) for the length metric and we can iterate to decrease λ to 1. The value of λ can thus be assumed to be 1 when X is proper. For this argument see [4] and also Section 6 below.

One should notice that the metric completion of X is always proper because of the doubling condition. Moreover, our Poincaré inequality and our doubling measure extend to this completion. On the other hand, the above Poincaré inequality does not require X to be proper or connected. Indeed, $(\mathbb{R}^2 \setminus S^1, d, \mu)$ supports the Poincaré inequality for all $p \geq 1$ when S^1 is the unit circle, d is the usual Euclidean distance and μ is the Lebesgue measure. This follows from the usual Poincaré inequality in $\mathbb{R}^2$.

It is immediate from Hölder's inequality that the Poincaré inequality for exponent p implies the corresponding inequality for all $q > p$. Thus the strongest of all these inequalities is the $(1,1)$-inequality. This is closely related to relative isoperimetric inequalities. Moreover, the inequality gets strictly weaker when the exponent p increases: given $1 \leq p < q$, one can construct a proper space (X, d, μ) so that the q-Poincaré inequality holds on X but the p-Poincaré inequality fails. Indeed, consider, for example, two closed cubes in the space $\mathbb{R}^n$. If one identifies an edge of each of the cubes (i.e. glues the cubes along an edge), one obtains a space for which the p-Poincaré inequality holds exactly when $p > n-1$. Here the measure is the usual Lebesgue measure and the distance the natural distance for the union. If the cubes get glued along a single vertex, then the p-Poincaré inequality holds exactly when $p > n$. One can also glue the cubes along other subsets, say along Cantor-sets. For example, when the gluing set is a copy of the usual Cantor ternary set, then the p-Poincaré inequality holds if and only if $p > n - \log 3/\log 4$. Notice that in all these examples the allowable range for the exponents is $p > n - \dim(E)$, where $\dim(E)$ is the dimension of the gluing set.

In the classical case of a Euclidean space, the Poincaré inequality (1) extends to hold for all functions in the Sobolev space $W^{1,p}$ when the Lipschitz constant on the right-hand side is replaced by the length of the gradient defined as a suitable limit. One can obtain a generalization of this by taking the completion of the collection of all Lipschitz functions with respect to the norm $\|u\| = \|u\|_p + \|\operatorname{Lip} u\|_p$; the concept of the length of a gradient needs then to be replaced with a more abstract notion for functions that are not Lipschitz. On the other hand, the partial derivatives of a Sobolev function exist almost everywhere and one can thus make sense of a pointwise (length of a) gradient. It turns out that even in our generality a version of this phenomenon persists. This will be given by the concept of an upper gradient.

Let $u : A \to \overline{\mathbb{R}}$, $A \subset X$. Any Borel function $g : A \to [0, \infty]$ such that, for every rectifiable path $\gamma : [0, l] \to A$, $0 < l < \infty$,

$$|u(\gamma(l)) - u(\gamma(0))| \leq \int_\gamma g\, ds$$

is called an *upper gradient* of u on A. Some comments on the above definition are perhaps in order. First of all, our version of the fundamental theorem of calculus requires that $|u(x) - u(y)|$ is well defined whenever the points x, y can be joined by a rectifiable curve, on which g is integrable. In fact, u has to be continuous on every such a curve. Secondly, if the space admits no rectifiable, non-constant curves, then the zero function is an upper gradient of every function u. Furthermore, the function $g \equiv \infty$ is always an upper gradient of every u.

We now define, for given p, $1 \leq p \leq \infty$,

$$N^{1,p}(X) = \{u \in L^p(X) : u \text{ has an upper gradient } g \in L^p(X)\},$$

where the L^p-spaces are taken with respect to our measure μ and the concept of an upper gradient is with respect to our metric d. As a norm on $N^{1,p}$ we take

$$\|u\|_{1,p} = \|u\|_p + \inf_{g_u} \|g_u\|_p,$$

where the two norms on the right are usual L^p-norms and the infimum is taken over all upper gradients g_u of u. As usual, one needs to consider equivalence classes in our definition in order to obtain a normed vector space. The necessary equivalence relation is given by setting $u \approx v$ if $\|u - v\|_{1,p} = 0$. Notice that the representatives are defined outside of a potentially very small set; recall that u is defined and continuous on every curve on which g_u is integrable. If $u \in N^{1,p}$ and $v = u$ almost everywhere, it is not necessarily true that $v \in N^{1,p}$. In the Euclidean setting it turns out that

$$N^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$$

as sets when both the metric and the measure are the usual Euclidean ones. The representatives arising from $N^{1,p}$ are in a sense “fine representatives” of

