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GENERALIZATIONS OF HANKEL OPERATORS

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To Eila with love

Preface

I became first interested in Hankel operators in March or April 1981. Before that period I had hardly even heard the name. The purpose of these notes is, roughly speaking, to narrate - without entering into too many details - what I have learned about them since then. In a way this is a sequel to my report to the '82 Edmonton meeting [27] (compare also [28], [29], [30], [31]) but possibly even narrower in scope. To be able to read this compilation an acquaintance with [27] is however not required, although we occasionally give some crossreferences. For a more comprehensive treatment of Hankel (and/or Toeplitz) operators we refer to any of the following excellent sources of survey character: [47], [23], especially App. 4, [24], [37], [25], [42], [43].

Looking back in time, there were two things which I learned more or less by accident but which turned out to be pivotal for me:

1° the new solution of the Littlewood conjecture [21], [22], [20] (of which I first heard via M. Cwikel and then E. Svensson, eventually leading to our joint paper [32]),

2° V. V. Peller's announcement of his now famous trace ideal criterion [33] (to which J. Bergh turned my attention),

because I saw how to combine the two. In retrospect however one sees now that at least the former has little to do with the core of the matter. Moral?

Of much greater importance, however, was my later association with J. Arazy, who emphasized the importance of Möbius invariant function spaces in (complex) Analysis, and S. Fisher (see [2], [3], [4], [5], [28], [29], [30]). At any rate, I am here expressing a most personal view of the state of affairs, not even always agreeing with that of my many coworkers, of which I would here especially like to mention: Jonathan Arazy, Stephen Fisher, Svante Janson and Richard Rochberg.

1. Classical material

A *Hankel matrix* is a matrix of the form

$$(a_{n+m}) \ .$$

By comparison a *Toeplitz matrix* is one of the form

$$(a_{n-m}) \ .$$

Hankel (and Toeplitz) matrices have a long history (see e.g. [27] where there is a short bibliographical sketch of the life of Hermann Hankel (1839-1873)).

EXAMPLE. The two Hilbert matrices

$$\left(\frac{1}{n \pm m}\right) \ .$$

Hilbert in his celebrated lectures (1906) on integral equations proved that both were bounded in the sequence space ℓ^2 .

To proceed further it is better to let them act on suitable function spaces. There are several options.

1) Let $H^2(\underline{T})$ be the classical Hardy class of analytic functions in the unit disk D of the complex plane $\underline{C}$, $\partial D = \underline{T}$ being the unit circumference:

$$f \in H^2(\underline{T}) \quad \text{iff} \quad \int_{\underline{T}} |f(z)|^2 |dz| < \infty ,$$

and let $H^2_-(\underline{T})$ $(= H^2(\underline{T})^\perp)$ be its complement in $L^2(\underline{T})$ - it consists of anti-analytic functions - denoting the corresponding projection by $P^\perp$. Then we define the *Hankel operator* $H_b : H^2(\underline{T}) \to H^2_-(\underline{T})$ with "symbol" b , usually an analytic function, by the formula

$$H_b f = P^\perp(\bar{b} f) \quad (f \in H^2(\underline{T})).$$

If we in $H^2(\underline{T})$ and $H^2_-(\underline{T})$ use the "natural" bases $(z^m)_{m \geq 0}$ and $(z^{-n})_{n>0}$ respectively we see that H_b is given by the matrix

$$\overline{\hat{b}(n+m)} \ ,$$

a Hankel matrix.

2) *Alternatively*, we may consider the operator $\tilde{H}_b : H^2(\underline{T}) \to H^2(\underline{T})$, likewise termed Hankel operator, defined by

$$\tilde{H}_b f = P(b\bar{f}) \ ,$$

where now P is projection onto $H^2(\underline{T})$ $(P + P^\perp = id)$; it is thus *anti-linear* over $\underline{C}$. Now the matrix is

$$\hat{b}(n + m + 1) \ .$$

[3) By comparison, a *Toeplitz operator* with symbol b (not necessarily analytic) is usually defined as an operator $T_b : H^2(\underline{T}) \to H^2(\underline{T})$ such that

$$T_b f = P(bf) \quad (f \in H^2(\underline{T})).$$

Its matrix is

$$\hat{b}(m - n) \ ,$$

a Toeplitz matrix.]

The two type of Hankel operators are formally connected by the identity

$$H_b f = z^{-1} \overline{\tilde{H}_c f} \quad \text{with} \quad b = \frac{c - c(0)}{z} \ .$$

As we shall see, on higher levels the theory usually bifurcates.

With the operator $\tilde{H}_b$ it is also natural to associate the bilinear form Γ_b, a *Hankel form*, defined by

$$\Gamma_b(f,g) = \int_{\underline{T}} \bar{b} fg |dz|/2\pi \quad (= \langle f, \tilde{H}_b g\rangle_{H^2(\underline{T})}).$$

NOTATION. If X is any Hilbert space, we denote the corresponding inner product by $\langle .,. \rangle_X$. If B is a Banach space the norm in B is written $||\cdot||_B$ (or $||\cdot||B$).

