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ASPECTS OF NONLINEAR SCATTERING THEORY

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1. Introduction

Scattering theory means different things to different people. Broadly speaking it can be thought of as the study of the interaction of an evolutionary process with a nonhomogeneous medium. Certainly, it has played a central role in mathematical physics over the years with perhaps the earliest investigation of such phenomena being attributed to Leonard da Vinci who studied the scattering of light into the geometrical shadow of an opaque body. Subsequently other scattering processes have been discovered and often utilised in such varied fields as acoustics, quantum mechanics and medical diagnosis.

If u, the total field describing the evolutionary process, is thought of as the sum of u_i, an incident field, and u_s, a scattered field, then two classes of problem present themselves. First there is the so called direct scattering problem in which the aim is to determine u_s from a knowledge of u_i, the equations governing the process and the various auxiliary conditions, such as boundary and initial conditions, which might be imposed on the process. The second and perhaps more challenging class of problems is the inverse scattering problem in which the intention is to determine the nature of the inhomogeneous medium from a, possibly incomplete, knowledge of the asymptotic form of u_s, that is to reconstruct the governing equation and its domain of definition from such data.

The above description is rather simplified and covers a vast range of physical and mathematical details. Clearly it would be impossible to do justice to them all in these lectures. The intention therefore will be to give a survey of the field, adopting a unified approach and concentrating mainly on direct scattering problems. Consequently, much of the material could be well known, particularly to those to whom it is well known. Nevertheless, it is hoped that these lectures will offer a sufficiently persuasive introduction for those whose primary research interests lie elsewhere to begin working in this very active area.

Mathematically, scattering processes can be realised as a consequence of perturbing either the symbolic form of an associated operator (so called potential scattering) or its domain (so called target scattering) or indeed both. A first step in developing a framework for investigating nonlinear problems is to examine and extend as far as possible the theory developed for corresponding linear problems. For this reason to begin with two typical problems in linear scattering theory are considered. First an initial value problem arising in potential scattering is examined; this will introduce many of the techniques used later. Next the rather more intuitively acceptable problem of the scattering of an acoustic wave by a rigid target is discussed; this initial boundary value problem offers different challenges and gives a first indication of the profound effect which the geometry of the medium can have on a propagating wave profile. With this introduction nonlinear scattering processes are then examined.

The results and ideas presented in the following sections are based on the research of a large number of authors. In this connection we would particularly mention the contributions of Kato, [4], Kuroda [6], Ikebe [2] and Wilcox [16] to linear scattering and of Reed and Simon [12], Segal [13] and Strauss [15] for nonlinear problems together with the references cited in these various works.

2. Potential scattering

As motivation for much of the detailed discussions which will follow consider the evolutionary process governed by the following model.

$$u_t(x,t) - u_x(s,t) = 0, \quad x \in \mathbb{R} = (-\infty.\infty), \quad t>0 \tag{2.1}$$

$$u(x,0) = f(x), \quad x \in \mathbb{R} \tag{2.2}$$

It will be convenient to analyse this problem in a Hilbert setting. The evolutionary process governed by (2.1) - (2.2) is then described by a curve $t \to u(.,t) \equiv u(t) \in H$ and has an associated operator realisation: determine $u \equiv u(.,t) \equiv H$ satisfying

$$u_t + iA_0u = 0, \quad u \in D(A_0) \tag{2.3}$$

$$u(.,0) \equiv u(0) = f \in H \tag{2.4}$$

$$A_0: H \to H \equiv L_2(\mathbb{R})$$

$$A_0u := L_0u := -i \frac{\partial u}{\partial x}, \quad u \in D(A_0)$$

$$D(A_0) = \{u \in H: L_0u \in H\}.$$

Proceeding formally, (2.3) has a solution in the form

$$u(x,t) = \exp(-itA_0)f(x) = : (U_tf)(x). \tag{2.5}$$

For such a solution to be of any use an interpretation of U_t is required. To this end the properties of the Fourier Transform in $\mathbb{R}$

$$(Fu)(k) := \hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} u(x)dx \tag{2.6}$$

indicate that

$$(F(A_0u))(k) = k\hat{u}(k) \tag{2.7}$$

$$(F(U_tu))(k) = \exp(-itx)\hat{u}(k) \tag{2.8}$$

$$(U_tf)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(ik(x-t))\hat{f}(k)dk = f(x-t). \tag{2.9}$$

Consequently we see that

$$u(x,t) = (U_tf)(x) = f(x-t)$$

a solution which could have determined directly by inspection of (2.1), (2.2). However, with a view to generalisations we notice

(i) F diagonalises A_0 in the sense that F enables A_0 on $u(x)$ to be replaced by multiplication of k on $\hat{u}(k)$.

(ii) A_0 is symmetric on H and consequently possesses self adjoint extensions.

(iii) A_0 is unbounded on H.

(iv) The spectrum of $A_0 := \sigma(A_0) = \sigma_c(A_0)$ = the continuous spectrum of A_0 $= (-\infty,\infty)$.

Now consider, instead of (2.3) - (2.4), the problem

$$v_t + iAv = 0, \quad v \in D(A), \quad v \equiv v(.,t) = v(t) \tag{2.10}$$

$$v(.,0) \equiv v(0) = g \tag{2.11}$$

where

$$A : H \to H \equiv L_2(\mathbb{R}) \tag{2.12}$$

$$Av = Lv$$

$$D(A) := \{v \in H : Lv \in H\}.$$

Here L may be regarded as perturbation of L_0 involving additional terms. Preceding formally (2.11) - (2.12) has a solution in the form

$$v(x,t) = \exp(-itA)g(x) := (V_t g)(x). \tag{2.13}$$

However, in general, since V_t depends crucially on the nature of L it is not so easy to determine as U_t. Nevertheless, the solution u of (2.3) - (2.4) and v of (2.11) - (2.12) can be compared provided the following assumptions hold.

<u>A1</u>: A is symmetric on it and has self adjoint extensions.

<u>A2</u>: A and A_0 are unitarily equivalent.

In this case a possible comparison is afforded by examining

$$\|v(.,t)-u(.,t)\|_H=\|v-u\|_H=\|V_t g-U_t f\|_H=\|g-V_t^* U_t g\|=:\|g-\Omega(t)f\|_H. \tag{2.14}$$

To remove the time dependence we take the limit as $t \to \pm\infty$ to obtain

$$\lim_{t\to\pm\infty} \|v-u\|_H = \lim_{t\to\pm\infty} \|g-\Omega(t)f\|_H = \|g-\Omega_\pm f\|_H \tag{2.15}$$

where in arriving at (2.15) we have assumed

A3: $\Omega_\pm f \equiv \Omega_\pm(A,A_0)f := \lim_{t\to\pm\infty} \Omega(t)f$

exists as an element of H for all $f \in H$.