Sobolev functions. We shall discuss these issues in more detail below. In the general setting, under the properness and Poincaré inequality assumptions, $N^{1,p}(X)$ coincides with the completion of the class of Lipschitz continuous functions with respect to the norm $\|\cdot\|$. The simple (Euclidean) example $(\mathbb{R}^2 \setminus S^1, d, \mu)$ discussed above shows that the properness assumption cannot be disposed of.

3. Sobolev inequalities

Suppose that X is a doubling space that supports a p-Poincaré inequality. In order to talk about Sobolev-type inequalities, we need a substitute for the dimension of X. This is obtained from the doubling condition: a simple iteration argument shows that there are constants $C > 0$ and $s > 0$ so that

$$\mu(B(x,r)) \geq C(r/R)^s \mu(B(x,R)) \tag{2}$$

for every $x \in X$ and all $0 < r < R$. Here $s = \log_2 C_d$. One more application of the doubling condition shows that this estimate also holds when $B(x,r)$ is replaced with $B(y,r) \subset B(x,R)$. In what follows, the exponent s can be chosen to be any number as in (2); it need not be the one obtained from iterating the doubling condition and the estimate is only needed up to the scale of the diameter of the ball in question. Notice, however, that for the Lebesgue measure in $\mathbb{R}^n$ the above argument gives $s = n$.

Theorem 3.1. *Suppose that (X, d, μ) is a doubling length space that supports a p-Poincaré inequality. Let B be a ball and $u \in N^{1,p}(B)$ with an upper gradient g. Suppose that $\mu(B(x,r)) \geq C_b(r/\operatorname{diam}(B))^s \mu(B)$ whenever $B(x,r) \subset B$.*

(i) *If $p < s$, then*

$$\|u - u_B\|_{L^{p^*}(B)} \leq C \operatorname{diam}(B)\mu(B)^{1/p^* - 1/p}\|g\|_{L^p(B)},$$

where $p^* = ps/(s-p)$.

(ii) *If $p = s$, then*

$$\fint_B \exp\left(\frac{C_1 \mu(B)^{1/s}|u - u_B|}{\operatorname{diam}(B)\|g\|_{L^s(B)}}\right)^{s/(s-1)} d\mu \leq C_2.$$

(iii) *If $p > s$, then $|u(x) - u_B| \in L^\infty(B)$ and*

$$\|u - u_B\|_{L^\infty(B)} \leq C \operatorname{diam}(B)\mu(B)^{-1/p}\|g\|_{L^p(B)}.$$

Here $C = C(\lambda, s, C_p, C_b, C_d)$, $C_i = C_i(\lambda, s, C_p, C_b, C_d)$.

The proof of this result is a combination of arguments from [5], see [6].

There is also a version of Theorem 3.1 without the length metric assumption. In this general setting one replaces the sets B of integration for g by the balls τB, where $\tau \geq 1$ is a constant depending on the given data.

We shall not prove Theorem 3.1 here. See however the next section that contains pointwise inequalities that are helpful. Especially, part (iii) can be proven with a slight modification of the arguments given in the next section.

One could also ask for Lorentz norm improvements of the inequalities in Theorem 3.1: The inequality (i) is generalized to the usually looking inequality

$$\inf_{c\in\mathbb{R}} \int_0^\infty t^{p-1} \big[\mu\big(\{x \in B : |u(x) - c| > t\}\big)\big]^{(s-p)/s}\, dt \leq C \int_B g^p(y)\, d\mu(y).$$

There is no Lorentz norm inequality for (iii) but for the Trudinger inequality (ii) one has the generalization

$$\inf_{c\in\mathbb{R}} \int_0^\infty t^{s-1} \log^{1-s}\Big(\frac{\mathrm{e}\,\mu(B)}{\mu\left(\{x \in B : |u(x) - c| > t\}\right)}\Big)\, dt \leq C \int_B g^s(y)\, d\mu(y).$$

These improvements on Theorem 3.1 are given in [10]; the latter inequality has its roots in [13].

