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A LA RECHERCHE DU SPECTRE PERDU: AN INVITATION TO NONLINEAR SPECTRAL THEORY

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Abstract. We give a survey on spectra for various classes of nonlinear operators, with a particular emphasis on a comparison of their advantages and drawbacks. Here the most useful spectra are the asymptotic spectrum by M. Furi, M. Martelli and A. Vignoli (1978), the global spectrum by W. Feng (1997), and the local spectrum (called "phantom") by P. Santucci and M. Väth (2000). In the last part we discuss these spectra for homogeneous operators (of any degree), and derive a discreteness result and a nonlinear Fredholm alternative for such operators. This may be applied to an eigenvalue problem for the p-Laplace operator which arises in various fields of applied mathematics, mechanics, and physics.

1. Linear spectral theory

The notion of spectrum is particularly important for linear operators but spectral ideas have also been used in the study of nonlinear operator equations. This has led to the development of various theories of spectrum for nonlinear operators which attempt to preserve the useful properties of the linear case, on the one hand, but to apply to a possibly large variety of nonlinear problems, on the other hand.

The purpose of these lectures is to discuss three particularly important spectra for nonlinear operators, viz. the "asymptotic" Furi-Martelli-Vignoli spectrum, the "global" Feng spectrum, and the "local" Väth phantom. Moreover, we show that these spectra coincide for certain classes of operators, and indicate how this may be used to prove some kind of "nonlinear Fredholm alternative" with applications to eigenvalue problems for the

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p-Laplace operator. The main emphasis, however, will be put on illuminating examples, rather than abstract theorems.

To put things in the right framework, let us first recall some properties of the "linear" spectrum. Given a Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and an operator $L \in \mathfrak{L}(X)$, the algebra of all bounded linear operators on X, the *resolvent set* of L is defined by

$$\rho(L) = \{\lambda \in \mathbb{K} : (\lambda I - L)^{-1} \in \mathfrak{L}(X)\}, \tag{1}$$

and the *spectrum* of L by

$$\sigma(L) = \mathbb{K} \setminus \rho(L). \tag{2}$$

This spectrum has some remarkable properties which we collect for further reference:

- $\sigma(L)$ is bounded, closed, and (in case $\mathbb{K} = \mathbb{C}$) nonempty,
- $\sigma(L)$ is at most countable (and so has empty interior) for L compact,
- $\sigma(L)$ is even finite in case $X = \mathbb{K}^n$ (pure eigenvalue spectrum),
- $\sigma(L)$ is bounded by $r(L) = \lim_{n\to\infty} \sqrt[n]{\|L^n\|}$ (Gel'fand's formula),
- $\sigma(p(L)) = p(\sigma(L))$ for any polynomial p (spectral mapping theorem),
- the map $\rho(L) \ni \lambda \mapsto (\lambda I - L)^{-1} \in \mathfrak{L}(X)$ is analytic,
- the multivalued map $\mathfrak{L}(X) \ni L \mapsto \sigma(L) \in 2^{\mathbb{K}}$ is upper semicontinuous.

In view of the last property, we remark that the multivalued map $\mathfrak{L}(X) \ni L \mapsto \sigma(L) \in 2^{\mathbb{K}}$ is in general not lower semicontinuous. This means that, roughly speaking, the spectrum cannot blow up, but it may very well shrink down when the operator changes continuously. Since this seems not to be well known, we present a simple example.

E x a m p l e 1. Let $X = l_1(\mathbb{Z}, \mathbb{C})$ be the Banach space of all summable complex sequences indexed with the integers, and let

$$\{\ldots, e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3, \ldots\}$$

denote the usual canonical basis in X. For $\varepsilon \in \mathbb{R}$, define $L_\varepsilon \in \mathfrak{L}(X)$ by $L_\varepsilon e_k = e_{k-1}$ for $k \neq 0$ and $L_\varepsilon e_0 = \varepsilon e_{-1}$. Then $\sigma(L_0) = \{z \in \mathbb{C} : |z| \leq 1\}$ but $\sigma(L_\varepsilon) = \{z \in \mathbb{C} : |z| = 1\}$ for $\varepsilon \neq 0$, and so the spectrum of L_ε "collapses" when leaving the value $\varepsilon = 0$.

In the next section we discuss three spectra for various classes of nonlinear operators which are modelled on the linear definition (2), and show that these spectra do not have "good" properties.

2. Three nonlinear spectra: the "naive" approach

If one wants to build a spectral theory in the nonlinear case, one is led to some "natural" requirements. So, given a continuous nonlinear operator $F : X \to X$, let us try to define a *spectrum* $\sigma(F)$ for F such that

- $\sigma(F)$ is the familiar spectrum for F bounded linear,
- $\sigma(F)$ has the usual properties (compact, nonempty etc.),
- $\sigma(F)$ contains the point spectrum $\sigma_p(F)$ (eigenvalues) of F,
- $\sigma(F)$ has applications (e.g., existence, uniqueness, bifurcation problems).

We remark that the *point spectrum* of F in the third requirement is defined precisely as in the linear case, i.e.

$$\sigma_p(F) = \{\lambda \in \mathbb{K} : F(u) = \lambda u \text{ for some } u \neq 0\}. \tag{3}$$

Some "naive" definitions in this spirit have been given in the last 30 years. The philosophy is simply to replace the algebra $\mathfrak{L}(X)$ in (1) by other classes of continuous (nonlinear) operators. For the set $\mathfrak{C}(X)$ of all continuous operators $F : X \to X$ this leads to the *Rhodius resolvent set*

$$\rho_R(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ homeomorphism}\} \tag{4}$$

and the *Rhodius spectrum* [23]

$$\sigma_R(F) = \mathbb{K} \setminus \rho_R(F). \tag{5}$$

Thus, a scalar λ belongs to $\sigma_R(F)$ if $\lambda I - F$ is not a bijection, or $(\lambda I - F)^{-1}$ exists but is discontinuous. However, the following very simple scalar examples show that the spectrum (5) does not have the familiar properties.

E x a m p l e 2. In $X = \mathbb{R}$, let $F(u) = u^n$. Then $\sigma_R(F) = \mathbb{R}$ if n is even and $\sigma_R(F) = (0, \infty)$ if n is odd. So the Rhodius spectrum may be unbounded or not closed.

E x a m p l e 3. In $X = \mathbb{R}$, let $F(u) = \sqrt{|u|}$. Then $\sigma_R(F) = \mathbb{R}$. Moreover, the point spectrum of F is here $\sigma_p(F) = \mathbb{R} \setminus \{0\}$, since every straight line $f(u) = \lambda u$ hits the graph of F for $\lambda \neq 0$ in some non-zero point u. This shows that, in contrast to the linear case, the point spectrum (3) is "too big" to contain any reasonable information on F.