To fix the ideas let us below only investigate the operators H_b

The natural question is now to try to relate the (smoothness) properties of the operator with the ones of its symbol. Thus, when is H_b of finite rank, bounded, compact, of class S_p ("smooth")? Here are the standard answers:

L. KRONECKER'S THEOREM (1881). H_b *is of finite rank iff* b *is a rational function (with all poles outside* $\overline{D}$).

Z. NEHARI'S THEOREM (1957). H_b *is bounded (as an operator acting from* $H^2(\underline{T})$ *into* $H^2_-(\underline{T})$) *iff* $b \in$ BMOA (bounded mean oscillation).

PH. HARTMAN'S THEOREM (1958). H_b *is compact (i.e.* $\in S_\infty$) *iff* $b \in$ VMOA (vanishing mean oscillation).

V. V. PELLER'S THEOREM (1980). $H_b \in S_p$ *iff* $b \in B_p^{1/p,p}A(\underline{T})$ (a Besov space).

COMMENT. This formulation is of course quite "unhistorical", because neither BMO nor VMO were invented at the time. Peller proved his theorem first in the case $1 \leq p < \infty$ ([34], research announcement in [33]) and the extension to the full range $0 < p < \infty$ was given only later by him [35] and, independently, by Semmes [48] (and, implicitly, by Pekarskiĭ [38]). If X is any space of functions or distributions on $\underline{T}$ we denote by XA the subspace of those elements of X which are (distributional) boundary values of functions analytic in D. Thus BMOA consists of the "analytic" functions in BMO etc. A good example for Nehari's theorem is otherwise $b = -\log(1-z)$, which gives the Hilbert matrix ($\hat{b}(n) = 1/n$). Let us also point out that the standard proof essentially depends on the idea of (weak) factorization (cf. especially [10]). Kronecker's theorem has a purely algebraic content (cf. the concluding paragraph of §5).

For completeness sake, we also recall the basic notions connected with the Schatten (-von Neumann) or trace classes (ideals) S_p. Let H and K be any two Hilbert spaces. If T is a compact linear operator from H into K, its Schmidt, s- or approximation numbers $s_n = s_n(T)$ ($n = 0,1,\ldots$) may be defined as the eigenvalues of the compact positive selfadjoint operator $(T^*T)^{1/2}$:

$$s_n(T) = \lambda_n\left((T^*T)^{1/2}\right) \quad \left(= \lambda_n\left((TT^*)^{1/2}\right)\right).$$

We say that $T \in S_p$, where $0 < p \leq \infty$, iff $(s_n)_0^\infty \in \ell_p$. The class S_2 consists of the Hilbert-Schmidt operators ($\sum |a_{nm}|^2 < \infty$ where (a_{nm}) is the matrix of T with respect to any two orthonormal bases in H and K respectively). $T \in S_1$ means that T is nuclear (and $T \in S_p$, $0 < p < 1$, that T is p-nuclear in Grothendieck's sense). S_∞ is the class of all compact operators. An extensive treatment of the Schatten classes can be found in [49] or in [12]. For the interpolation theory see the relevant divisions in [8] or [54].

Returning to the Hankel operators a remarkable fact is that the entire theory is *conformally invariant*. Indeed, let the group SU(1,1) (the factor group PSU(1,1) = SU(1,1)/center is the *Möbius group* of all conformal selfmaps of D) act on $L^2(\underline{T})$ via the formula

$$f(z) \to U_\phi f(z) = f(\phi z)(cz+d)^{-1},$$
$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1).$$

Then both $H^2(\underline{T})$ and its complement $H_-^2(\underline{T})$ are left invariant under this action. It is an easy excercise to show that then symbols of Hankel operators transform according to the rule

$$b(z) \to b(\phi z) \quad (= (b \circ \phi)(z)).$$

More precisely, one has the formula:

$$H_b U_\phi = U_\phi H_{b \circ \phi} .$$

REMARK. In all what we say the Möbius group plays just the rôle of the "model" case. To some extent similar considerations can be made with other (semi-) simple Lie groups G of non-compact type such that the Lie algebra of its maximal compact subgroup K has a non-zero center; this makes it possible to introduce a complex structure on the associated symmetric space G/K (which takes the place of the unit disk in the general case).

The above is reflected by the fact that the symbol classes appearing in the above theorems (Kronecker etc.) likewise display this in invariance. Conversely, as I observed (in the summer of 1983?), the invariance can be used to give a *very simple proof* ("by handwaving") of Peller's theorem at least in the special case $1 < p < \infty$. (Peller's own proof was quite complicated ("hard analysis")!) For more or less detailed accounts we refer to [28], [29], [30], [31]. Here we restrict ourselves to giving just the main idea.