4. The Sobolev space

We now state the equivalence of the intrinsic definition of the Sobolev space with the "completion" definition.

Theorem 4.1. *Let X be a proper doubling space that supports a p-Poincaré inequality, $p \geq 1$. Then $N^{1,p}(X)$ consists precisely of those functions in $L^p(X)$ that are L^p-limits of sequences of Lipschitz functions for which also the sequence of the pointwise Lipschitz norms converges in $L^p(X)$. Moreover, when $p > 1$, the space $N^{1,p}(X)$ is reflexive.*

The approximation result here is essentially due to N. Shanmugalingam (see [16]) and the reflexivity was proven by J. Cheeger in [1]. The reflexivity seems to be a very subtle issue: the related space defined using pointwise inequalities similar to those in the next section need not be reflexive. For this see [14]. Cheeger's argument is based on showing that the norm is equivalent to a uniformly convex one. To prove this he constructs in [1] a differential for every Lipschitz function. This then leads to yet another characterization of the Sobolev class as those L^p-functions whose differentials also belong to $L^p(X)$. For details see [6] and also [7] for related results.

Now that the first order Sobolev class has been defined, one could also aim for the higher order theory. It is not clear if upper gradients and pointwise Lipschitz constants could somehow be used as a basis for the theory. A more promising approach is to look at the Poincaré inequality directly: one can show that functions $u \in L^p(X)$ for which there is a function $g \in L^p(X)$ so that the p-Poincaré inequality holds for this pair and all balls are precisely the members of $N^{1,p}(X)$, see [2]. In the Euclidean setting, this generalizes to higher order spaces. Indeed, we can characterize the second order Sobolev space if instead of using averages we use a Poincaré inequality type approximation by linear functions. For general orders one then uses polynomials. There does not seem to be a natural way to construct substitutes for polynomials in the abstract setting of a metric space. When X has group structure, an approach of this kind has been used by Y. Liu, G. Lu and R. L. Wheeden in [11], [12].

5. Pointwise inequalities

It is often convenient to know that the Poincaré inequality can be characterized by a pointwise inequality. We recall that for every $R > 0$ the restricted maximal operator M_R is defined by

$$M_R u(x) = \sup_{0<r<R} \fint_{B(x,r)} |u(y)|\, d\mu,$$

where u is a measurable function. Because the proof of the pointwise inequality is somewhat easier when $p > 1$ and works for pairs of functions, not only pairs of functions and upper gradients, we first only state this case.

Lemma 5.1. *Let (X, d, μ) be a doubling space, u be locally integrable and $g \geq 0$ measurable. If $p > 1$, then the following conditions are quantitatively equivalent:*

(i) *There exist $C > 0$ and $\lambda \geq 1$ such that*

$$\fint_B |u - u_B|\, d\mu \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p} \tag{3}$$

for every ball B.

(ii) *There exist $C > 0$ and $\tau > 0$ such that*

$$|u(x) - u_B| \leq C \operatorname{diam}(B) \left(M_{\tau \operatorname{diam}(B)} g^p(x) \right)^{1/p}$$

for every ball B and almost every $x \in B$.

(iii) *There exist* $C > 0$ *and* $\sigma > 0$ *such that*

$$|u(x) - u(y)| \leq Cd(x,y)\left(M_{\sigma d(x,y)}g^p(x) + M_{\sigma d(x,y)}g^p(y)\right)^{1/p}$$

for almost every $x, y \in X$.

Moreover, even when $p = 1$, *condition* (i) *implies condition* (ii) *which yields condition* (iii).

The proof is rather simple and thus we give it below.