E x a m p l e 4. In $X = \mathbb{R}$, take $F(u) = 0$ for $u \leq 1$, $F(u) = u - 1$ for $1 < u < 2$, and $F(u) = 1$ for $u \geq 2$. Then $\sigma_R(F) = [0, 1]$, but $\sigma_R(F^2) = \{0\}$, since $F(F(u)) \equiv 0$. This shows that neither a Gel'fand type formula nor a spectral mapping theorem for polynomials can hold for the Rhodius spectrum (5).

Example 5. Let $F:\mathbb{C}\to\mathbb{C}$ be defined by $F(z)=\min\{1,|z|\}\,e^z$. Then F is onto, but for all $\varepsilon>0$ there exists some map $G:\mathbb{C}\to\mathbb{C}$ such that $|G(z)|\le\varepsilon$, but $F+G$ is not onto. (To see this consider, e.g., $G(z)=\max\{\varepsilon(1-|z|),0\}$.) So surjectivity is an unstable property.

Example 6. Let $F:\mathbb{C}^2\to\mathbb{C}^2$ be defined by $F(z,w)=(\overline{w},i\overline{z})$. Then $\lambda I-F$ is a homeomorphism for *each* $\lambda\in\mathbb{C}$; in fact, the inverse

$$(\lambda I-F)^{-1}(z,w)=\left(\frac{\overline{\lambda}z+\overline{w}}{i+|\lambda|^2},\frac{\overline{\lambda}w+i\overline{z}}{i-|\lambda|^2}\right)\tag{6}$$

is a continuous bijection for any $\lambda\in\mathbb{C}$. So $\sigma_R(F)=\emptyset$ in this example. (Observe, however, that $F^2(z,w)=(-iz,iw)$ is *linear* with $\sigma_R(F^2)=\sigma(F^2)=\{\pm i\}$!)

The maps discussed in Examples 2, 3 and 5 have essentially "nonlinear growth". So one might guess that a more reasonable spectral theory could be obtained by restricting the class of operators to those which are "closer to being linear". One possible choice is that of *Lipschitz continuous* maps $F:X\to X$, i.e., those whose minimal Lipschitz constant

$$[F]_{Lip}=\sup_{u\neq v}\frac{\|F(u)-F(v)\|}{\|u-v\|}\tag{7}$$

is finite. Let us call a map $F:X\to X$ a *lipeomorphism* if F is bijective and both $[F]_{Lip}<\infty$ and $[F^{-1}]_{Lip}<\infty$. In analogy to (4) and (5) we put

$$\rho_K(F)=\{\lambda\in\mathbb{K}:\ \lambda I-F\ \text{lipeomorphism}\}$$

and

$$\sigma_K(F)=\mathbb{K}\setminus\rho_K(F),\tag{8}$$

and call these sets the *Kachurovskij resolvent set* and the *Kachurovskij spectrum*, respectively (see [17]). The authors of [18] have studied this spectrum, being obviously unaware of the paper [17]; in particular, they obtained the following result.

Theorem 1. *The Kachurovskij spectrum* (8) *is always compact; moreover, the inclusion*

$$\sigma_K(F)\subseteq\{\lambda\in\mathbb{K}:|\lambda|\le[F]_{Lip}\}\tag{9}$$

holds.

The proof of Theorem 1 simply follows from the fact that, by the Banach contraction mapping principle, the operator $u\mapsto(F(u)+v)/\lambda$ has a unique fixed point for each $v\in X$ and $|\lambda|>[F]_{Lip}$, and that $[(\lambda I-F)^{-1}]_{Lip}\le 1/(|\lambda|-[F]_{Lip})$ for such λ. Since $[L]_{Lip}=\|L\|$ for $L\in\mathfrak{L}(X)$, the inclusion (9) generalizes the classical estimate of the spectrum of a bounded linear operator through its norm.

In [18] the authors also show that $\sigma_K(F) \neq \emptyset$ in the one-dimensional case $X = \mathbb{C}$, and pose the question whether or not this is also true in higher dimensions. Example 6 gives a negative answer to this question, since the inverse operator (6) is in fact a lipeomorhism for any $\lambda \in \mathbb{C}$, and so $\sigma_K(F) = \emptyset$. However, one may easily show (see [2]) that $0 \in \sigma_K(F)$ if X is infinite dimensional and F is in addition compact. This is of course analogous to the linear case.

There is another class of maps which gives an interesting spectrum, viz. that of *continuously differentiable* operators. Recall that $F : X \to X$ has a Fréchet derivative $F'(x_0) \in \mathfrak{L}(X)$ at $x_0 \in X$ if

$$F(x_0 + h) - F(x_0) - F'(x_0)h = o(h) \quad \text{as} \quad h \to 0.$$

As usual, by $\mathfrak{C}^1(X)$ we denote the linear space of all continuously differentiable operators $F : X \to X$. In this case we are led to the *Neuberger resolvent set*

$$\rho_N(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ diffeomorphism}\}$$

and the *Neuberger spectrum* ([21]),

$$\sigma_N(F) = \mathbb{K} \setminus \rho_N(F), \tag{10}$$

respectively. For instance, for F in Example 2 we get $\sigma_N(F) = \mathbb{R}$ if n is even, but $\sigma_N(F) = [0, \infty)$ if n is odd (because for $n = 3, 5, 7, \ldots$ the mapping $u \mapsto u^n$ is a homeomorphism, but not a diffeomorphism). So also the Neuberger spectrum may be unbounded. Simple examples show that it need not be closed either. On the other hand, the following remarkable property of $\sigma_N(F)$ was proved in [21].

Theorem 2. *The Neuberger spectrum* (10) *is always nonempty in the case* $\mathbb{K} = \mathbb{C}$.

Theorem 2 does not contradict Example 6 because the operator $F(z, w) = (\overline{w}, i\overline{z})$ is not differentiable at any point.

Since Fréchet differentiability is related to linear approximation, one could expect that there is some link between the Neuberger spectrum of an operator $F \in \mathfrak{C}^1(X)$, on the one hand, and the (classical) spectra of its derivatives $F'(x)$, on the other hand. This is in fact true. We recall a representation formula for the Neuberger spectrum from [2] whose proof relies on the classical Banach-Mazur lemma which states that a continuous map is a global homeomorphism if and only if it is a local homeomorphism and *proper* (i.e., the preimage of any compact set is compact).

Theorem 3. *For $F \in \mathfrak{C}^1(X)$, the representation*

$$\sigma_N(F) = \pi(F) \cup \bigcup_{x \in X} \sigma(F'(x))$$

holds, where $\pi(F)$ denotes the set of all $\lambda \in \mathbb{K}$ such that $\lambda I - F$ is not proper, and $\sigma(F'(x))$ is the usual spectrum of the bounded linear operator $F'(x)$.