Thus in terms of strong limits we can write

$$\Omega_\pm := \operatorname*{s-lim}_{t\to\pm\infty} \Omega(t), \quad \Omega(t) := V_t^* U_t. \tag{2.16}$$

To fix ideas even further consider the specific case when

$$Lv(x,t) := -i\frac{\partial v}{\partial x}(x,t) + q(x)v(x,t) = (L_0+q(x)))v(x,t) \tag{2.17}$$

where q is real valued and bounded.

Introduce

$$W : H \to H \tag{2.18}$$

$$Wf(x) := \exp(-i\rho(x))f(x), \quad f \in H$$

where ρ is a sufficiently smooth function yet to be determined.

Clearly W is a unitary operator and direct calculation yields

$$A = WA_0W^{-1}$$

that is, A and A_0 are unitarily equivalent. Further calculation indicates

$$(FW^{-1}Af)(k) = k(FW^{-1}f)(k). \tag{2.19}$$

Consequently, introducing the "generalised" Fourier transform

$$T := FW^{-1} \tag{2.20}$$

we see that T replaces A on f by multiplication of k on $\tilde{f} =: Tf$, that is T diagonalises A whereas F alone was required to diagonalise A_0.

With

$$U_t := \exp(itA_0), \quad V_t := \exp(itA) \tag{2.21}$$

the quantities $\Omega_\pm$ can be computed to yield

$$(\Omega_\pm f)(x) = \lim_{t\to\pm\infty} (V_tU_tf)(x) = \lim_{t\to\pm\infty} (\Omega(t)f(x)$$

$$= \exp\left\{i\int_x^{\pm\infty} q(\xi)d\xi\right\}f(x) \tag{2.22}$$

and we see that in this particular case $\Omega_\pm$ are operators of multiplication by the quantity $\exp\left\{i\int_x^{\pm\infty} q(\xi)d\xi\right\}$. Furthermore they are unitary operators and provide the result

$$A = \Omega_{\pm} A_0 \Omega_{\pm}^{-1}. \tag{2.23}$$

Moreover, direct calculation shows that

$$F\Omega_{\pm}^{-1} Af = kF\Omega_{\pm}^{-1} f. \tag{2.24}$$

Setting

$$G := F\Omega_{\pm}^{-1}$$

we see from (2.24) that

$$GAf = kGf \tag{2.25}$$

which implies that G is a "generalised" Fourier transform which diagonalises A.

Interesting though this example may be a more systematic approach is required in order to tackle more general problems especially those involving boundary as well as initial conditions. However, before moving in this direction we notice from the above example that in any such approach we shall be required to perform at least the following steps.

(a) Provide an operator realisation of the scattering process in which A_0 and A are self adjoint.

If A were not self adjoint then V_t in (2.13) would not be unitary. Consequently the quantities $\Omega_{\pm}$ would not provide a unitary equivalence between A_0 and A and the above method would fail to provide a means of obtaining the spectral properties of A from those of A_0.

(b) Define the spectrum of a self adjoint operator and investigate the existence of its various subdivisions.

(c) Establish the spectral properties of A_0.

(d) Prove the existence of the quantities $\Omega_{\pm}$ and investigate their properties.

(e) Obtain the spectral properties of A.

If this programme can be carried through then expansion theorems can be established for problems involving A_0 and A. In addition information regarding the asymptotic behaviour of solutions can be obtained and compared.

With the above example as a guide it would appear that evolutionary processes can conveniently be studied in terms of

A "free" system modelled by

$$u_t = iA_0u, \quad u \in D(A_0) \subset H$$
$$u(0) = f \in H \tag{2.26}$$

and

A "perturbed" system modelled by

$$v_t = -iAv, \quad v \in D(A) \subset H$$
$$v(0) = g \in H \tag{2.27}$$

where

u,v = state (vectors) of the system

f,g = initial state (vectors) of the system

A_0,A = linear, self-adjoint operators on a Hilbert space H.

For definiteness we shall think of the operators as having the symbolic forms

$$A_0 = -\Delta \qquad A = -\Delta + V$$

where V, the perturbation of A_0, will be referred to as a potential. A reason for this terminology and also for certain other concepts introduced later can be given by considering a particle moving in one direction, interacting with an obstacle and being deflected to move off in another direction. If the obstacle were not present the motion (evolution) of the particle would be governed by the free system (2.26) whilst in the presence of an obstacle its evolution is governed by the perturbed system (2.27). A characteristic feature of scattering in this sense is that if the history of the particle is traced back in time then it moves further from the obstacle (the scattering centre). Therefore it seems plausible that it ought to behave more like a free particle the further back in time we go. Mathematically this can be expressed by requiring that there should exist $f_- \in H$ such that the free process, as governed by (2.26),

$$u(t) = e^{-itA_0}f_-$$

and the perturbed scattering process, as governed by (2.27),

$$v(t) = e^{-itA}g$$

should satisfy

$$\lim_{t\to -\infty} \|e^{-itA}g - e^{-itA_0}f_-\|_H = 0 \qquad (2.28)$$

We shall regard (2.28) as the mathematical expression for the intuitive idea that originally the particle moves freely.

Similarly we can argue that eventually the particle should also move freely, a fact that we can express mathematically by requiring that there should exist $f_+ \in H$ such that

$$\lim_{t\to+\infty} \|e^{-itA}g - e^{itA_0}f_+\|_H = 0. \qquad (2.29)$$

In an actual experiment involving the scattering of a particle it is very difficult, if indeed possible, to work with the initial condition g, Instead a particle is prepared a long way from the obstacle in what is considered to be a good approximation to free motion. It is then allowed to evolve in the given system and, after a long time, it is again detected in a state which is regarded to be a good approximation to free motion. Therefore the quantities in the experiment with which there are direct contact are

the prepared state: $e^{-itA_0}f_-$

the detected state: $e^{-itA_0}f_+$

Thus the initial condition $f_- \in H$ in (2.26) determines the initial free motion of the particle and $f_+ \in H$ in (2.26) determines the detected or eventual free motion of the particle.

Scattering theory is concerned with the problem of being able to predict what will be measured knowing what had been prepared. Mathematically this means scattering theory must indicate the conditions under which each $f_- \in H$ uniquely determines an $f_+ \in H$ and whenever possible provide a method for computing f_+ given f_-. In many cases it turns out that for a suitable class of potentials this problem is solvable in the form

$$f_+ = Sf_- \tag{2.30}$$

where S, the scattering operator, is unitary. In this case it is convenient to discuss the problem in two parts.