Consider the class X, say, of symbols b such that $H_b \in S_1$. By Kronecker's theorem certainly $X \neq 0$. We equip X with the norm $||b||_X = ||H_b||S_1$, which is *Möbius invariant*. But by the basic facts on Möbius invariant spaces of holomorphic functions ([4]; for this theory see also [2], [3], [45], [11], [28], [29], [30]) the Besov space $B_1^{1,1}A(\underline{T})$ is the *minimal* such space. This gives us at once one endpoint result, viz. $b \in B_1^{1,1}A(\underline{T}) \Rightarrow H_b \in S_1$, the other one coming from Nehari's theorem: $b \in BMOA(\underline{T}) \Rightarrow H_b$ bounded. From there half of the theorem (for $1 < p < \infty$) follows by routine application of suitable *interpolation theorems*. The other half follows by duality (as in Peller). This again resides basically on an essentially group (representation) theoretic fact. As $B_2^{2,2}A(\underline{T})$ is the (up to a factor in the metric) unique Hilbert space among all Möbius invariant spaces [3], we see that (up to a factor) the two norms $||b||B_2^{2,2}$ and $||H_b||S_1$ must agree. This gives us an isometric imbedding $I : B_2^{2,2}A(\underline{T}) \to S_2$. The adjoint $J = I^* : S_2 \to B_2^{2,2}A(\underline{T})$ then formally satisfies $J \circ I = id$ so we have a *retraction* of the map which to a symbol assigns the corresponding Hankel operator. This suffices to complete the proof.

The virtue of this proof is that it is so general and in the sequel we shall see ample illustrations of it at the hand of a large variety of "trace ideal criteria" for various generalizations of Hankel operators.

As a first instance let us mention - a rather easy extension - the case of the *weighted Bergman spaces* $A^{\alpha 2}(D)$, $\alpha > -1$:

$$f \in A^{\alpha 2}(D) \quad \text{iff} \quad \iint_D |f(z)|^2 (1 - |z|^2)^\alpha \, d\sigma(z) < \infty ,$$

which as a limiting case ($\alpha \to -1$) comprise the Hardy space $H^2(\underline{T})$ (if $\alpha \to -1$ the 2-dimensional measure $(1 + \alpha)(1 - |z|^2)^\alpha \, d\sigma(z)$ - here as well as in the sequel $d\sigma(z) = 1/\pi \cdot dxdy$ is the normalized area measure - formally tends to the 1-dimensional (boundary) measure $|dz|/2\pi$); another important limiting case ($\alpha \to \infty$), to be treated later (§4), corresponds to the *Fock space*.

It is convenient now to work with $\tilde{H}_b$, not with H_b (cf. however §5). The only major difference is then (what the trace ideal criterion goes) that the Besov spaces experience a shift: $\tilde{H}_b \in S_p$ as an operator on $A^{\alpha 2}(\underline{T})$ iff $b \in B_p^{1/p+\alpha+1,p}A(\underline{T})$. To put the above machinery at work one has however to invoke another "symbol" than the previous b , namely the "true" symbol B , which transforms nicely under the group. The relation between b and B is most simply written in terms of the associated bilinear form Γ_b :

$$\Gamma_b(f,g) = \iint_D \bar{B} fg (1 - |z|^2)^\beta \, d\sigma(z) , \quad \beta = 2\alpha + 2 .$$

(This will be explained later (see §4).) For details about this theory see [26], [28], [29] and also [1], where an extension to the case of the unit ball in $\underline{C}^d$ is briefly outlined.

2. P a r a c o m m u t a t o r s

In this § we discuss a recent quite general extension of the theory of Hankel operators (forms) as outlined in the previous §. For full details we refer to [16]. Related (parallel) work can be found in [51], [52], [39], [40], [41]. (See also [53] where the corresponding set-up over *local fields* is investigated.)

To arrive at this generalization in a natural ("historical") order let us for a moment return to the Hankel operators H_b as defined in §1. Because of conformal equivalence we may replace the unit disk D ($\partial D = \underline{T}$) by the upper (Poincaré) halfplane U in $\underline{R}^2$ ($\partial U = \underline{R}$). Then the study of Hankel operators is essentially equivalent to the study of the commutator

$$C_b \overset{\mathrm{def}}{=} [M_b, P] ,$$

where now P is orthogonal projection in $L^2(\underline{R})$ onto $H^2(\underline{R})$ and M_b multiplication with b , $M_b f = bf$. This connection, apparently, was first (in 1976) pointed out by Coifman-Rochberg-Weiss [10] and goes as follows. If as before $H_b : H^2(\underline{R}) \to (H^2(\underline{R}))^{\perp}$ is defined via

$$H_b f = P^{\perp}(\bar{b}f) \quad (f \in H^2(\underline{R}))$$

we can also write this as (as $Pf = f$)

$$H_b f = \bar{b}f - P(\bar{b}f) = \bar{b}Pf - P(\bar{b}f) = [M_{\bar{b}}, P]f ,$$

i.e. we have

$$H_b = C_{\bar{b}} | H^2(\underline{T}) ,$$

this formally also for *non-analytic* symbols b . As plainly $C_b^* = -C_{\bar{b}}$ it follows that C_b in the decomposition $L^2(\underline{R}) = H^2(\underline{R}) \oplus H^2(\underline{R})^{\perp}$ is given by the "block matrix"