Of course, Theorem 3 implies Theorem 2, since $\sigma(F'(x)) \neq \emptyset$ for all $x \in X$ if $\mathbb{K} = \mathbb{C}$. We illustrate Theorem 3 by a simple example for a mildly nonlinear integral operator which is also taken from [2].

E x a m p l e 7. Let $X = C[0,1]$ be the space of all real continuous functions on $[0,1]$, and let $F : X \to X$ be the Hammerstein integral operator defined by

$$Fx(t) = \int_0^1 t^\alpha s^\beta \sin x(s)\, ds \quad (\alpha, \beta \geq 0).$$

The operator may be written as composition $F = KN$ of the nonlinear Nemytskij operator $Nx(t) = \sin x(t)$ and the linear Fredholm integral operator

$$Ky(t) = \int_0^1 t^\alpha s^\beta y(s)\, ds.$$

From classical differentiability criteria of Nemytskij operators it follows that the Fréchet derivative $F'(x) = KN'(x)$ of F at $x \in X$ is given by

$$F'(x)h(t) = \int_0^1 t^\alpha s^\beta \cos x(s)\, h(s)\, ds \quad (h \in X),$$

i.e., it is a linear integral operator with degenerate kernel function $k_x(t,s) = t^\alpha s^\beta \cos x(s)$. A direct calculation shows that this operator has precisely one eigenvalue

$$\lambda = \lambda_x = \int_0^1 k_x(t,t)\, dt \in \left[-\frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}\right]. \tag{11}$$

Conversely, it is easy to see that every λ from the interval on the right-hand side of (11) is an eigenvalue of $F'(x)$ for suitable $x \in X$. Moreover, being a compact operator, F cannot be proper. On the other hand, $\lambda I - F$ is proper for $\lambda \neq 0$, and so $\pi(F) = \{0\}$. From Theorem 3 we conclude that

$$\sigma_N(F) = \left[-\frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}\right]$$

in this example. We point out that a direct calculation of the Neuberger spectrum in this case would have been more complicated.

For the sake of completeness, we collect the three spectra discussed so far, as well as the point spectrum (3) for the operators from Examples 2–6 in a table.

Operator F	$\sigma_R(F)$	$\sigma_K(F)$	$\sigma_N(F)$	$\sigma_p(F)$
Example 2	$\mathbb{R}$ / $(0,\infty)$	——	$\mathbb{R}$ / $[0,\infty)$	$\mathbb{R}\setminus\{0\}$ / $(0,\infty)$
Example 3	$\mathbb{R}$	——	——	$\mathbb{R}\setminus\{0\}$
Example 4	$[0,1]$	$[0,1]$	——	$[0,\frac{1}{2}]$
Example 5	$\mathbb{C}$	——	——	$\mathbb{C}$
Example 6	$\emptyset$	$\emptyset$	——	$\emptyset$

Our examples show that the definitions (5), (8) and (10) are in a certain sense too "naive" to be really useful for building a reasonable nonlinear spectral theory. For instance, we have seen that no spectrum defined in this way satisfies the four requirements given at the beginning of this section. (It is easy to see, however, that they all contain the point spectrum (3) and reduce to the familiar spectrum for linear operators.) A detailed discussion of the advantages and drawbacks of these and other spectra may be found in the survey article [1]. One might therefore try to define nonlinear spectra by a completely different method, and so we will do in the next section.

3. Three nonlinear spectra: a different approach

An alternative approach to define "reasonable" spectra consists in decomposing the spectrum into several *subspectra*. Recall that a classical (not necessarily disjoint) decomposition of the spectrum $\sigma(L)$ in the linear case is

- $\lambda I - L$ is not onto (the "defect spectrum" $\sigma_\delta(L)$),
- $\lambda I - L$ is not 1–1 (the "point spectrum" $\sigma_p(L)$),
- $\lambda I - L$ is not proper (the "compression spectrum" $\sigma_{co}(L)$).

Unfortunately, there is no canonical analogue to this in the nonlinear case. However, one may "imitate" these properties in different ways. To this end we have to recall some definitions. The (Kuratowski) *measure of noncompactness* $\alpha(M)$ of a bounded set $M \subset X$ is defined as infimum of all $\delta > 0$ such that M may be covered by finitely many sets of diameter not greater than δ. So we have $\alpha(M) = 0$ if and only if M has compact closure (which motivates the name). Now, given a nonlinear operator F between two Banach spaces X and Y, the two conditions

$$\alpha(F(M)) \leq k\alpha(M) \quad (M \subset X \text{ bounded}) \tag{12}$$

and

$$\alpha(F(M)) \geq \ell\alpha(M) \quad (M \subset X \text{ bounded}) \tag{13}$$

are crucial. The smallest constant k in (12) will be denoted by $[F]_A$, and the largest constant ℓ in (13) by $[F]_a$ in what follows. So $[F]_A = 0$ if and only if F is compact, and $[F]_a > 0$ implies that F is proper on closed bounded sets. Moreover, one always has the estimate $[F]_A \leq [F]_{Lip}$, with $[F]_{Lip}$ given by (7); in particular, $[L]_A \leq \|L\|$ for $L \in \mathfrak{L}(X,Y)$.

For $F : X \to Y$ we call the two characteristics

$$[F]_Q = \limsup_{\|u\|\to\infty} \frac{\|F(u)\|}{\|u\|} \tag{14}$$

and

$$[F]_q = \liminf_{\|u\|\to\infty} \frac{\|F(u)\|}{\|u\|} \tag{15}$$

the *upper and lower quasinorm* of F, respectively. Following M. Furi, M. Martelli and A. Vignoli [14] we call a continuous operator $F : X \to Y$ *stably solvable* if, for any compact operator $G : X \to Y$ satisfying $[G]_Q = 0$, the coincidence equation

$$F(u) = G(u) \tag{16}$$

has a solution $u \in X$. Taking $G(u) \equiv v$ for fixed $v \in Y$, one readily sees that stable solvability implies surjectivity. The converse is not true: the map $F(u) = u/\sqrt{1 + |u|}$ is a homeomorphism in $X = \mathbb{R}$, but not stably solvable (consider $G(u) = F(u) + 1$). On the other hand, one may show (see [14]) that for $L \in \mathfrak{L}(X,Y)$ the stable solvability is *equivalent* to the surjectivity of L.

Before introducing the first "non-standard" spectrum, we make a general remark. The examples in the preceding section show that the definition (3) of the point spectrum is not suitable for nonlinear operators F. As a matter of fact, there is absolutely no reason to compare a nonlinear operator F with the *identity* I because this is too much modelled on the linear case. Instead, it is a useful device to replace the identity by some other "well-behaved" *nonlinear* operator J which takes into account the analytical and topological properties of the given operator F. So from now on, we shall define spectra for *pairs of operators* (F, J) between two Banach spaces X

and Y, although the original definitions have been given only in case $X = Y$ and $J = I$.