Part 1:

Given $f_- \in H$ determine $g \in H$ such that

$$\lim_{t\to -\infty} \|e^{-itA}g - e^{-itA_0}f_-\| = \lim_{t\to -\infty} \|V_t g - U_t f_-\| = 0.$$

This means that for any $f_- \in H$ we expect to be able to prepare $e^{-itA_0}f_-$, an initial state of the system, which initiates an evolutionary process in the system, of the form $e^{-itA}g$, which converges strongly to the initial state $e^{-itA_0}f_-$ as $t \to -\infty$. Of course the question remains as to whether or not there exists a state $e^{-itA}g$ which agrees asymptotically with $e^{-itA_0}f_-$ as $t \to -\infty$. An argument suggesting that this should be the case can be given by considering again the motion of a particle; for large enough time (in the past) the particle is in a region of space where the effects of the obstacle are negligible and the motion of the particle can be satisfactorily approximated by the state function $e^{-itA_0}f_-$.

Part 2:

Given $g \in H$, determined in Part 1, find $f_+ \in H$ such that

$$\lim_{t\to+\infty} \|e^{-itA}g - e^{-itA_0}f_+\| = \lim_{t\to+\infty} \|V_t g - U_t f_+\| = 0.$$

This implies that the evolutionary process, $e^{-itA}g$, defined by the quantity g determined in Part 1 as a consequence of an initial or free process $e^{-itA_0}f_-$ once again becomes a free process described by $e^{-itA}f_+$.

Whether or not the problems presented in Part 1 and Part 2 are solvable depends on the nature of the potential V. Results in this direction have been obtained by Cook [1] and Kuroda [6,7].

The arguments used above for describing particle motions will be used for more general evolutionary processes. Consequently we introduce

Definition 2.1 (Asymptotic Condition)

As $t \to \pm\infty$ perturbed states behave, in the sense described in Parts 1 and 2, as free states.

Definition 2.2 (Scattering states)

Scattering states are elements $g \in H$ such that the Asymptotic Condition holds.

We notice that if $g \in H$ is an eigenvector of A then there exists $\mu \in \mathbb{R}$ such that $Ag = \mu g$. Consequently

$$v = V_t g = \exp(-itA)g = \exp(it\mu)g$$

and v is simply a multiple of the eigenvector g and as such describes a bound (non-scattering) state. Therefore it is to be expected that scattering states are associated with $\sigma_c(A)$.

Definition 2.3

A family of operators $\{U_t\}$ defined on a Hilbert space H by

$$U_t : \mathbb{R} \to B(H), \qquad t \in \mathbb{R}$$

and satisfying

(i) Strong Continuity:

$$s - \lim_{\tau \to 0}(U_{t+\tau} - U_t) = 0, \qquad t \in \mathbb{R}$$

(ii) Unitarity:

$$U_t^* = U_t^{-1}, \qquad t \in \mathbb{R}.$$

(iii) Group property:

$$U_t U_s = U_s U_t = U_{t+s}, \qquad s,t \in \mathbb{R}$$

$$U_0 = I$$

is called a strongly continuous, one parameter, unitary group of operators.
We shall also need the result

Theorem 2.4 (Stone's Theorem) [14]

Let

(i) $\{U_t\}$, $-\infty < t < \infty$, a strongly continuous, one parameter unitary group of operators on a Hilbert space H.

(ii) A be a linear operator (an infinitesimal generator of $\{U_t\}$) satisfying

(a) $D(A) := \{f \in H: \text{s-lim}[i\tau^{-1}(U_\tau - I)f]$ exists as $\tau \to 0$

(b) $Af = \text{s-lim}[i\tau^{-1}(U_\tau - I)f], \quad f \in D(A).$

Then

(iii) $D(A)$ is dense in H

(iv) A is self adjoint.

Corollary 2.5

Let

(i) $\Phi_t(\lambda) := \exp(-it\lambda), \quad \lambda, t \in \mathbb{R}.$

(ii) $A: H \to H$, densely defined, self adjoint.

then $\{\Phi_t\}$, $-\infty < t < \infty$ is a strongly continuous, one-parameter unitary group with infinitesimal generator A.

Furthermore we shall assume that with each operator $A : H \to H$ there can be associated a set of scattering states, $M(A)$, satisfying

(i) $M(A) \subset H$ is a subspace.

(ii) $M(A)$ is invariant with respect to the group $\{-itA)\}$.

We can now say that scattering is associated with

an "initial" state $f_- \in M(A_0)$ at $t = 0$

a "final" state $f_+ \in M(A_0)$ at $t = 0$

The asymptotic condition together with the property of unitarity implies

$$\lim_{t \to -\infty} \|V_t g - U_t g_-\| = \lim_{t \to -\infty} \|g - V_t^* U_t f_-\| = 0 \tag{2.31}$$

and this requires that

$$\text{s} - \lim_{t \to -\infty} V_t^* U_t$$

should exist on $M(A_0)$.

A similar result involving f_+ and the limit $t \to +\infty$ also obtains. This leads to

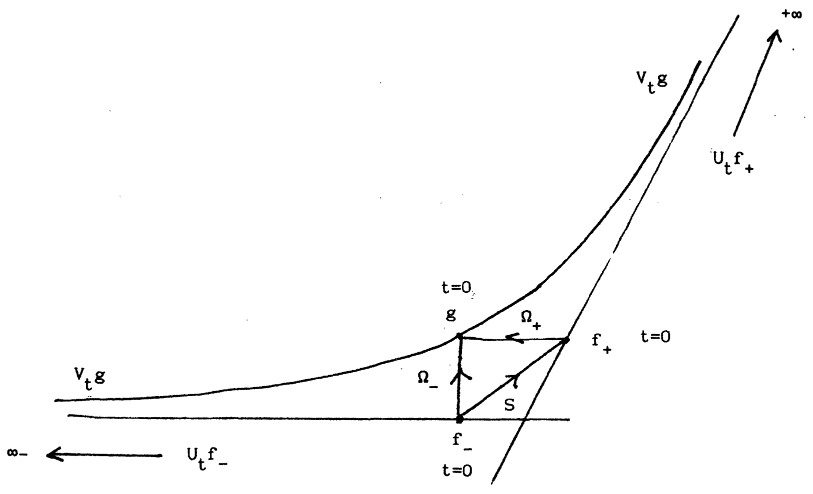
<u>Definition 2.6</u> (Wave operators)

$$\Omega_\pm \equiv \Omega_\pm(A,A_0) := s - \lim_{t\to\pm\infty} V_t^* U_t P(A_0)$$

where

$$P(A_0) : H \to M(A_0) \qquad P(A_0) = P(A_0)^* = P(A_0)^2.$$

The concept of a scattering process introduced above can be conveniently thought of as trajectories in H as illustrated in the diagram.



Straight lines = free evolutions.

Curved line = perturbed evolution.

The asymptotic condition, of which (2.31) is a typical part indicates that the wave operator Ω_- (Ω_+) applied to $f_- \in M(A_0)$ at $t = 0$ yields an initial state $g \in M(A)$ at $t = 0$ for the perturbed system which will evolve from a prepared state $[\exp(-itA_0)]f_-$ in the remote past and which will converge, in the distant future, to the state $[\exp(-itA_0)]f_+$.