$$\begin{pmatrix} 0 & -H_b^* \\ H_{\bar{b}} & 0 \end{pmatrix} .$$

The paper [10] likewise initiated the study of analogous commutators

$$C_b = [M_b, K] ,$$

where K is a *Calderón-Zygmund* operator in $\underline{R}^d$ (i.e., convolution with a kernel k which is homogeneous of degree $-d$ and has vanishing spherical averages; if $d = 1$ there is up to a factor only one Calderón-Zygmund operator, namely the Hilbert transform $H = 2P - \mathrm{id}$). As far as boundedness and compactness (even in $L^p(\underline{R}^d)$, $p \neq 2$) goes, their theory was later completed by Janson [14] and Uchiyama [55]. A trace ideal criterion was then proved by Janson and Wolff [19]: Unless $K \neq 0$ (and $d \neq 1$) one has $C_b \in S_p$ iff $b \in B_p^{d/p,p}(\underline{R}^d)$ for $p > d$; if $p \leq d$ then $C_b \in S_p \Rightarrow b$ constant. (The characterization of symbols b with $C_b \in S_{d\infty}$ (Lorentz-Schatten class) seems to be open even in this simple case.) A new phenomenon is thus the complete failure of Kronecker in higher dimensions (no finite rank operators at all!). The proofs in [19] were quite complicated, especially for the converse part.

In [15] a new approach to this theory was given. It went via *higher order* commutators of the type

$$C_b^N = \left[\cdots\left[[M_b, K_1], K_2\right], \ldots, K_N\right] ,$$

where each K_i is as before a Calderón-Zygmund operator. Then the limit $p > d$ in the single commutator case gets replaced by $p > d/N$: $C_b^N \in S_p$ iff $b \in B_p^{d/p,p}(\underline{R}^d)$ in the latter hypothesis. The virtue of this is that by taking N sufficiently large one could include the limit $p = 1$ in the "good" parameter-range and so the pattern in Peller's proof (see §1) could be more or less copied. What is relevant here is the *minimality* of the Besov spaces $B_1^{s,1}(\underline{R}^d)$ with respect to the *affine group* (translations + dilations); the complete statement can be found in e.g. [29]. (Invariance with respect to the full Möbius group §1 was therefore not that vital after all!) It then became clear that in the same way one could treat a much more general class of operators (or forms), which were termed *paracommutators*, because of the apparent analogy with the paraproducts of Bony (see e.g. [9]; see also [50] for a more informal account). Recall that Bony applies his theory to analyze the singularities of semi-linear P.D.E.; roughly speaking, paramultiplication helps to "linearize" the problem, the paraproduct containing all the essential information about the singularity.

Leaving history here is, finally, the formal definition. A paracommutator is an operator P_b, usually acting on $L^2(\underline{R}^d)$, of the form

$$P_b f(\xi) = (2\pi)^{-d} \int_{\underline{R}^d} \hat{b}(\xi - \eta) A(\xi,\eta) \hat{f}(\eta)\, d\eta$$

where b is the *symbol* of the paracommutator and A its *Fourier kernel;* A is usually considered as "fixed". If $A(\xi,\eta) \equiv 1$ one recovers the multiplication operator M_b. The idea is thus that in the general case P_b is a multiplication operator perturbed by a Schur multiplier A (on the Fourier side). To get something which resembles a Hankel operator A must drop off sufficiently fast on the diagonal $\{(\xi,\eta) : \xi = \eta\}$; otherwise one gets something which is more like a Toeplitz operator.

<u>EXAMPLE</u>. The previous higher commutators C_b^N arise if we take

$$A(\xi,\eta) = \prod_{j=1}^{N} \left(\hat{k}_j(\xi) - \hat{k}_j(\eta)\right)$$

where k_j is the kernel (a homogeneous function of degree $-d$ with vanishing spherical means) associated with the operator K_j ($j = 1,\dots,N$) .

The theory in [16] is quite technical, mainly due to the rather complicated assumptions one has to impose on A ; indeed, there appears a whole series on conditions labelled A0 through A8 . The essential ingredients of the proof are however already present in [15] (the case of higher commutators). As far as the trace ideal criterion goes, the main result, however, still can be condensed into the simple implication:

<u>"THEOREM".</u> $P_b \in S_p$ *iff* $b \in B_p^{d/p,p}(\underline{R}^d)$ *for* $p > d/N$.

This, in suitable assumptions on A ; in particular, the number N gives the order of vanishing of A on the diagonal. In the model case of the example (the higher commutators) N is in general the number of factors K_j .

<u>REMARK.</u> For technical reason it is also convenient to consider more general operators $P_b^{s,t}$, which are defined by an analogous formula with $A(\xi,\eta)$ replaced by $A(\xi,\eta)|\xi|^s|\eta|^t$. (<u>N.B.</u> - $A(\xi,\eta)$ should be thought of as homogeneous of degree 0 ; this is precisely the content of A0 .) Alternatively, one considers P_b as an operator from $B_2^{-t,2}(\underline{R}^d)$ into $B_2^{s,2}(\underline{R}^d)$. One can also formulate everything in terms of certain bilinear forms. More precisely, let us put

$$\Pi_b^{s,t}(f,g)$$
$$= (2\pi)^{-d} \iint_{\underline{R}^d\times\underline{R}^d} A(\xi,\eta)|\xi|^s|\eta|^t\hat{b}(\xi+\eta)\hat{f}(\xi)\hat{g}(\eta)\,d\xi d\eta \ ;$$

such forms will be termed paracommutators too. Finally, this suggests that one can treat similar *multilinear* forms too (cf. [29], Lect. 5, for some more hints on this).