The *Furi-Martelli-Vignoli spectrum* of $F, J : X \to Y$ (see [15]) is defined as the union

$$\sigma_{FMV}(F, J) = \sigma_{ss}(F, J) \cup \sigma_q(F, J) \cup \sigma_a(F, J) \tag{17}$$

of the three subspectra

$$\begin{aligned} \sigma_{ss}(F, J) &= \{\lambda \in \mathbb{K} : \lambda J - F \text{ is not stably solvable}\}, \\ \sigma_q(F, J) &= \{\lambda \in \mathbb{K} : [\lambda J - F]_q = 0\} \end{aligned} \tag{18}$$

and

$$\sigma_a(F, J) = \{\lambda \in \mathbb{K} : [\lambda J - F]_a = 0\}.$$

By what we have observed before, for $L \in \mathfrak{L}(X)$ we get the relations

$$\sigma_{ss}(L, I) = \sigma_\delta(L), \quad \sigma_q(L, I) \supseteq \sigma_p(L), \quad \sigma_a(L, I) \subseteq \sigma_{co}(L).$$

We collect some important properties of the spectrum (17) in the following theorem (cf. [15]) which is similar to Theorem 1.

Theorem 4. *Suppose that $J : X \to Y$ is stably solvable with $[J]_Q < \infty$. Then the Furi-Martelli-Vignoli spectrum* (17) *is closed and upper semicontinuous. Moreover,*

$$\sigma_{FMV}(F, J) \subseteq \left\{ \lambda \in \mathbb{K} : |\lambda| \le \max \left\{ \frac{[F]_A}{[J]_a}, \frac{[F]_Q}{[J]_q} \right\} \right\}, \tag{19}$$

and so $\sigma_{FMV}(F, J)$ is compact if $[F]_A < \infty$, $[F]_Q < \infty$, $[J]_a > 0$ and $[J]_q > 0$. Finally, $\sigma_{FMV}(L, I)$ coincides with the familiar spectrum (2) *for $L \in \mathfrak{L}(X)$.*

We remark that Theorem 4 was proved for $X = Y$ and $J = I$ in [15]. In this case we have $[I]_Q = [I]_q = [I]_a = 1$, and so $\sigma_{FMV}(F, I)$ is compact if just $[F]_A < \infty$ and $[F]_Q < \infty$.

So we see that the Furi-Martelli-Vignoli spectrum (17) has many good properties in common with the usual linear spectrum. However, it has the series flaw of *not containing the point spectrum* (3). In fact, for the map F from Example 3 we already know that $\sigma_p(F) = \mathbb{R} \setminus \{0\}$. On the other hand, from (19) and the trivial relations $[F]_A = [F]_Q = 0$ we get $\sigma_{FMV}(F, I) = \{0\}$. Thus, in this example the spectrum and the point spectrum are even *disjoint*, and this is of course in sharp contrast to what we want.

A scrutiny of the situation shows that this unpleasant phenomenon is due to the "bad" definition of the point spectrum (3) which is simply inadequate for the Furi-Martelli-Vignoli spectrum. Indeed, the definition (3) is "global", while the definition (17) is "asymptotic". Consequently, one has to replace the inappropriate definition (3) by the *asymptotic point spectrum* (18) which, by (17), is always a part of the spectrum $\sigma_{FMV}(F)$. For instance, for the map F from Example 3 we have $\sigma_q(F, I) = \{0\}$ because $[\lambda I - F]_q = |\lambda|$.

Let us show a more interesting example how to calculate the Furi-Martelli-Vignoli spectrum in an infinite dimensional space.

E x a m p l e 8. Let X be an infinite dimensional real Banach space, and let $F : X \to X$ be defined by

$$F(u) = \|u\|u.$$

It is not hard to see that F is a proper homeomorphism with the inverse

$$F^{-1}(v) = \frac{v}{\sqrt{\|v\|}}.$$

We claim that $\sigma_{FMV}(F, I) = [0, \infty)$. First of all, the eigenvalue equation $F(u) = \lambda u$ has a nontrivial solution u with $\lambda = \|u\|$ if and only if $\lambda > 0$, and so $\sigma_p(F, I) = (0, \infty)$. On the other hand, it is easy to see that F has no asymptotic eigenvalue at all, i.e. $\sigma_q(F, I) = \emptyset$.

Taking as M the closed ball $B_{1/n} = \{u \in X : \|u\| \leq 1/n\}$ in (13) shows that $[F]_a = 0$. Likewise, the fact that $\lambda I - F$ maps, for $\lambda > 0$, the sphere $\{u \in X : \|u\| = \lambda\}$ into $\{0\}$ implies that $[\lambda I - F]_a = 0$ for such λ. Now fix $\lambda < 0$ and consider the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(t) = \frac{\sqrt{\lambda^2 + 4t} + \lambda}{2t} \quad (0 < t < \infty).$$

L'Hospital's rule shows that f admits a continuous extension to 0 by putting $f(0) = 1/|\lambda|$. A straightforward computation shows that

$$(\lambda I - F)^{-1}(v) = f(\|v\|)v$$

for every $v \in X$; for $\lambda = 0$ this coincides of course with the formula for F^{-1} above. It is not hard to see that f is bounded on $[0, \infty)$, say $|f(t)| \leq M_0$.

Moreover, the derivative of f has the property that $t|f'(t)|$ remains bounded as $t \to 0$ or $t \to \infty$. For $u, v \in X$, $\|u\| \leq \|v\|$, we have

$$\begin{aligned}\big\|f(\|u\|)u - f(\|v\|)v\big\| &= \big\|[f(\|u\|) - f(\|v\|)]u + f(\|v\|)(u-v)\big\| \\ &\leq \big|f'(\tau)(\|u\| - \|v\|)\big|\,\|u\| + M_0\|u-v\|,\end{aligned}$$

where $\|u\| \leq \tau \leq \|v\|$. Since the map $t \mapsto t|f'(t)|$ is bounded, say $t|f'(t)| \leq M_1$, we conclude that

$$\|(\lambda I - F)^{-1}(u) - (\lambda I - F)^{-1}(v)\| \leq (M_0 + M_1)\|u - v\|.$$

Interchanging the role of u and v, we get the same estimates when $\|v\| \leq \|u\|$, and so we have proved that $(\lambda I - F)^{-1}$ is Lipschitz continuous on X for $\lambda < 0$. Consequently, $[\lambda I - F]_a = [(\lambda I - F)^{-1}]_A^{-1} \geq [(\lambda I - F)^{-1}]_{Lip}^{-1} > 0$, and so $\sigma_a(F, I) = [0, \infty)$.