Conditions ensuring the existence of the wave operators $\Omega_\pm$ can be found in the works of Cook [1], Jauch [3], Kato [4] and Kuroda [7]. Furthermore, it is sometimes possible to prove that the wave operators have identical ranges, that is

$$R(\Omega_+) = R(\Omega_-).$$

In this case the wave operators are said to be asymptotically complete. Results in this connection can be found in Kato [4], Kato and Kuroda [5].

3. Target scattering

In this section we consider the scattering of an acoustic wave by a rigid obstacle. This process can be modelled by the d'Alembert equation

$$\left[\frac{\partial^2}{\partial t^2} - \Delta\right]u(\underset{\sim}{x},t) = 0, \quad \underset{\sim}{x} \in \mathbb{R}^n \tag{3.1}$$

together with initial and boundary conditions as appropriate. This particular problem has been comprehensively analysed by Wilcox [16] and much of the presentation here is based on this work.

We consider first the case when the target is absent; that is, the free problem. This will be analysed in $L_2(\mathbb{R}^n)$ with the usual structure denoted by $(.,.)$. It will be convenient to introduce

$$L_2^m(\mathbb{R}^n) := L_2(\mathbb{R}^n) \cap \{u : D^\alpha u \in L_2(\mathbb{R}^n), |\alpha| \leq m\} \tag{3.2}$$

$$L_2(\Delta,\mathbb{R}^n) := L_2(\mathbb{R}^n) \cap \{u : \Delta u \in L_2(\mathbb{R}^n)\} \tag{3.3}$$

the latter being endowed with the inner product

$$(u,v)_\Delta := (u,v) + (\Delta u,\Delta v) \tag{3.4}$$

where $(.,.)$ denotes the usual $L_2(\mathbb{R}^n)$ inner product.

If we define

$$A_0 : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \tag{3.5}$$

$$Au = -\Delta u, \quad u \in D(A_0) = L_2(\Delta,\mathbb{R}^n)$$

then the free problem can be cast in the form: determine $u_0(.,t) \equiv u_0 \in L_2(\mathbb{R}^n)$ satisfying

$$(D_0^2+A_0)u_0 = 0, \qquad t \in \mathbb{R}, \qquad D_0 = \partial/\partial t \tag{3.6}$$

$$u_0(0) = f \in L_2(\mathbb{R}^n)$$

$$D_0u_0(0) = g \in L_2(\mathbb{R}^n).$$

Formally, this has a solution

$$u_0(t) = (\cos A_0^{1/2})f + (A_0^{-1/2}\sin t\, A_0^{1/2})g. \tag{3.7}$$

This solution is interpreted by means of the spectral theorem since

Theorem 3.1 [16]

The operator A_0 is self adjoint and non-negative on $L_2(\mathbb{R}^n)$.

When the scattering obstacle is immersed in the incident field boundary conditions have to be allowed for.

Let $B \subset \mathbb{R}^n$ denote the open region occupied by the target and let $\Omega = \mathbb{R}^n\backslash\bar{B}$ denote the open region exterior to B and $\partial\Omega$ its boundary. We shall assume that $\bar{B} = B \cup \partial\Omega$ is bounded and that $\partial\Omega$ is smooth.

The target scattering (perturbed) problem will be analysed in $L_2(\Omega)$ endowed with the usual inner product. Furthermore we introduce

$$L_2^m(\Omega) := L_2(\Omega) \cap \{u:D^\alpha u \in L_2(\Omega), \quad |\alpha|\leq m\} \tag{3.8}$$

$$L_2(\Delta,\Omega) := L_2(\Omega) \cap \{u:\Delta u \in L_2(\Omega)\} \tag{3.9}$$

$$L_2^1(\Delta,\Omega) := L_2^1(\Omega) \cap L_2(\Delta,\Omega) \tag{3.10}$$

with the following inner products

$$L_2^m(\Omega): \qquad (u,v)_m := \sum_{|\alpha|\leq m} (D^\alpha u,D^\alpha v)$$

$$L_2(\Delta,\Omega): \qquad (u,v)_\Delta := (u,v) + (\Delta u,\Delta v)$$

$$L_2^1(\Delta,\Omega): \qquad (u,v)_{1,\Delta} := (u,v)_1 + (\Delta u,\Delta v)$$

where

$$(u,v) = \int_\Omega u(x)\bar{v}(x)dx$$

is the usual inner product in $L_2(\Omega)$.

An element $u \in L_2^1(\Delta,\Omega)$ is said to satisfy a generalised Neumann condition if

$$\int_\Omega \{v(x)\Delta u(x) + \nabla v(x).\nabla v(x)\}d\tau = 0, \qquad v \in L_2^1(\Omega). \tag{3.11}$$

Consequently we shall define

$$L_2^N(\Delta,\Omega) := L_2^1(\Delta,\Omega) \cap \{u: u \text{ satisfies } (3.11)\} \tag{3.12}$$

Furthermore if we define

$$\begin{gathered} A: L_2(\Omega) \to L_2(\Omega) \\ Au = -\Delta u, \qquad u \in D(A) := L_2^N(\Delta,\Omega) \end{gathered} \tag{3.13}$$

then the perturbed problem can be stated in the form: determine $u(.,t) \equiv u(t) \in L_2(\Omega)$ satisfying

$$\begin{gathered} (D_0^2+A)u = 0 \quad \text{in } \Omega, \quad t \in \mathbb{R} \\ u \in D(A) = L_2^N(\Delta,\Omega) \\ u(0) = f \in L_2(\Omega), \quad D_0u(0) = g \in L_2(\Omega). \end{gathered} \tag{3.14}$$

Corresponding to Theorem 3.1 there can be established

Theorem 3.2 [16]

A is a non negative, self-adjoint operator on $L_2(\Omega)$. Consequently a solution of (3.14) written in the form

$$u(t) = \{\text{cost}A^{1/2})f + (A^{-1/2}\sin t\ A^{1/2})g \tag{3.15}$$

can be made meaningful by using the spectral theorem to interpret the solution operators $(\text{cost } A^{1/2})$ and $(A^{-1/2}\text{sint } A^{1/2})$. However, by virtue of the generality of this theorem it is often the case that very little specific information can be obtained about the wave function $u(x,t)$. This can be overcome by developing an eigenfunction expansion for A. Then, if required the relation between the spectral family and eigenfunctions of A can be developed.