<u>3. Higher weights</u>

Let us return to the original Hankel situation of §1, especially to the "group theoretic" proof of Peller's theorem given there. At the end of that § we also indicated that the same proof works in the more general case of Hankel forms over the weighted Bergman spaces $A^{\alpha,2}(D)$, $\alpha > -1$. This due to the fact that the "true" symbol B of the form in question transforms in a nice way under the group.

If $\phi = \binom{a\ b}{c\ d} \in SU(1,1)$ then $B(z) \to B(\phi z)(cz + d)^{-(2\alpha+4)}$ for $f \in A^{\alpha,2}(D)$; more about this in the next §. (If α is not an integer one has to pass to the universal covering group of $SU(1,1)$.)

We now give the above another twist. For convenience, let us introduce the following notation:

$$G = SU(1,1) ,$$
$$V = A^{\alpha 2}(D) ,$$

where in the sequel we let α be an *integer*. Let us put

$$\nu = \alpha + 2 .$$

Then G has a natural action on the Hilbert tensor product $V \otimes V$, which may be viewed as the space of Hilbert-Schmidt forms on V . What are the irreducible constituents of $V \otimes V$ under this action? They were determined in [17] (cf. also [44]). It turns out that they are labeled by their lowest weights 2ν, $2\nu + 2$, $2\nu + 4,\ldots$ (in the sense of É. Cartan's theory) and that the component of lowest weight 2ν corresponds precisely to the Hankel forms. It likewise turns out that the forms belonging to a given component may be treated as special paracommutators (after a conformal transplantation to the halfplane) so the theory in [18] is applicable (§2). In particular, for each (irreducible) component one has a theory entirely parallel to the usual theory of Hankel forms, especially including a Peller's theorem.

Rather than carrying out this spectral analysis in detail (for this see [17]) let us start from scratch.

Exploiting the aforementioned conformal invariance we may as well replace the disk D by the upper (Poincaré) halfplane U and the unit circumference $\underline{T} = \partial D$ by the real line $\underline{R} = \partial U$ (in the usual identifications).

Let f_k ($k = 1,2$) transform according to $f_k(z) \to f_k(\phi z)(cz + d)^{-\nu}$ (ν integer ≥ 1). Then we have the following result, the proof of which we leave as an excercise to the reader (else see [17]). Set

$$J = \sum_{u=0}^{s} (-1)^{s-u} \binom{s}{u} \frac{1}{(\nu)_u (\nu)_{s-u}} D^u f_1 D^{s-u} f_2 \quad \left(D = \frac{d}{dx} \right).$$

LEMMA. *J transforms according to the rule* $J(z) \to J(\phi z)(cz + d)^{-2r}$ *where* $r = s + \nu$.

We can then define a *generalized Hankel form* (of weight $2r$) by the formula

$$\Gamma_g(f_1,f_2) = \int_{\underline{R}} \bar{g}J\,dx .$$

As for $s = 0$ clearly $J = f_1 f_2$, this is consistent with the previous notation. It is easy to see that this is formally a paracommutator Π_g (see §2) with

$$A(\xi,\eta) = \begin{cases} J(\xi,\eta) & \text{if } \xi > 0 , \quad \eta > 0 \\ 0 & \text{else} \end{cases} ,$$

where $J(\xi,\eta)$ is the symbol of the "bi-PDO" J . Therefore the theory of [18] (see §2) is in principle applicable. Here is the result; for the details of the proof we refer to [17].

<u>THEOREM</u>. Γ_g *is* a) *bounded (in* $A^{\alpha 2}(U)$ *) iff* $g \in I^{r-1}BMOA(\underline{R})$ *(where* $I = D^{-1}$ *) and* b) *in* S_p , $1 < p < \infty$, *iff* $g \in B_p^{r-1+1/p}A(\underline{R})$.

<u>REMARK</u>. For technical reasons it is necessary to replace the previous Fourier kernel A by the Fourier kernel B given by

$$B(\xi,\eta) = A(\xi,\eta)/(\xi + \eta)^s$$

[this in order to maintain the "homogeneity"] and g by $h = D^s g$. Then we apply the theory in [18] to the paracommutator $\Pi_h^{\lambda,\lambda}$, where $\lambda = (\alpha + 1)/2$ ($= (\nu - 1)/2$).