It remains to calculate the subspectrum $\sigma_{ss}(F, I)$. Suppose that $G : X \to X$ is compact with the quasinorm $[G]_Q = 0$; in particular, the last condition implies that G maps a closed ball B_R for sufficiently large $R > 0$ into itself. Since $\|(\lambda I - F)(u)\| \geq \|u\|$, for any λ, hence $\|(\lambda I - F)^{-1}(v)\| \leq \|v\|$, the (compact!) operator $(\lambda I - F)^{-1}G$ also maps the ball B_R into itself, and so has a fixed point $\hat{u} \in B_R$, by Schauder's fixed point theorem. But then $\lambda\hat{u} - F(\hat{u}) = G(\hat{u})$ which shows that $\lambda I - F$ is stably solvable. We conclude that $\sigma_{ss}(F, I) = \emptyset$, and so $\sigma_{FMV}(F, I) = [0, \infty)$ as claimed.

The drawback of the spectrum $\sigma_{FMV}(F, I)$ not to contain the eigenvalues of F in the sense of (3) has motivated W. Feng [11] to define another spectrum $\sigma_F(F, J)$ (for $J = I$) which is quite similar to the spectrum $\sigma_{FMV}(F, J)$ but contains the classical point spectrum

$$\sigma_p(F, J) = \{\lambda \in \mathbb{K} : F(u) = \lambda J(u) \text{ for some } u \neq 0\}.$$

To describe this spectrum, again some definitions are in order. First, we have to replace the "asymptotic" characteristics (14) and (15) by the "global" characteristics

$$[F]_B = \sup_{u \neq 0} \frac{\|F(u)\|}{\|u\|}$$

and

$$[F]_b = \inf_{u \neq 0} \frac{\|F(u)\|}{\|u\|}.$$

It is clear that $[F]_b \leq [F]_q \leq [F]_Q \leq [F]_B$, and so $\sigma_b(F, J) \supseteq \sigma_q(F, J)$, where

$$\sigma_b(F, J) = \{\lambda \in \mathbb{K} : [\lambda J - F]_b = 0\}.$$

In what follows, we denote by $\mathfrak{O}(X)$ the family of all open, bounded, connected subsets Ω of a Banach space X with $0 \in \Omega$; as a model example one may think of the open ball $\Omega = B_r^o$ for $r > 0$. Given $\Omega \in \mathfrak{O}(X)$, a continuous map $F : \overline{\Omega} \to Y$ is called *epi* (see [16]) if, first,

$$F(u) \neq 0 \quad (u \in \partial\Omega) \tag{20}$$

and, second, the coincidence equation (16) has a solution $u \in \Omega$ for every compact operator $G : \overline{\Omega} \to Y$ satisfying $G(u) \equiv 0$ on $\partial\Omega$. Thus, epi operators have a similar meaning on small balls as stably solvable operators on large spheres. It is not hard to see that, if an operator $F : X \to Y$ is stably solvable on X with $[F]_b > 0$, then F is epi on every $\Omega \in \mathfrak{O}(X)$. In fact, the condition $[F]_b > 0$ implies that $F(u) \neq 0$ for $u \neq 0$. Moreover, if $G : \overline{\Omega} \to Y$ is compact and satisfies $G(u) \equiv 0$ on $\partial\Omega$, we may extend G by continuity to the whole space X putting $G(u) \equiv 0$ outside $\overline{\Omega}$. Then trivially $[G]_Q = 0$, and so the equation (16) has a solution $\hat{u} \in X$, by assumption. Clearly, we must have $\hat{u} \in \Omega$ since $F(u) \neq 0 = G(u)$ outside Ω, and so we see that F is epi on Ω.

Now, the *Feng spectrum* of $F, J : X \to Y$ (see [11]) is defined as the union

$$\sigma_F(F, J) = \sigma_e(F, J) \cup \sigma_b(F, J) \cup \sigma_a(F, J), \tag{21}$$

where

$$\sigma_e(F, J) = \{\lambda \in \mathbb{K} : \lambda J - F \text{ is not epi on some } \Omega \in \mathfrak{O}(X)\}.$$

Our previous discussion shows that the inclusion $\sigma_{FMV}(F, J) \subseteq \sigma_F(F, J)$ is true. Moreover, we have the following result which is parallel to Theorem 4.

Theorem 5. *Suppose that $J : X \to Y$ is epi on every $\Omega \in \mathfrak{O}(X)$ with $[J]_B < \infty$. Then the Feng spectrum* (21) *is closed and upper semicontinuous. Moreover, the inclusion*

$$\sigma_F(F, J) \subseteq \left\{ \lambda \in \mathbb{K} : |\lambda| \leq \max \left\{ \frac{[F]_A}{[J]_a}, \frac{[F]_B}{[J]_b} \right\} \right\}$$

holds, and so $\sigma_F(F, J)$ is compact if $[F]_A < \infty$, $[F]_B < \infty$, $[J]_a > 0$, and $[J]_b > 0$. Finally, $\sigma_{FMV}(L, I)$ coincides with the familiar spectrum (2) *for $L \in \mathfrak{L}(X)$.*

Again, Theorem 5 essentially simplifies for $X = Y$ and $J = I$. In this case we have $[I]_B = [I]_b = [I]_a = 1$, and so $\sigma_F(F, I)$ is compact if just $[F]_A < \infty$ and $[F]_B < \infty$.

Consider, for example, the map F from Example 3 (and $J = I$). We have seen that $\sigma_{FMV}(F, I) = \{0\}$ in this case. On the other hand, from the relation $\sigma_p(F, I) = \mathbb{R} \setminus \{0\}$ and the closedness of the Feng spectrum in case $J = I$ we deduce that $\sigma_F(F, J) = \mathbb{R}$. This shows that the Feng spectrum takes into account the global behaviour of F, while the Furi-Martelli-Vignoli spectrum reflects only the asymptotic properties of F.

On the other hand, for the operator F from Example 8 (and $J = I$) the Feng spectrum and the Furi-Martelli-Vignoli spectrum are the same. To see this, fix $\lambda < 0$ and $\Omega \in \mathfrak{O}(X)$, and let $G : \overline{\Omega} \to X$ be compact with $G(u) \equiv 0$ on $\partial\Omega$. The operator $(\lambda I - F)^{-1}G : \Omega \to (\lambda I - F)^{-1}G(\Omega)$ is then a compact homeomorphism which vanishes on the boundary $\partial\Omega$, and so has a fixed point in Ω. We conclude that $\lambda I - F$ is epi for $\lambda < 0$, and so $\sigma_F(F, I) = [0, \infty)$.

M. Väth ([24]–[26]) proposed yet another approach which goes as follows. Let $\Omega \in \mathfrak{O}(X)$ and let $F : \overline{\Omega} \to Y$ be continuous with

$$\inf \{\|F(u)\| : u \in \partial\Omega\} > 0. \tag{22}$$

Then F is called *strictly epi* if there exists some $k > 0$ such that, for any operator $G : \overline{\Omega} \to Y$ satisfying $G(u) \equiv 0$ on $\partial\Omega$ and $[G]_A \leq k$, the coincidence equation (16) has a solution in Ω. Thus, strictly epi maps are stronger than epi maps in two ways: first, condition (20) is replaced by the stronger condition (22), and, instead of compact perturbations G in (16), one allows also "slightly noncompact" right-hand sides.