Proof. (i) $\Rightarrow$ (ii). Let $x \in X$ be a Lebesgue point of u; by the Lebesgue differentiation theorem, this is true for almost all points. Write $r = \operatorname{diam}(B)$ and $B_i = B(x, r_i) = B(x, 2^{-i}r)$ for every non-negative integer i. Then u_{B_i} tends to $u(x)$ as i tends to infinity. We have

$$|u(x) - u_B| \leq |u(x) - u_{B_0}| + |u_{B_0} - u_B|,$$

and now, using the doubling property of μ and (i), we conclude that

$$\begin{aligned}
|u(x) - u_{B_0}| &\leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\
&\leq \sum_{i=0}^{\infty} \fint_{B_{i+1}} |u - u_{B_i}|\, d\mu \\
&\leq C_d \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu \\
&\leq C \sum_{i=0}^{\infty} \operatorname{diam}(B_i) \left(\fint_{\lambda B_i} g^p\, d\mu\right)^{1/p} \\
&\leq C \sum_{i=0}^{\infty} \operatorname{diam}(B_i)\left(M_{\lambda r} g^p(x)\right)^{1/p} \\
&= C \operatorname{diam}(B)\left(M_{\lambda r} g^p(x)\right)^{1/p}.
\end{aligned}$$

Here C depends on C_d and the constant in inequality (3).

Furthermore,

$$|u_{B_0} - u_B| \leq |u_{B_0} - u_{2B_0}| + |u_{2B_0} - u_B|$$

and

$$\begin{aligned}|u_{2B_0} - u_B| &\le \fint_B |u - u_{2B_0}|\,d\mu \\ &\le \frac{\mu(2B_0)}{\mu(B)} \fint_{2B_0} |u - u_{2B_0}|\,d\mu \\ &\le C\operatorname{diam}(B_0)\left(\fint_{2\lambda B_0} g^p\,d\mu\right)^{1/p} \\ &\le C\operatorname{diam}(B)\big(M_{2\lambda r} g^p(x)\big)^{1/p}.\end{aligned}$$

The same estimate holds for $|u_{B_0} - u_{2B_0}|$, and now (ii) easily follows.

(ii) ⇒ (iii). Given a ball B and $x, y \in B$ such that $d(x,y) \ge \operatorname{diam}(B)/10$ write

$$|u(x) - u(y)| \le |u(x) - u_B| + |u(y) - u_B|$$

and apply (ii).

(iii) ⇒ (i). Applying Cavalieri's principle and the weak-type estimate for the maximal function, we obtain

$$\begin{aligned}\fint_B |u - u_B|\,d\mu &\le \fint_B \fint_B |u(x) - u(y)|\,d\mu(x)\,d\mu(y) \\ &\le C\operatorname{diam}(B) \fint_B \big(M(g^p\chi_{3\sigma B})\big)^{1/p}\,d\mu \\ &= C\operatorname{diam}(B)\mu(B)^{-1} \int_0^\infty \mu\big(\{x \in B : M(g^p\chi_{3\sigma B}) > t^p\}\big)\,dt \\ &\le C\operatorname{diam}(B)\mu(B)^{-1} \left(\int_0^{t_0} \mu(B)\,dt + \int_{t_0}^\infty \Big(\frac{C}{t^p}\int_{3\sigma B} g^p\,d\mu\Big)\,dt\right) \\ &= C\operatorname{diam}(B)\mu(B)^{-1}\left(t_0\mu(B) + Ct_0^{1-p}\int_{3\sigma B} g^p\,d\mu\right).\end{aligned}$$

The claim follows when we choose $t_0 = \left(\mu(B)^{-1}\int_{3\sigma B} g^p\,d\mu\right)^{1/p}$. □

The previous characterization also holds for $p = 1$ when we consider pairs of functions and their upper gradients (or Lipschitz functions and their pointwise Lipschitz constants) — see the following Lemma 5.2. For a proof of this fact see [4], [6]. The main idea here goes back to V. Maz'ya: conditions (ii) and (iii) give us weak-type inequalities which can be shown to imply the Poincaré inequality using truncation arguments.