With a view to developing eigenfunction expansions we notice that

$$w_0(x,p) := \frac{1}{(2\pi)^{n/2}} e^{ix.p} \qquad ,x \in \mathbb{R}^n \tag{3.16}$$

satisfies

$$(\Delta+|p|^2)w_0(x,p) = 0, \quad x,p \in \mathbb{R}^n.$$

Hence

$$A_0w_0(x,p) = -\Delta w_0(x,p) = |p|^2w_0(x,p) \tag{3.17}$$

and we conclude that $w_0(x,p)$ in (3.16) is an eigenfunction of A_0. However it is a generalised eigenfunction since

$$w_0(.,p) \notin L_2(\mathbb{R}^n) \tag{3.18}$$

Nevertheless, the Plancherel theory of Fourier Transforms yields the following results:

$$\hat{f}(p) = \lim_{M\to\infty} \int_{|x|\leq M} \overline{w_0(x,p)}\, f(x)dx =: (Ff)(p) \tag{3.19}$$

$$f(x) = \lim_{M\to\infty} \int w_0(x,p)\hat{f}(p)dp =: (F^*\hat{f})(x) \tag{3.20}$$

$$\Phi(A_0)f(x) = \lim_{M\to\infty} \int_{|p|\leq M} w_0(x,p)\Phi(|p|^2\hat{f}((p)dp \tag{3.21}$$

where the limit must be interpreted in the $L_2(\mathbb{R}^n)$ sense. Thus we have obtained a spectral representation for A_0 as required.

The quantity $w_0(x,p)$ can also be interpreted as a steady state solution of the d'Alembert equation

$$\left[\frac{\partial^2}{\partial t^2} - \Delta\right]u(x,t:|p|) = 0 \tag{3.22}$$

if we assume a time dependence in the form

$$u(x,t:|p|) = e^{i|p|t}w_0(x,p). \tag{3.23}$$

Thus $w_0(x,p)$ can be interpreted as a plane wave propagating in the direction p.

For the perturbed problem a corresponding eigenfunction for the operator A would then be a steady wave $w(x,p)$ which is produced when the obstacle B is immersed in the plane wave $w_0(x,p)$. Mathematically this means that $w(x,p)$ should satisfy

$$(\Delta+|p|^2)w(x,p) = 0, \qquad x \in \Omega \tag{3.24}$$
$$w \in L_2^N(\Delta,\Omega).$$

In this Neumann problem w represents the total wave and is assumed to have the form

$$w(x,p) = w_0(x,p) + w'(x,p) \tag{3.25}$$

where

w_0 = the incident wave

w' = the scattered wave.

The scattered wave w' will be required to satisfy either an outgoing or an incoming radiation condition. Depending on which radiation condition is employed the total wave w, also called a distorted plane wave, will be called appropriately either outgoing or incoming and will be denoted by w_+ and w_- respectively.

Employing the Plancherel theory once again it is possible to introduce generalised Fourier transforms

$$(F_\pm f)(p) = \hat{f}_\pm(p) = \lim_{R\to\infty} \int_{B(R)} \overline{w_\pm(x,p)} f(x)dx \tag{3.26}$$
$$(F_\pm^* \hat{f})(x) = f(x) = \lim_{k\to\infty} \int_{|p|\le R} w_\pm(x,p)\hat{f}(p)dp \tag{3.27}$$

where

$$B(R) = \Omega \cap \{x:|x|<R\}$$

and the limits are to be interpreted in the $L_2(\Omega)$ sense. Furthermore, it can be shown that if ϕ is Lebesgue measurable then

$$\Phi(A)f(x) = \lim_{R\to\infty} \int_{|p|\le R} w_\pm(x,p)\Phi(|p|^2)\hat{f}_\pm(p)dp. \tag{3.28}$$

Consequently (3.26) - (3.27) provide the required spectral representation for A.

In order to investigate the asymptotic properties of the free and perturbed waves as $t \to \infty$ it will be convenient to rewrite the solutions (3.7) and (3.15). To this end set

$$h_0 = f_0 + iA_0^{-1/2}g_0, \qquad h = f + iA^{-1/2}g$$

$$v_0(.,t) = \{\exp(-itA_0^{1/2})\}h_0(.), \quad v(.,t) = \{\exp(-itA^{1/2})\}h(.) \tag{3.29}$$

then a straightforward calculation allows the solutions (3.7) and (3.15) to be written in the form

$$u_0(x,t) = R_e(v_0(x,t)) \tag{3.30}$$

$$u(x,t) = R_e(v(x,t)). \tag{3.31}$$

Arguments based on the local decay of energy [16] suggest that all the distorted waves should tend asymptotically to a free wave, that is

$$\lim_{t\to\infty}\|e^{-itA^{1/2}}h - e^{-itA_0^{1/2}}h_0\|_{L_2(\Omega)} = 0 \tag{3.32}$$

or equivalently, since $h_0 \in L_2(\mathbb{R}^n)$

$$\lim_{t\to\infty}\|J_\Omega e^{-itA^{1/2}}h - e^{-itA_0^{1/2}}h_0\|_{L_2(\mathbb{R}^n)} \tag{3.33}$$

where

$$J_\Omega: L_2(\Omega) \to L_2(\mathbb{R}^n)$$

$$J_\Omega u(x) = \begin{cases} u(x), & x \in \mathbb{R}^n \\ 0, & x \in \mathbb{R}^n\backslash\Omega. \end{cases}$$

Recognising the unitarity of the operators involved (3.33) can be written

$$\lim_{t\to\infty}\|e^{itA_0^{1/2}}J_\Omega e^{-itA^{1/2}}h - h_0\|_{L_2(\mathbb{R}^n)} = 0. \tag{3.34}$$

Furthermore, (3.34) is equivalent to requiring that

$$W_+ \equiv W_+(A_0^{1/2},A^{1/2},J_\Omega) := s - \lim_{t\to\infty} e^{-itA_0^{1/2}}J_\Omega e^{-itA^{1/2}} \tag{3.35}$$

should exist. W_+ is called the wave operator associated with the triple $(A_0{}^{1/2},A^{1/2}J_\Omega)$ and its existence is entirely equivalent to the asymptotic condition (3.33) with

$$h_0 = W_+h. \tag{3.36}$$

Similarly, we can define the wave operator

$$W_- \equiv W_-(A_0^{1/2},A^{1/2},J_\Omega) := s - \lim_{t\to-\infty} e^{itA_0^{1/2}}J_\Omega e^{-itA^{1/2}} \tag{3.37}$$

The wave operators

$$W_{\pm}: L_2(\Omega) \to L_2(\mathbb{R}^n) \tag{3.38}$$

are unitary and provide the unitary equivalence between A and A_0.
Furthermore it can be shown [16] that

$$W_+ = F^*F_-, \qquad W_- = F^*F_+. \tag{3.39}$$

This construction can be carried out for more general equations and more general initial and boundary conditions [8,9].

This we see that in scattering theory there would appear to be four basic problems; one free problem and three perturbed problems with the following typical structure

operator	conditions
A_0	(ic)
$A_1=A_0+V$	(ic)
$B_0=A_0$	(ic)+(bc)
$B_1=A_1$	(ic)+(bc)

and related by means of the various wave operators according to the diagram

$$\begin{tikzcd} A_0 \arrow[r, "W(B_0,A_0)"] \arrow[d, "\Omega(A_1,A_0)"'] & B_0 \arrow[d, "\Omega(B_1,B_0)"] \\ A_1 \arrow[r, "W(B_1,A_1)"'] & B_1 \end{tikzcd}$$

4. Nonlinear scattering theory

As we have seen scattering theories investigate the behaviour in the distant past and in the distant future of a system evolving in time. In practice this is done by comparing the behaviour of the given system with an associated free system. The process is called nonlinear if a system evolves in a nonlinear manner. It turns out that the nonlinear effects can often be quite well accounted for by taking as the free problem an associated linear problem which has been fully analysed by the methods outlined in the previous sections.