The *Väth phantom* of $F, J : X \to Y$ (cf. [24]) is defined by

$$\phi(F, J) = \bigcap_{\Omega \in \mathfrak{O}(X)} \phi(F, J; \Omega), \tag{23}$$

where

$$\begin{aligned}\phi(F, J; \Omega) = \{\lambda \in \mathbb{K} : \lambda J - F \text{ is not strictly epi on } \Omega\} \\ \cup \{\lambda \in \mathbb{K} : F(u) = \lambda J(u) \text{ for some } u \in \partial\Omega\}.\end{aligned}$$

There is also an analogue of Theorems 4 and 5 for the phantom (23): If J is strictly epi on some $\Omega \in \mathfrak{O}(X)$, then $\phi(F, J)$ is closed and upper semicontinuous (with respect to a suitable topology), and even compact if

$$[F|_{\overline{\Omega}}]_A < \infty, \quad [J|_{\overline{\Omega}}]_a > 0, \quad \sup_{u \in \partial\Omega} \|F(u)\| < \infty, \quad \inf_{u \in \partial\Omega} \|J(u)\| > 0.$$

In case $J = I$ these conditions are automatically satisfied on each $\Omega \in \mathfrak{O}(X)$. We remark that $\phi(L, I)$ also coincides with the usual spectrum (2) in the linear case.

As for the other spectra, we still have to introduce an appropriate notion of the point spectrum associated to the phantom (23). Let us call $\lambda \in \mathbb{K}$ a *connected eigenvalue* of the pair (F, J) if the solution set of the generalized eigenvalue equation

$$F(u) = \lambda J(u) \tag{24}$$

contains an unbounded connected subset C with $0 \in C$. We write $\phi_p(F, J)$ for the set of all connected eigenvalues and call this set the *point phantom* of (F, J). Of course, for $L \in \mathfrak{L}(X)$ and $J = I$ we simply have $\phi_p(L, I) = \sigma_p(L)$, since the eigenvectors of $\lambda \in \sigma_p(L)$ form a linear space.

The following table gives a comparison of the three spectra introduced above.

Author	Spectrum	Point spectrum	Character
Furi-Martelli-Vignoli (1978)	FMV-spectrum $\sigma_{FMV}(F, J)$	asymptotic eigenvalues $\sigma_q(F, J)$	asymptotic $(\|u\| \to \infty)$
Feng (1997)	Feng spectrum $\sigma_F(F, J)$	classical eigenvalues $\sigma_p(F, J)$	global $(u \in X)$
Väth (2000)	phantom $\phi(F, J)$	connected eigenvalues $\phi_p(F, J)$	local $(u \in \overline{\Omega})$

To close this section, we give some relations between all the spectra and point spectra, and calculate them for the operators F from Examples 3 and 8 (with $J = I$). First of all, we remark that the Väth phantom $\phi(F, J)$ is always contained in the Furi-Martelli-Vignoli spectrum $\sigma_{FMV}(F, J)$. In fact, for $\lambda \notin \sigma_{FMV}(F, J)$ we have, in particular, $[\lambda J - F]_q > 0$, and this implies that $\|\lambda J(u) - F(u)\| \geq [\lambda J - F]_q \|u\|$ for $\|u\| \geq R$ with $R > 0$ sufficiently large. Extending then $G : B_R \to Y$ with $G(u) \equiv 0$ to be zero outside B_R and arguing as before, one sees that the equation $\lambda J(u) - F(u) = G(u)$ has a solution in the interior of B_R.

So for general operators $F, J : X \to Y$ we get the following relations.

$$\begin{array}{ccccc} \phi(F,J) & \subseteq & \sigma_{FMV}(F,J) & \subseteq & \sigma_F(F,J) \\ \cup\!| & & \cup\!| & & \cup\!| \\ \phi_p(F,J) & \subseteq & \sigma_q(F,J) & & \sigma_p(F,J) \end{array}$$

F, J nonlinear

In the linear case $L \in \mathfrak{L}(X)$ (and $J = I$) this table simplifies. Here all the spectra in the first row coincide with the usual spectrum $\sigma(L)$, and both the point spectrum $\sigma_p(L,I)$ and point phantom $\phi_p(L,I)$ coincide with the usual point spectrum $\sigma_p(L)$.

$$\begin{array}{ccccc} \phi(L,I) & = & \sigma_{FMV}(L,I) & = & \sigma_F(L,I) \\ \cup\!| & & \cup\!| & & \cup\!| \\ \phi_p(L,I) & \subseteq & \sigma_q(L,I) & \supseteq & \sigma_p(L,I) \end{array}$$

L linear

We still have to calculate the phantom and point phantom for the maps F from Examples 3 and 8. Since F is not strictly epi for F as in Example 3, we have $0 \in \phi(F,I)$. From the inclusion $\phi(F,I) \subseteq \sigma_{FMV}(F,I)$ it follows that $\phi(F,I) = \{0\}$. Moreover, it is trivial that F has no connected eigenvalues, and so $\phi_p(F,I) = \emptyset$. So for Example 3 we get the following table which also shows, by the way, that there is no relation between $\sigma_p(F,J)$ and $\sigma_q(F,J)$.

$$\begin{array}{ccccc} \{0\} & = & \{0\} & \subset & \mathbb{R} \\ \cup & & \| & & \cup \\ \emptyset & \subset & \{0\} & & \mathbb{R} \setminus \{0\} \end{array}$$

$F(u) = \sqrt{|u|}$ in $X = \mathbb{R}$

Now consider $F : X \to X$ from Example 8. From $\phi_p(F,I) \subseteq \sigma_q(F,I) = \emptyset$ it follows that F has no connected eigenvalues (which may also be verified directly). We claim that $\phi(F,I) = \{0\}$. Indeed, for $\lambda \neq 0$ and $\Omega = \{u \in X : \|u\| < |\lambda|/2\}$ the restriction $(\lambda I - F)|_{\overline{\Omega}} : \overline{\Omega} \to X$ is open and injective with $0 \in (\lambda I - F)(\overline{\Omega})$ and $[(\lambda I - F)|_{\overline{\Omega}}]_a > 0$. This implies that $\lambda I - F$ is strictly epi on Ω, and so $\lambda \notin \phi(F, I)$. On the other hand, M. Furi quite recently proved in [13] that the operator F is epi but *not* strictly epi, and so $0 \in \phi(F, I)$.

Thus we may summarize our discussion for the operator F from Example 8 in the following table.