Lemma 5.2. *Let (X, d, μ) be a proper doubling space. Then the validity of conditions* (i), (ii) *and* (iii) *of Lemma 5.1 for all functions $u \in N^{1,1}_{\mathrm{loc}}(X)$ and their upper gradients are quantitatively equivalent.*

Notice that already the simple results Lemma 5.1 and Lemma 5.2 indicate that the Poincaré inequality implies a better inequality: estimate (ii) and the weak boundedness of the maximal operator in L^1 show that every function that satisfies (3) with some non-negative $g \in L^p$ belongs to the weak L^p.

6. Quasiconvexity

We say that X is quasiconvex if there exists a constant $C \geq 1$ such that every pair $x, y \in X$ can be joined with a rectifiable curve γ such that

$$\mathrm{length}(\gamma) \leq C d(x, y).$$

The Poincaré inequality does not even require the connectivity of X as the standard example $(\mathbb{R}^2 \setminus S^1, d, \mu)$ shows. When X is assumed to be proper (or, equivalently, complete), we can conclude the quasiconvexity from the Poincaré inequality.

Lemma 6.1. *Assume that (X, d, μ) is a doubling space that supports a p-Poincaré inequality and that X is proper. Then X is quasiconvex.*

Remark 6.2. The proof below shows that it suffices to assume the p-Poincaré inequality for Lipschitz functions and their continuous upper gradients.

Proof. Fix $y \in X$, and let $k \geq 1$. We say that $y = x_1, x_2, \ldots, x_l = x$ is a k-chain from y to x if $d(x_{i+1}, x_i) \leq 1/k$ for every i, $1 \leq i \leq l-1$. Consider the set U_y consisting of all points $x \in X$ for which there exists a k-chain from y to x and the residual set V consisting of those points $x \in X$ for which no such chain exists. Then both U_y and V are clearly open. Moreover, U_y is non-empty as $B(y, 1/k) \subset U_y$.

We claim that $V = \emptyset$. Suppose the opposite. Choose $r > 1/k$ so that $B(y, r) \cap V \neq \emptyset$. Define $u = \chi_{U_y}$. Then u is Lipschitz and $g = 0$ is an upper gradient of u as there can be no curve joining a point of U_y to V; in fact $d(U_y, V) \geq 1/k$. On the other hand, the doubling condition and the facts that $B(y, r) \cap V \neq \emptyset$ and V is open, guarantee that $u = 0$ in a set of positive measure in $B(y, r)$. This contradicts the Poincaré inequality because $\mu(B(y, 1/k)) > 0$ and $u = 1$ on $B(y, 1/k)$. It follows that, for every $k \geq 1$, there is a k-chain to every point $x \in X$.

We now define a function $u_k : X \to [0, \infty)$ by setting

$$u_k(x) = \inf \Big\{ \sum_1^{l-1} d(x_{i+1}, x_i) : \\ y = x_1, x_2, \dots, x_l = x \text{ is a } k\text{-chain from } y \text{ to } x \Big\}.$$

Here the infimum is taken over the k-chains of all lengths. Then u_k is locally 1-Lipschitz and the constant function $g(x) = 1$ is an upper gradient of u_k. We would now like to apply the Poincaré inequality to u_k to obtain a suitable k-chain. The problem is that it is not clear if u_k is Lipschitz. However, it is easy to see that the restriction of u_k to any ball B is Lipschitz. Thus, multiplying u by a suitable Lipschitz cut-off function that equals to 1 in B and with support in $2B$, we may assume that u_k is Lipschitz when we make estimates in εB, where $\varepsilon > 0$ is a small number determined by the data (as the constant σ below). Thus we may assume that the pair u, g satisfies a p-Poincaré inequality and so the pointwise inequality

$$u_k(x) = |u_k(x) - u_k(y)| \le C d(x, y)\big(M_{\sigma d(x,y)} g^p(x) + M_{\sigma d(x,y)} g^p(y)\big)^{1/p}$$

obtained from Lemma 5.1 shows that $u_k(x) \le C d(x, y)$.