<u>4. Fock space</u>

Let us write down the definition of weighted Bergman space for the case of a concentric disk D_R with radius R :

$$f \in A^{\alpha 2}(D_R) \quad \text{iff} \quad \iint_{D_R} |f(z)|^2 (1 - |z|^2/R^2)^\alpha \, d\sigma(z) < \infty .$$

If we write $\alpha = \lambda R^2$ and if we let $R \to \infty$, what evolves is formally the famous *Fock space:*

$$f \in \Phi^{\lambda 2}(\underline{C}) \quad \text{iff} \quad \iint_{\underline{C}} |f(z)|^2 e^{-\lambda |z|^2} \, d\sigma(z) < \infty .$$

The role of the semi-simple group $SU(1,1)$ is now taken on by its "contraction" the equally famous *Heisenberg group*, a nilpotent Lie group, the corresponding representation of it in $\Phi^{\lambda 2}(\underline{C})$ being known as the *Bargman-Segal representation,* very familiar in quantum field theory.

We now wish to outline a theory of Hankel forms which applies to both the weighted Bergman space and as a limiting case the Fock space. As the theory of Besov spaces is not available in the latter case, we must proceed differently. Another bonus is that we at the same time capture the extension to several (complex) variables. For details we refer again to [18].

Let Ω be any domain in $\underline{C}^d$ equipped with the measure μ (supp $\mu \subset \Omega$ or, possibly, $\bar{\Omega}$), which we assume to be absolutely continuous with respect to Lebesgue (volume) measure, and define $H^2(\Omega,\mu)$ (also written $A^2(\Omega,\mu)$ in the literature) to be the Hilbert space of square integrable (with respect to μ) analytic functions in Ω . Let $K(z,\bar{w})$ (also written $K_w(z)$) be the corresponding reproducing kernel:

$$f(z) = \langle f,K_z\rangle \quad \text{for} \quad z \in \Omega \ , \quad f \in H^2(\Omega,\mu) \ .$$

Along with μ we now associate the measure ν defined by

$$d\nu(z) \overset{\text{def}}{=} \frac{d\mu(z)}{K(z,\bar{z})} \ .$$

Denote by $L(z,\bar{w})$ the reproducing kernel in the space $H^2(\Omega,\nu)$. Then we make the assumption that

(V) $$L(z,\bar{z}) = c\big(K(z,\bar{z})\big)^2$$

where c is a constant ≥ 1 (independent of $z \in \Omega$). In this assumption it is possible to develop a satisfactory theory of Hilbert forms on $H^2(\Omega,\mu)$. As a definition of the latter we take

$$\Gamma_b(f,g) = \int_\Omega \overline{b(z)}\ f(z)\ g(z)\ d\nu(z)$$

where b (an analytic function) again is termed the "symbol" of the form Γ_b . Writing $\omega(z) = 1/K(z,\bar{z})$ consider the weighted L^p-space

$$L^p_\omega(\Omega,\nu) = \{f \mid \int_\Omega |f(z)|^p\ \omega(z)^{p-2}\ d\nu(z) < \infty\}$$

and let $H^p_\omega(\Omega,\nu)$ be the subspace of $L^p_\omega(\Omega,\nu)$ consisting of analytic functions. Then we can at least formulate the main result in [18] to which paper we have to refer for proofs and further details and examples.

<u>THEOREM</u>. *In the assumption* (V) (*and some other* (*more*) *natural supplementary assumptions*) *one has*

a) Γ_b *is bounded (on* $H^2(\Omega,\mu)$ *) iff* $b \in H^\infty_\omega(\Omega,\nu)$.

b) Γ_b *is in* S_p , $1 \leq p < \infty$ *iff* $b \in H^p_\omega(\Omega,\nu)$.

(Note especially that Γ_b is in S_2 (Hilbert-Schmidt!) iff $b \in H^2(\Omega,\nu)$. The case $0 < p < 1$ is still open.)

The assumption (V) is fulfilled in all cases when the set-up is invariant for a sufficiently large group of "symmetries". Are there any other cases? This we do not know.

EXAMPLE. In the case of the weighted Bergman space $A^{\alpha 2}(D)$ ($d = 1$) we have

$$K(z,\bar{w}) = (1 - z\bar{w})^{-(\alpha+2)} ,$$

if we normalize the measure properly ($d\mu = (\alpha + 1)(1 - |z|^2)^\alpha \, d\sigma(z)/\pi$). It follows that the associated measure is $d\nu = (\alpha + 1)(1 - |z|^2)^{2\alpha+2} \, d\sigma(z)$ so that, with $\beta = 2\alpha + 2$, upto normalization

$$L(z,\bar{w}) = (1 - z\bar{w})^{-(\beta+2)} .$$

As $2(\alpha + 2) = \beta + 2$ we see by inspection that (V) indeed is fulfilled. Similar considerations can be made in the limiting case of the Fock space $\Phi^{\lambda,2}(\underline{C})$. One finds:

$$K(z,\bar{w}) = e^{\lambda z\bar{w}} , \quad L(z,\bar{w}) = e^{2\lambda z\bar{w}} ,$$

$$d\mu = \lambda/\pi \cdot e^{-\lambda|z|^2} \, d\sigma(z) , \quad d\nu = 2\lambda/\pi \cdot e^{-2\lambda|z|^2} \, d\sigma(z) .$$