$\{0\}$	$\subset$	$[0,\infty)$	$=$	$[0,\infty)$
$\cup$		$\cup$		$\cup$
$\emptyset$	$=$	$\emptyset$		$(0,\infty)$

$F(u) = \|u\|u$ for $\dim X = \infty$

4. Special classes of operators

As one could expect, more can be said about these spectra if one restricts the class of operators in consideration. We restrict ourselves to the case of τ*-homogeneous operators* F and J, i.e.

$$F(tu) = t^\tau F(u), \quad J(tu) = t^\tau J(u) \quad (t > 0,\, u \in X). \tag{25}$$

The following two theorems have been proved in [3].

Theorem 6 (coincidence theorem). *Let X and Y be infinite dimensional Banach spaces, and suppose that $F, J : X \to Y$ satisfy* (25) *for some $\tau > 0$. Then*

$$\sigma_{FMV}(F, J) = \sigma_F(F, J) = \phi(F, J), \quad \sigma_q(F, J) \supseteq \sigma_p(F, J) = \phi_p(F, J).$$

Theorem 7 (discreteness theorem). *Let X and Y be infinite dimensional Banach spaces, and suppose that $F, J : X \to Y$ are odd, $[F]_A = 0$ (i.e., F is compact), and $[J]_a > 0$. Then*

$$\sigma_{FMV}(F, J) \setminus \{0\} \subseteq \sigma_q(F, J) \quad \sigma_F(F, J) \setminus \{0\} \subseteq \sigma_p(F, J),$$

and

$$\phi(F, J) \setminus \{0\} \subseteq \phi_p(F, J).$$

Theorem 7 shows that, if F is compact and odd and J is “regular” and odd, then each non-zero spectral value is actually an eigenvalue (in a sense to be made precise). For F compact and linear and $J = I$ this is a classical fact.

To illustrate how these theorems apply to nonlinear problems, we consider the *eigenvalue problem for the p-Laplacian* which consists in finding solutions $u \not\equiv 0$ of

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = \mu |u(x)|^{p-2}u(x) & \text{in } G, \\ u(x) \equiv 0 & \text{on } \partial G, \end{cases} \tag{26}$$

where $G \subset \mathbb{R}^n$ is a bounded domain. Although this problem makes sense for $1 < p < \infty$, we restrict ourselves to the case $2 \le p < \infty$. The problem (26) may be reformulated as equivalent operator equation in weak form

$$F_p(u) = \lambda J_p(u), \tag{27}$$

where $\lambda = 1/\mu$, and $F_p, J_p : W_0^{1,p}(G) \to W^{-1,p'}(G)$ $(p' = p/(p-1))$ are defined by $F_p(u) = |u|^{p-2}u$ and

$$\langle J_p(u), v\rangle = -\int_G (|\nabla u(x)|^{p-2}\nabla u(x), \nabla v(x))\, dx \quad (u, v \in W_0^{1,p}(G)), \tag{28}$$

respectively. Equation (27) is of the form (24) and has been studied by many authors, e.g. by P. Drábek et al. in [4]–[10]. Interestingly, the eigenvalue theory for the problem (26) has many features in common with the classical *linear* eigenvalue problem $-\Delta u(x) = \mu u(x)$, which is a special case of (26) for $p = 2$. For instance, the first eigenvalue μ_1 of (26) is always positive and simple and may be "calculated" as the Rayleigh quotient

$$\mu_1 = \inf_{\substack{u \in W_0^{1,p}(G) \\ u \neq 0}} \frac{\int_G |\nabla u(x)|^p\, dx}{\int_G |u(x)|^p\, dx}. \tag{29}$$

Moreover, the corresponding eigenfunction $u_1 \in W_0^{1,p}(G)$ is positive on G and simple (in the sense that any other eigenfunction is a scalar multiple of u_1). This function has the same "variational characterization" as in the linear case $p = 2$: It minimizes the functional $\Psi_p : W_0^{1,p}(G) \to \mathbb{R}$ defined by

$$\Psi_p(u) = \frac{1}{p}\int_G |\nabla u(x)|^p\, dx,$$

subject to the constraint

$$\frac{1}{p}\int_G |u(x)|^{p-2}u(x)\, dx = 1.$$

Finally, we point out that there is a famous so-called *nonlinear Fredholm alternative* (see [12], [20], [22]) which implies that the operator $J_p - \mu F_p = \mu(\lambda J_p - F_p)$ is *onto* for $\mu < \mu_1$, while it is *not onto* for $\mu = \mu_1$.

However, the coincidence and discreteness theorems given above allow us a more precise statement. The following is just a reformulation of Theorems 6 and 7.

Theorem 8 (nonlinear Fredholm alternative). *Suppose that $J : X \to Y$ is an odd τ-homogeneous homeomorphism with $[J]_a > 0$, and $F : X \to Y$ is odd, τ-homogeneous and compact. Let $\lambda \neq 0$. Then the following four assertions are equivalent.*

(a) *The eigenvalue problem* (24) *has only the trivial solution $u = 0$.*
(b) *The operator $\lambda J - F$ is stably solvable, $[\lambda J - F]_a > 0$ and $[\lambda J - F]_q > 0$.*
(c) *The operator $\lambda J - F$ is epi on each $\Omega \in \mathfrak{O}(X)$, $[\lambda J - F]_a > 0$ and $[\lambda J - F]_b > 0$.*
(d) *The operator $\lambda J - F$ is strictly epi on some $\Omega \in \mathfrak{O}(X)$ and*

$$\inf\{\|\lambda J(u) - F(u)\| : u \in \partial\Omega\} > 0.$$

We claim that the operators F_p and J_p satisfy the hypotheses of Theorem 8 in the spaces $X = W_0^{1,p}(G)$ and $Y = X^* = W^{-1,p'}(G)$. In fact, since $J_p : X \to Y$ is continuous, strictly monotone, coercive (it is here that we use the restriction $p \geq 2$!), odd, and $(p-1)$-homogeneous, it is an *isomorphism*, by Minty's celebrated theorem (see [19]). Moreover, the coercivity also implies that $[J_p]_a > 0$. Finally, the operator $F_p : X \to Y$ is continuous, compact (by Krasnosel'skij's theorem and the compactness of the imbedding $X \subset L_p(G)$), odd, and also $(p-1)$-homogeneous. So Theorem 8 implies that, whenever μ is not a classical eigenvalue of (26), the operator $J_p - \mu F_p$ is not only onto but even stably solvable and epi. This makes possible to obtain existence, uniqueness, and stability results for nonlinear perturbations of (26).

We do not want to describe this in detail but rather pose two open problems which seem to be of interest for applications of the p-Laplacian.

P r o b l e m 1. In order to apply Minty's theorem and to guarantee that $[J_p]_a > 0$, we must know that the operator J_p given by (28) is coercive. We have been able to prove this only for $p \geq 2$, and so the case $1 < p < 2$ (which is particularly important in view of applications) remains open.