As a typical nonlinear system consider the following initial value problem for the nonlinear Klein-Gordon equation.

$$u_{tt}(x,t) - \Delta u(x,t) + m^2 u(x,t) = -q|u(x,t)|^{p-1}u(x,t) \tag{4.1}$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

An abstract setting for this problem can be obtained by first writing it as a first order system. Let

$$v(x,t) = u_t(x,t)$$

then (4.1) reduces to

$$\phi'(t) - L\phi(t) = N(\phi(t)) \tag{4.2}$$

where

$$\phi(t) = \begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}, \qquad N(\phi(t) = \begin{bmatrix} 0 \\ -q|u(x,t)|^{p-1} \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & I \\ \Delta - m^2 & 0 \end{bmatrix}$$

and the initial condition

$$\phi(0) = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} \equiv \phi_0$$

The operator on $L_2(\mathbb{R}^n)$ defined by

$$B^2 = (-\Delta + m^2)$$

is positive and self-adjoint. Furthermore the operator L is self-adjoint on the Hilbert space

$$H_B = D(B) \times L_2(\mathbb{R}^n) \tag{4.3}$$

endowed with the scalar product

$$((f_1,f_2),(g_1,g_2))_B := (Bf_1,Bg_1) + (f_2,g_2). \qquad (4.4)$$

Now let

$$A := iL \qquad (4.5)$$

then it can be verified that A is symmetric on H_B with domain $D(A) = D(B^2) \times D(B)$. Furthermore A is closed since B and B^2 are closed. Therefore the original problem has been reformulated as follows: Find $\phi(t) \in H_B$, $T \in \mathbb{R}$ satisfying

$$\frac{d\phi(t)}{dt} = -iA\phi(t) + N(\phi(t)) \qquad (4.6)$$

$$\phi(0) = \phi_0. \qquad (4.7)$$

This is the form of the abstract problem with which we shall be mainly concerned in the following sections. A scattering theory for a nonlinear system of the form (4.6) - (4.7) can be developed by comparing its evolution with that of the free system

$$\frac{d\phi(t)}{dt} = -iA\phi(t) \qquad (4.8)$$

$$\phi(0) = \tilde{\phi}_0 \qquad (4.9)$$

for large positive and negative times. However, the abstract theory is not so satisfactorily developed as in the linear case. Furthermore the applications are not so well understood; like all nonlinear problems the equations must, to a large extent, be dealt with individually in order to make full use of their various special properties. Nevertheless the abstract theory, which originated in the work of Segal [13], does form the basis for much of the recent work in nonlinear scattering theory. As might be expected many problems remain both in the abstract and particular settings.

To develop a scattering theory for a system of the form

$$\phi'(t) = -iA\phi(t) + N(\phi(t)) \qquad (4.10)$$

$$\phi(0) = \phi_0 \qquad (4.11)$$

the approach will be to reformulate (4.10), (4.11) as an equivalent integral equation. Direct integration yields

$$\phi(t) = e^{-itA}\phi_0 + \int_0^t U(t-s)N(\phi(s))ds. \tag{4.12}$$

We notice

(i) $\phi(t)$ given by (4.12) satisfies (4.10) and (4.11)

(ii) the first term in (4.12) is a solution of the free problem (4.8), (4.9) with data ϕ_0 at $t = 0$.

Similarly (4.10) can be integrated to yield

$$\phi(t) = U(t)\phi_\tau + \int_\tau^t U(t-s)N(\phi(s))ds. \tag{4.13}$$

We notice that

(i) $\phi(t)$ given by (4.13) satisfies (4.10) with data $U(\tau)\phi_\tau$ at $t = \tau$.

(ii) the first term in (4.13) satisfies the free problem (4.8) with data ϕ_τ at $t = 0$.

(iii) from (4.13)

$$\phi(0) := \phi_0 = \phi_\tau + \int_\tau^0 U(-s)N(\phi(s))ds. \tag{4.14}$$

In developing linear scattering theory we looked for conditions under which the solutions $\phi(t)$ of the "perturbed" system were related to the solutions $\psi_\pm(t) = U(t)f_\pm$ under the asymptotic condition

$$\|\phi(t)-\psi_\pm(t)\| = \|\phi(t)-U(t)f_\pm\| \to 0 \text{ as } t \to \pm\infty. \tag{4.15}$$

The same approach will be adopted for nonlinear problems. Motivation for this can be seen by formally taking the limits $\tau = \pm\infty$ in (4.13). This yields the two integral equations

$$\phi(t) = U(t)\phi_\pm + \int_{\pm\infty}^t U(t-s)N(\phi(s))ds. \tag{4.16}$$

We notice

(i) the first term on the right hand side of (4.16) is a "free" solution with data $\phi_\pm$ at $t = 0$.

(ii) for t very large negative we have the possibility of approaching the "prepared" state $U(t)\phi_-$.

(iii) for t very large positive we have the possibility of approaching the "eventual" state $U(t)\phi_+$.

The problems centred on (4.16) are as follows:

Let $M(A) \subset H$ denote a set of initial data for (4.16). Determine conditions which ensure

(i) (4.16) has a unique global solution for $\phi_\pm \in M(A)$.

(ii) $\phi(t) \in M(A)$ for each value of t.

(iii) if $\phi(t)$ satisfies

$$\phi(t) = U(t)\phi_- + \int_{-\infty}^{t} U(t-s)N(\phi(s))ds$$

then

$$\|\phi(t) - U(t)\phi_-\| \to 0 \text{ as } t \to -\infty.$$

(iv) there exists $\phi_+ \in M(A)$ such that for $\phi(t)$ as in (iii)

$$\|\phi(t) - U(t)\phi_+\| \to 0 \text{ as } t \to +\infty.$$

(v) the scattering operator

$$S: \phi_- \to \phi_+ \tag{4.17}$$

exists and is (1-1) and continuous.

(vi) the wave operators

$$\Omega_\pm: \phi_\pm \to \phi_0 \tag{4.18}$$

exist and are (1-1).

(vii) range of $\Omega_+ := R(\Omega_+) = R(\Omega_-)$.

This is the so-called asymptotic completeness of the wave operators. Complete results in this connection are known in only a few cases. However, if the initial condition ϕ_- is small enough global existence of the (4.16) can be settled. A scattering theory for the abstract problem (4.10), (4.11) can then be developed provided the solutions $U(t)\phi_-$ of the free problem decay sufficiently rapidly and N has a high enough degree.