Clearly again (V) is fulfilled. In this case it is thus question of the form

$$H^{2\lambda}_b(f,g) = 2\lambda \iint_{\underline{C}} \bar{b} \, f \, g \, e^{-2\lambda|z|^2} \, d\sigma(z) .$$

Actually, we can also treat the more general forms H^τ_b with τ arbitrary (not coupled to λ). We prove that

$$H^\tau_b \in S_p \quad (\text{on } \Phi^{\lambda,2}(\underline{C})) \text{ iff } b \in \Phi^{\lambda^2/2\tau,p}(\underline{C}) .$$

The special case $\tau = 2\lambda$ gives then

$$H^{2\lambda}_b \in S_p \text{ iff } b \in \Phi^{2\lambda,p}(\underline{C}) .$$

Both examples generalize to higher dimensions, the space $A^{\alpha,2}(B_d)$, where B_d is the unit ball in $\underline{C}^d$ (the "Rudin ball"), and $\Phi^{\lambda,2}(\underline{C}^d)$.

But there is a deeper reason why a condition like (V) indeed turns up at all. Namely, the whole theory is invariant for certain "supersymmetries" informally termed (*generalized*) *gauge transformations*. Let us briefly indicate what this is about. (For a related point of view, see the work of Berezin on "quantization", e.g. the expository paper [7].)

More precisely, the idea is the following. Consider, quite generally, a closed subspace H of $L^2(\Omega,\mu)$ consisting of *continuous* functions, where Ω is any locally compact space equipped with a measure μ. Let $K = K(z,w) = K_w(z)$ be the corresponding reproducing kernel. (As we are not any longer dealing with (necessarily) analytic functions, we drop the bar in the notation for "anti-analytic" arguments.) We argue that if we simultaneously replace f by ϕf and μ by $|\phi|^2 \cdot \mu$ where ϕ is any non-vanishing continuous function, we get an equivalent theory. Especially, H gets replaced by a space denoted H_ϕ with the reproducing kernel $\phi(z)\overline{\phi(w)}K(z,w)$. This is what we mean by "change of gauge", the philosophy being that we should deal only with gauge invariant quantities.

More generally, one can allow a homeomorphism γ of Ω followed by a gauge transformation in the previous sense, corresponding to a continuous function ϕ. We refer to such object as generalized gauge transformation; a generalized gauge transformation is thus determined by a pair (ϕ,γ).

Indeed, there is a canonical (up to sign unique) gauge in which the reproducing kernel is identically one on the diagonal $\{z = w\} \subset \Omega \times \Omega$: take $|\phi|^2 = \omega$ where as before $\omega = 1/K(z,z)$. It follows that the measure $\lambda = \mu/\omega$, which is the given measure μ transformed to the canonical gauge, has a gauge invariant meaning. In situation, when one has a sufficiently large group of automorphisms (consisting of generalized gauge transformations), it turns out that this is the usual invariant measure (the Poincaré measure in the special case of the unit disk D).

We can now also understand the meaning of the definition of the "associated" measure ν and condition (V): ν transforms according to the rule $\nu \to |\phi|^{-4}\nu$ (whereas $\mu \to |\phi|^{-2}\mu$). Consider the corresponding Hilbert space $\tilde{H}$ generated by products $f \cdot g$ where $f, g \in H$ and let L be its reproducing kernel. Then $L(z,w) \to \phi(z)^2\overline{\phi(w)}^2 L(z,w)$ (whereas, as we already know, $K(z,w) \to \phi(z)\overline{\phi(w)}K(z,w)$) so (V) has indeed an invariant meaning. Symbols of Hankel operator pertain to $\tilde{H}$, not H (because a symbol b transforms as a product $f \cdot g$: $b \to b\phi^2$).

REMARK. Above we raised the question whether the hypothesis (V) can be true in other than "group theoretic" cases. Let us write down what condition (V) means in a simple case. Take $d = 1$ and a simply connected domain Ω . Assume that $H^2(\Omega,\mu)$ admits a one parameter group of automorphisms induced by conformal selfmaps of Ω fixing a point. Thus we have virtually the case $\Omega = D$, μ = a *radial* measure. (If the fixed point sits at the boundary one has a corresponding "continuous" situation, with the Fourier transform instead of the Taylor development.) Then (V) becomes a condition of the moments $\gamma_k = \int_D |z|^k d\mu$ $(k = 0,1,2,\dots)$. Indeed, it is question of solving an infinite system of equations:

$$\sum_{n+m=p} \frac{1}{\gamma_n\gamma_m} \sum_{k=0}^{\infty} \delta_k\gamma_{p+k} = c \quad (= \text{constant}),$$

$$\sum_{k=0}^{p} \frac{\delta_k}{\gamma_{p-k}} = \begin{cases} 1 & p = 0 \\ 0 & p \neq 0 \end{cases}.$$

(The numbers γ_k determine the δ_k uniquely.) Thus, are there other solutions than the ones which come from weighted Bergman spaces (limiting cases included) given by

$$\gamma_k = \text{const}\cdot \frac{k!}{\Gamma(k + \alpha + 2)} R^k \ ?$$

(The letter Γ now stands for Euler's gamma function, of course.)