We would like to claim that the usual Arzelà-Ascoli argument guarantees that a subsequence of these k-chains converges to a rectifiable curve of length no more than $Cd(x, y)$. This conclusion turns out to be true, but we have to handle a technical problem, caused by the fact that it is not clear how to extend these chains to curves.

We construct a new metric space $\widehat{X}$ by attaching to the space X infinitely many Euclidean segments. We do this as follows. For given k, we choose a k-chain $y = x_{k,1}, \dots, x_{k,l(k)} = x$ so that $\Sigma_1^{l-1} d(x_{i+1}, x_i) \le 2Cd(y, x)$. We glue to the space X a straight Euclidean segment $I_{k,i}$ of length $d(x_{k,i}, x_{k,i+1})$ joining the points $x_{k,i}, x_{k,i+1}$, for $1 \le i \le l(k) - 1$. We repeat this for every $k \ge 1$. The space $\widehat{X}$ is equipped with a natural metric which is induced from the Euclidean metric in every segment and the metric d in X. We denote the metric in $\widehat{X}$ by $\widehat{d}$. Then, for given $k \ge 1$, we obtain a rectifiable curve γ_k joining y to x in $\widehat{X}$ by following the Euclidean segments. Moreover, the length of this curve is no more than $L := 2Cd(x, y)$. Because the lengths of the segments $I_{k,i}$ tend to zero when k tends to infinity, it is easy to check that $(\widehat{X}, \widehat{d})$ is proper. We may assume that each γ_k is parametrized by arc-length, and extending γ_k as $\gamma_k(\text{length}(\gamma_k)) = x$ to $[\text{length}(\gamma_k), L]$ if necessary, we may assume that $\gamma_k : [0, L] \to \widehat{X}$ is 1-Lipschitz.

The Arzelà-Ascoli theorem shows that a subsequence of the curves γ_k converges uniformly to a 1-Lipschitz curve γ of length no more than $2Cd(x,y)$ that joins y to x in $\widehat{X}$. Since the end-points of the segments $I_{k,i}$ lie in the proper space X, and their distance is at most $1/k$, it easily follows that γ lies in X. Then $\gamma : [0,L] \to X$ is also 1-Lipschitz because $\widehat{d} \le d$ on X. The claim follows. □

The previous result allows one to replace the metric of a proper space that supports a Poincaré inequality with a bi-Lipschitz equivalent path metric.

Corollary 6.3. *Suppose that (X, d, μ) supports a p-Poincaré inequality and that X is proper. Define $\widehat{d}(x,y) = \inf_\gamma \operatorname{length}(\gamma)$, where the infimum is taken over all curves that join x and y. Then $\widehat{d}$ is a geodesic metric and there exists a constant C so that*

$$\frac{1}{C} d(x,y) \le \widehat{d}(x,y) \le C d(x,y)$$

for all $x, y \in X$.

The quasiconvexity is crucial in the proof of the fact due to S. Keith (see [8]) that the Poincaré inequality for Lipschitz functions and their continuous upper gradients implies the Poincaré inequality for all functions in $N^{1,p}(X)$ (or in $N^{1,p}_{\mathrm{loc}}(X)$). It then easily follows that this also holds when X supports a Poincaré inequality.

Here is a sketch of a proof of such a result. By the Vitali-Carathéodory theorem, we may assume that the upper gradient g in the Poincaré inequality is lower semicontinuous. Thus there is an increasing sequence of continuous functions g_j that converges to g. Fixing a ball B, we may further assume that $g_j \ge \delta > 0$. By Lemma 5.1 and Lemma 5.2, it suffices to show that, for a.e. x, y,

$$|u(x) - u(y)| \le C d(x,y) \big(M_{\sigma d(x,y)} g^p(x) + M_{\sigma d(x,y)} g^p(y) \big)^{1/p}.$$

We may assume that $u(x) = 1$ and $u(y) = 0$. We construct a new Lipschitz function u_j by setting

$$u_j(z) = \inf_{\gamma_z} \int_{\gamma_z} g_j \, ds,$$

where the infimum is taken over all curves that join z to x in CB. Then u_j is Lipschitz and g_j is a continuous upper gradient of u_j. Applying Lemma 5.1 to the pairs u_j, g_j and using the fact that $g_j \le g$, we notice that it suffices to prove that $u_j(y) \to 1$ for a (sub)sequence. This can be shown with some work by using the Arzelà-Ascoli theorem and the lower semicontinuity of g; notice that if the numbers $u_j(y)$ were small, we would find a sequence of curves of uniformly bounded length joining y to x (recall that $g_j \ge \delta > 0$).