P r o b l e m 2. The term "discreteness" in Theorem 7 does not mean that the point spectrum $\sigma_p(F, J)$ for compact F consists only of a sequence of eigenvalues (as in the linear case). In fact, in [24] the authors give an example of a 1-homogeneous compact operator in $X = \mathbb{C}$ such that $\phi_p(F, I) = \sigma_p(F, I) = [0, 1]$. Of course, one may generate a sequence of eigenvalues by iterating the variational formula (29) but it is not known if there are other eigenvalues which cannot be obtained in this way.

References

[1] J. Appell, E. De Pascale and A. Vignoli: *A comparison of different spectra for nonlinear operators.* Nonlinear Anal., Theory Methods Appl. **40A** (2000), 73–90. Zbl 0956.47035, MR 2001g:47117.

[2] J. Appell and M. Dörfner: *Some spectral theory for nonlinear operators.* Nonlinear Anal., Theory Methods Appl. **28** (1997), 1955–1976. Zbl 0876.47042, MR 98e:47098.

[3] J. Appell, E. Giorgieri and M. Väth: *Nonlinear spectral theory for homogeneous operators.* Nonlinear Funct. Anal. Appl. **7** (2002), 589–618.

[4] P. A. Binding, P. Drábek and Y. X. Huang: *On the Fredholm alternative for the p-Laplacian.* Proc. Amer. Math. Soc. **125** (1997), 3555–3559. Zbl 0882.35049, MR 98b:35058.

[5] M. del Pino, P. Drábek and R. Manásievich: *The Fredholm alternative at the first eigenvalue for the one dimensional p-Laplacian.* J. Differ. Equations **151** (1999), 386–419. Zbl 0931.34065, MR 99m:34042.

[6] P. Drábek: *On the Fredholm alternative for nonlinear homogeneous operators.* In: Applied nonlinear analysis (A. Sequeira et al., eds.). Kluwer Academic/Plenum Publishing, New York, 1999, 41–48. Zbl 0956.47036, MR 1 727 439.

[7] P. Drábek: *Analogy of the Fredholm alternative for nonlinear operators.* RIMS Kokyuroku **1105** (1999), 31–38. Zbl 0951.47503.

[8] P. Drábek: *Fredholm alternative for the p-Laplacian: yes or no?* In: Function Spaces, Differential Operators and Nonlinear Analysis. Proceedings of the conference, Syöte, Finland, June 10–16, 1999 (V. Mustonen and J. Rákosník, eds.). Math. Inst. Acad. Sci. Czech Rep., Prague, 2000, 57–64. Zbl 0966.34012, MR 2000m:34042.

[9] P. Drábek, P. Girg and R. Manásievich: *Generic Fredholm alternative-type results for the one dimensional p-Laplacian.* Nonlinear Differential Equations Appl. **8** (2001), 285–298. Zbl pre01652489, MR 2002f:34027.

[10] P. Drábek and G. Holubová: *Fredholm alternative for the p-Laplacian in higher dimensions.* J. Math. Anal. Appl. **263** (2001), 182–194. Zbl 1002.35046, MR 2002h:35083.

[11] W. Feng: *A new spectral theory for nonlinear operators and its applications.* Abstr. Appl. Anal. **2** (1997), 163–183. Zbl 0952.47047, MR 99d:47061.

[12] S. Fučík: *Fredholm alternative for nonlinear operators in Banach spaces and its applications to differential and integral equations.* Commentat. Math. Univ. Carol. **11** (1970), 271–284. Zbl 0995.42801, MR 42 #909.

[13] M. Furi: *Stably solvable maps are unstable under small perturbations.* Z. Anal. Anwend. **21** (2002), 203–208. Zbl pre01779543, MR 1 916 412.

[14] M. Furi, M. Martelli and A. Vignoli: *Stably solvable operators in Banach spaces.* Atti Accad. Naz. Lincei, VIII. Ser., Rend. Cl. Sci. Fis. Mat. Nat. **60** (1976), 21–26. Zbl 0361.47024, MR 58 #7251.

[15] M. Furi, M. Martelli and A. Vignoli: *Contributions to the spectral theory for nonlinear operators in Banach spaces.* Ann. Mat. Pura Appl., IV. Ser. **118** (1978), 229–294. Zbl 0409.47043, MR 80k:47070.

[16] M. Furi, M. Martelli and A. Vignoli: *On the solvability of nonlinear operator equations in normed spaces.* Ann. Mat. Pura Appl., IV. Ser. **128** (1980), 321–343. Zbl 0456.47051, MR 83h:47047.

[17] R. I. Kachurovskij: *Regular points, spectrum and eigenfunctions of nonlinear operators* (Russian). Dokl. Akad. Nauk SSSR **188** (1969) 274–277. English transl. in Soviet Math. Dokl. **10** (1969), 1101–1105. Zbl 0197.40402.

[18] I. J. Maddox and A. W. Wickstead: *The spectrum of uniformly Lipschitz mappings.* Proc. Royal Irish Acad., Sect. A **89** (1989), 101–114. Zbl 0661.47048, MR 90k:47120.

[19] G. Minty: *Monotone (nonlinear) operators in Hilbert space.* Duke Math. J. **29** (1962), 341–346. Zbl 0111.31202, MR 29 #6319.

[20] J. Nečas: *Sur l'alternative de Fredholm pour les opérateurs non linéaires avec applications aux problèmes aux limites.* Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat., III. Ser. **23** (1969), 331–345. Zbl 0187.08103, MR 42 #2332.

[21] J. W. Neuberger: *Existence of a spectrum for nonlinear transformations.* Pacific J. Math. **31** (1969), 157–159. Zbl 0182.47203, MR 41 #4329.

[22] S. I. Pokhozhaev: *Solvability of nonlinear equations with odd operators* (Russian). Funkts. Anal. Prilozh. **1** (1967), 66–73. Zbl 0165.49502, MR 36 #4396. English transl. in Funct. Anal. Appl. **1** (1967), 227–233. Zbl 0165.49502.

[23] A. Rhodius: *Über numerische Wertebereiche und Spektralwertabschätzungen.* Acta Sci. Math. **47** (1984), 465–470. Zbl 0575.47005, MR 86i:47005.

[24] P. Santucci and M. Väth: *On the definition of eigenvalues for nonlinear operators.* Nonlin. Anal., Theory Methods Appl. **40A** (2000), 565–576. Zbl 0956.47038, MR 2001g:47118.

[25] P. Santucci and M. Väth: *Grasping the phantom: a new approach to nonlinear spectral theory.* Ann. Mat. Pura Appl. **180** (2001), 255–284.

[26] M. Väth: *The Furi-Martelli-Vignoli spectrum vs. the phantom.* Nonlinear Anal., Theory Methods Appl. **47** (2001), 2237–2248.