Typical result which can be obtained for the problems (i) - (vii) listed above are outlined in the next three sections.

5. Nonlinear scattering with small data

Let H be a Hilbert space with norm $\|.\|$ and A a linear, self adjoint operator on H. Further, let $\|.\|_1$ and $\|.\|_2$ denote two other "norms" defined on H which satisfy all the usual norm axioms except that $\|\phi\|_1 = 0$ need not imply $\phi = 0$ and $\|.\|_2$ might possibly be unbounded. We shall assume

A1: There exist constants $c > 0$, $d > 0$ so that for $\phi \in H$

$$\|\phi\|_1 \leq c\|\phi\| \quad , \quad \phi \in H.$$

A2: There exist constants $c_1 > 0$, $d > 0$ so that for $\phi \in H$

$$\|U(t)\phi\|_1 \leq c_1 t^{-d}\|\phi\|_2 \quad , \ |t| \geq 1.$$

A3: There exist $\beta > 0$, $\delta > 0$ and $q \geq 1$ with $dq > 1$ so that

$$\|N(\phi_1)-N(\phi_2)\| \leq \beta\{\|\phi_1\|_1+\|\phi_2\|_1\}^q\|\phi_1-\phi_2\|$$

$$\|N(\phi_1)-N(\phi_2)\|_2 \leq \beta\{[\|\phi_1\|_1+\|\phi_2\|_1]^{q-1}\|\phi_1-\phi_2\|_1+[\|\phi_1\|_1+\|\phi_2\|_1]^q\|\phi_1-\phi_2\|\}$$

for all $\phi_i \in H$ satisfying $\|\phi_i\| \leq \delta$, $i = 1,2$.

In order to define scattering states and an associated norm we introduce

$$\|\phi\|_3 = \sup_{-\infty\leq t\leq\infty} \{\|\phi(t)\| + (1+|t|)^d\|\phi(t)\|_1\}.$$

We now define (with a slight abuse of notation)

$$M(A) = \{\phi \in H: \|U(t)\phi\|_3 < \infty\}$$

$$\|\phi\|_M = \|U(t)\phi\|_3.$$

Thus we see that the scattering states are those which have a certain acceptable decay under free propagation.

The following results can now be established [12], [13], [15].

Theorem 5.1

Let

(i) A: H $\to$ H be a self-adjoint, linear operator.

(ii) N: H $\to$ H a nonlinear operator.

(iii) $\|\cdot\|_1$, $\|\cdot\|_2$ be two auxiliary norms on H so that

A_1, A_2, A_3 hold.

Then there exists $\eta_0 > 0$ such that for all $\phi_- \in M(A)$ with $\|\phi_-\|_M \leq \eta_0$, the integral equation

$$\phi(t) = U(t)\phi_- + \int_{-\infty}^{t} U(t-s)N(\phi(s))ds \qquad (5.1)$$

has a unique global solution ϕ which is continuous and satisfies $\|\phi\|_3 \le 2\eta_0$. Furthermore

(a) $\phi(t) \in M(A)$ for each t.

(b) $\|\phi(t) - U(t)\phi_-\| \to 0$ as $t \to -\infty$.

Theorem 5.1 settles global existence for small data. Corresponding results for the scattering operator are given by [15].

Theorem 5.2

Assume all the hypotheses of Theorem 5.1 and let $\phi(t)$ be a solution of (5.1) corresponding to $\phi_- \in M(A)$ with $\|\phi_-\|_M \le \eta_0$. Then for sufficiently small η_0

(a) there exists $\phi_+ \in M(A)$ with $\|\phi_+\|_M \le 2\eta_0$ such that

$$\|\phi(t) - U(t)\phi_+\| \to 0 \text{ as } t \to +\infty.$$

(b) $S: \phi_- \to \phi_+$ is (1-1) and continuous, with respect to $\|.\|$, mapping of $\{\psi \in M: \|\psi\|_M \le \eta_0\}$ into $\{\psi \in M: \|\psi\|_M \le 2\eta_0\}$.

(c) for $q \neq 1$

$$\|U(-t)\phi(t) - \phi_+\|_M \to 0 \text{ as } t \to +\infty.$$

(d) S is continuous with respect to $\|.\|_M$.

Two aspects of these Theorems should be noted: the hypotheses of the theorems required neither a priori estimates on the solutions of the non linear equations nor the use of "energy" inequalities. All that was required was that the solution of the associated free equation should decay sufficiently rapidly, and that the degree of non linearity in the given problem be sufficiently high.

This development of a scattering theory for small data was initiated by the pioneering work of Segal [13] in his investigations of the Klein-Gordon equation. Subsequently Strauss [15] simplified much of Segal's work and presented an abstract formulation of the problem.

6. Existence of wave operators

Let Y be a Banach space and I an interval of real numbers. We introduce the notation

$L_p(I,Y)$ = Lebesgue space of functions: $I \to Y$

$C(I,Y)$ = space of strongly continuous functions: $I \to Y$

$B(I,Y)$ = space of bounded functions: $I \to Y$

H = Hilbert space with norm $\|.\| = (H,\|.\|)$

H_1 = Hilbert space H with norm $\|.\|_1 = (H,\|.\|_1)$

H_2 = Hilbert space H with norm $\|.\|_2 = (H_1,\|.\|_2)$

Z = $L_{p+1}(\mathbb{R},H_1) \cap B(\mathbb{R},H_1)$.

We also make the following assumptions.

A4: Let G denote a functional which maps a neighbourhood of zero in H_1 into $\mathbb{R}$. The functional G is lower semicontinuous and continuous at the zero element.

A5: Whenever I is a time interval, $\tau \in I$, $f \in H$ and $\phi \in Z$, with $\|\phi\|_Z$ sufficiently small, a solution of

$$\phi(t) = U(t)\phi_\tau + \int_\tau^t U(t-s)N(\phi(s))ds \tag{6.1}$$

then $\phi \in C(I,H)$ with

$$\frac{1}{2}\|\phi(t)\|^2 + G(\phi(t))$$

independent of t.

With this preparation the following result can be stated, [15].

Theorem 6.1

Assume A1 to A5. If $\phi_- \in M \subset H$ then there exists a time $T > -\infty$ and a unique solution of ϕ of

$$\phi(t) = U(t)\phi_- + \int_{-\infty}^t U(t-s)N(\phi(s))ds \tag{6.2}$$

in the time interval $I := (-\infty,T)$ such that

$$\phi \in C(I,H) \cap L_{p+1}(I,H_1)$$

and

$$\|\phi(t) - U(t)\phi_-\| \to 0 \quad \text{as } t \to -\infty \tag{6.3}$$

$$\frac{1}{2}\|\phi(t)\|^2 + G(\phi(t)) = \frac{1}{2}\|\phi_-\|^2. \tag{6.4}$$

Furthermore the mapping

$$\Omega_- : \phi_- \to \phi(0)$$

is a (1-1) mapping of $M(A)$ into $M(A)$ which is uniformly continuous on compacta in $M(A)$.