7. Poincaré inequality revisited

There is yet another way to characterize the Poincaré inequality. Following S. Semmes (see [15]), we define, for given $\varepsilon > 0$ and measurable $u : X \to \overline{R}$,

$$D_\varepsilon u(x) = \sup_{y \in B(x,\varepsilon)} \frac{|u(x) - u(y)|}{\varepsilon},$$

for every $x \in X$.

The following result is due to S. Keith and K. Rajala (cf. [9]).

Theorem 7.1. *Let X be a proper doubling space. Then X supports a p-Poincaré inequality, $p \geq 1$, if and only if there are constants C and λ such that*

$$\fint_B |u - u_B|\, d\mu \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} (D_\varepsilon u)^p \, d\mu \right)^{1/p},$$

for every ε and every ball $B \subset X$ of diameter at least 2ε and all u.

The point behind the theorem is essentially that, under the properness assumption, it suffices to consider Lipschitz functions instead of general ones. For such functions the claim is somewhat more transparent than for general functions.

8. Stability

We have already mentioned one form of stability for the Poincaré inequality: bi-Lipschitz changes of the metric do not destroy the inequality. The Poincaré inequality turns out also to persist in convergence of spaces. For this, we need the concept of (pointed) measured Gromov-Hausdorff convergence.

First of all, the Hausdorff distance of two compact subsets X, Y of a metric space Z is the infimum of numbers $\varepsilon > 0$ for which X is contained in the ε-neighborhood $Y_\varepsilon = \{z \in Z : d(z, Y) < \varepsilon\}$ of Y and for which Y is contained in the ε-neighborhood of X. Then the Gromov-Hausdorff distance of compact metric spaces X, Y is the infimum over the Hausdorff distances of all isometric embeddings of X and Y into a common metric space Z. The concept of convergence in this case should now be obvious. One can also make sense of this for unbounded (proper) spaces: one fixes points $x_i \in X_i$ and $y \in Y$ and essentially requires that the isometric embeddings take x_i to the image of y and that, for every $r > 0$, the closed balls centered at x_i

converge to $\overline{B}(y,r)$. We refer the reader to [6] for a more detailed discussion. The convergence of measures is also a natural concept: one requires that the measures $\mu_i(B(\widehat{x}_i,r))$ converge to $\mu(B(\widehat{y},r))$ whenever the balls $B(\widehat{x}_i,r)\subset X_i$ converge to $B(\widehat{y},r)\subset Y$.

We can now state the stability result.

Theorem 8.1. *Suppose that* $\{(X_i,x_i,d_i,\mu_i)\}_i$ *is a sequence of geodesic, pointed, proper doubling spaces so that every space is doubling with the same constant* C_d *and so that every of them supports a p-Poincaré inequality with fixed constants* C_p, λ. *If this sequence converges in the pointed, measured Gromov-Hausdorff sense to a proper space* (X,x,d,μ), *then* (X,d,μ) *is a doubling space that supports a p-Poincaré inequality. Moreover,* (X,d,μ) *is geodesic.*

There is also an associated precompactness result. If we are given a sequence as above, then automatically a subsequence will converge in the pointed Gromov-Hausdorff sense, but the measures need not converge: consider $(\mathbb{R}^2,d,\mu_i)$, where d is the usual Euclidean metric, $\mu_i=i\mu$, and μ the Lebesgue area. One can, however, guarantee the convergence of the measures by a simple modification: we replace μ_i by $\widehat{\mu}_i$ that is defined as $\mu_i(A)=\widehat{\mu}_i(A)/\mu_i(B(x_i,1))$. For all this see [6].

The stability results presented here are essentially due to J. Cheeger (see [1]); the general version given here was proven by S. Keith in [8].

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