Similarly, for $\phi_+ \in M \subset H$

$$\|\phi(t) - U(t)\phi_+\| \to 0 \quad \text{as } t \to +\infty \tag{6.5}$$

$$\frac{1}{2}\|\phi(t)\|^2 + G(\phi(t)) = \frac{1}{2}\|\phi_+\|^2 \tag{6.6}$$

and the mapping

$$\Omega_+ : \phi_+ \to \phi(0)$$

is a (1-1) mapping of $M(A)$ into $M(A)$ which is uniformly continuous on compacta in $M(A)$.

7. Asymptotic completeness

We assume that the hypotheses of the previous two sections hold. Consequently, we can assert that the wave operators

$$\Omega_\pm : \phi_\pm \to \phi(0) \tag{7.1}$$

exist as (1-1) continuous maps of $M(A)$ into itself.

If we can now prove that

$$R(\Omega_+) = R(\Omega_-) \tag{7.2}$$

then it will be possible to define the scattering operator

$$S = (\Omega_+)^{-1}\Omega_- : \phi_- \to \phi_+. \tag{7.3}$$

An indication of how the asymptotic completeness problem posed by (7.2) may be approached can be given as follows.

Let $\phi_0 \in R(\Omega_-)$ then there exists a $\phi_- \in M(A)$ such that $\Omega_-\phi_- = \phi_0$ and a $\phi(t)$ satisfying

$$\phi(t) = U(t)\phi_- + \int_{-\infty}^{t} U(t-s)N(\phi(s))ds \qquad (7.4)$$

so that

$$\phi(0) = \phi_0$$

and

$$\|\phi(t) - U(t)\phi_-\| \to 0 \quad \text{as } t \to -\infty.$$

It is worth remarking that (7.4) can be rewritten in the form

$$\phi(t) = U(t)\left\{\phi_- + \int_{-\infty}^{t} U(-s)N(\phi(s))ds\right\} + \int_0^t U(t-s)N(\phi(s)ds \qquad (7.5)$$

from which we can conclude

$$\phi(0) = \phi_0 = \left\{\phi_- + \int_{-\infty}^{0} U(-s)N(\phi(s))ds. \qquad (7.6)\right.$$

What now has to be done is prove that there exists $\phi_+ \in M(A)$ such that for $\phi(t)$ defined by (7.4)

$$\|\phi(t) - U(t)\phi_+\| = \|U(-t)\phi(t) - \phi_+\| \to 0 \quad \text{as } t \to +\infty \qquad (7.7)$$

the equality following by virtue of the unitarity of U(t).

We now notice that (7.7) holds if and only if

$$\phi_+ = \lim_{t\to+\infty} U(-t)\phi(t) \qquad (7.8)$$

and hence the existence of ϕ_+ is settled.

With a view to establishing (7.8) consider the Cauchy sequence, with respect to t, generated by the right hand side of (7.8). Using A3 we obtain

$$\begin{aligned}\|U(t_1)\phi(t_1) - U(-t_2)\phi(t_2)\| &= \left\|\int_{t_1}^{t_2} e^{isA}N(\phi(s))ds\right\| \\ &\leq \int_{t_1}^{t_2} \|N(\phi(s))\|ds \\ &\leq \beta\int_{t_1}^{t_2} \|\phi\|_1^q\|\phi\|ds. \qquad (7.9)\end{aligned}$$

In general an upper bound for $\|\phi\|$ appearing in (7.9) can be obtained by means of an energy argument. Therefore, to ensure that the right hand side of (7.9) tends to zero, as required, as $t \to \infty$ we need to be able to show that

$$\int_{-\infty}^{\infty} \|\phi\|_1^q ds < \infty. \tag{7.10}$$

That is we need an a priori estimate on the solutions of the nonlinear equations which indicate that the solutions decay sufficiently rapidly in the $\|\cdot\|_1$ norm as $t \to \pm\infty$. If, in addition, we want to establish the continuity of the scattering operator then we will need to be able to estimate the decay constants. In general no such a priori results are available. However, for particular equations some progress has been made and one of the most complete results is the following due to Morawetz and Strauss [10].

Theorem 7.1

Let

(i) $u(x,t)$ denote the solution of the free problem

$$u_{tt}(x,t) - \Delta u(x,t) + m^2 u(x,t) = 0, \qquad x \in \mathbb{R}^3 \tag{7.11}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x). \tag{7.12}$$

(ii) F denote the closure of $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_4$ defined by

$$\|(f,g)\|_4 := \sup_t \left\{ \int_{\mathbb{R}^3} \left[u_t^2 + |\nabla u|^2 + m^2 u^2 \right] dx \right\} +$$
$$\sup_t \sup_x \left\{ |u(x,t)|^2 \right\} + \int_{-\infty}^{\infty} \sup_x \left\{ |u(x,t)|^2 dt \right\}. \tag{7.13}$$

Then the scattering operator for

$$u_{tt}(x,t) - \Delta u(x,t) + m^2 u(x,t) = -u^3(x,t), \qquad x \in \mathbb{R}^3 \tag{7.14}$$

with data (7.12) exists and is a (1-1) continuous mapping of F into itself.

If $(f,g) \in F$ and ∇f has finite energy then the solution $\tilde{u}(x,t)$ of (7.14) with data (f, g) satisfies

$$\|\tilde{u}(x,t)\|_\infty \le c(1+|t|)^{-3/2} \tag{7.15}$$

8. Concluding remarks

This has been a very rapid tour through a technically demanding area. Nevertheless, it is hoped that just the simple statements of some of the results which can be obtained will be sufficiently attractive to persuade people to work in the field. There are considerable opportunities for making significant contributions to the area. For instance in the development of an abstract scattering theory without the restriction of small data and high degree of nonlinearity. This could then be followed by application to specific equations and systems of equations, of particular interest would be the behaviour of the wave and scattering operators as functions of the various parameters. The question of asymptotic completeness will always be a difficult one and any progress in this direction would always be welcome. There are two areas which are perhaps more promising than most for making early contributions. The first concerns the influence of boundary conditions. A considerable amount of work has been done in this direction for linear scattering theory and in this connection we would cite the work of Wilcox [16] and his co-workers. There would seem to be good prospects for performing a similarly detailed investigation for nonlinear scattering problems. The second centres on inverse scattering problems. If the "free" or linear problem is described in terms of a group $V_0(t)$ and the "perturbed" or nonlinear problem is described in terms of a group $V(t)$ then the inverse problem is concerned with the determination of $V(t)$ from a knowledge of $V_0(t)$ and S. There are indications that for nonlinear scattering problems certain results can be obtained sometimes more readily than for linear scattering problems. Results in this direction have been obtained by Morawetz and Strauss [11]. However, there still remains the detailed considerations of boundary effect in this class of problem.

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