Finaly, concluding this Section, let us mention that in [18] there is also a very general Kronecker's theorem, essentially without "any" assumption on the measure μ (we are now thinking of the "analytic" case, of course). This is more or less an excercise in commutative algebra (Hilbert's "Nullstellensatz", primary ideal decomposition and all that). Indeed, *modulo* some simple functional analysis, it is just question of the structure of ideals of finite codimension in a polynomial ring. As a young student I bought a copy of [56]; finally, after so many years, I found a use for it ... Moral?

5. Axler

In this § we shall mostly deal with the weighted Bergman space $A^{\alpha 2}(D)$, $\alpha > -1$ (especially we are back in the case $d = 1$) and we shall report on the recent paper [5], which in turn is a sequel to Sheldon Axler's work [6] (the case $\alpha = 0$ (Bergman's own space!) and boundedness and compactness criteria only). In §1 already in the case of the Hardy classes $H^2(\mathbb{T})$ we indicated two different (but essentially equivalent) ways of defining Hankel operators. Especially, we defined the anti-$\underline{C}$-linear operator

$$\tilde{H}_b f = P(b\bar{f}) .$$

In the Bergman case one might want to consider instead

$$\tilde{H}_b^\alpha f = P^\alpha(b\bar{f}) ,$$

where P^α is the *Bergman projection* (orthogonal projection onto $A^{\alpha 2}(D)$ in the Hilbert space $L^2(D,\mu_\alpha)$). The operators H_b and $\tilde{H}_b^\alpha$ are essentiallly the same from the formal point of view, because there is a simple relation between those symbols. That this is the case is easily seen if one notes that the associated bilinear forms, viz.

$$\langle g,\tilde{H}_b f\rangle_{H^2(\underline{T})} \quad \text{and} \quad \langle g,\tilde{H}_b^\alpha f\rangle_{A^{\alpha 2}(D)}$$

respectively, both are Hankel forms in the sense that they only depend on the combination $f\cdot g$. However, if we instead use the complementary projection $P^{\alpha\perp} = id - P^\alpha$ mapping $L^2(D,\mu_\alpha)$ onto the orthogonal complement $A^{\alpha 2}(D)^\perp$, one gets an entirely different theory. We thus set

$$H_b^\alpha f = P^{\alpha\perp}(\bar{b}f) ,$$

where b is an analytic function (cf. however *infra*); H_b^α will be considered as an operator from $A^{\alpha 2}(D)$ into $A^{\alpha 2}(D)^\perp$. An important fact about this definition is that it is conformally invariant. The main result in [5] can now be formulated as follows.

THEOREM. (a) H_b^α *is bounded iff* b *is in the Bloch space* $B_\infty^{0\infty}A(\underline{T})$ *and compact iff* b *is in the "little" Bloch space* $b_\infty^{0\infty}A(\underline{T})$ (= the closure of analytic polynomials in the Bloch metric).

(b) H_b^α *is in* S_p *where* $1 < p < \infty$ *iff* $b \in B_p^{1,1/p}A(\underline{T})$.

COMMENT. Again we shall not enter into the details of the proof but content ourselves with some comment. First of all part (a) is an easy extension of Axler's result [6] (the case $\alpha = 0$). As for part (b), one might also think that it is a simple variant of Peller's theorem (§1). But this is not the case, because the result breaks completely down in the end point case $p = 1$. The only thing one can prove is that $b \in B_1^{1,1}A(\underline{T}) \Rightarrow H_b^\alpha \in S_\Omega$, where S_Ω is the Macaev ideal (the smallest normed ideal containing the Marcinkiewicz (quasi-normed) ideal $S_{1\infty}$; $T \in S_{1\infty}$ iff $s_n(T) = O(1/n)$ so that $T \in S_\Omega$ iff $\sum_{k=1}^n s_k(T) = O(\log n)$; see e.g. [12], [49]) and also that if H_b^α is in any smaller normed ideal S_E (i.e. the sequence space E is *strictly* contained in the "Macaev sequence space" Ω) then $b \equiv$ constant. So one must proceed differently. Passing to the upper halfplane one readily sees that the operators H_b^α formally may be viewed as special paracommutators (§2) but with a *vector valued* Fourier kernel (functions of the variable $z = x + iy$ are then viewed as functions of x whose values are functions of y). As there is (not yet) a theory of "vector valued" paracommutators, the results of [16] (see §2) do not apply directly. (Obvious challenge: Develop such a theory! In this connection it is perhaps pertinent to recall Peller's paper [36] on vector valued (ordinary) Hankel operators.) It is however easy to give an *ad hoc* construction which somehow mimicks the proofs in the paracommutator case; in one way or other one has to imbed the given object into a one parameter family so that one can do the interpolation, which breaks down if one tries to proceed in the naive way. Roughly speaking, the idea is the following. One knows that if $(e_n)_{n=1}^\infty$ is any orthonormal basis and $1 \leq p \leq 2$, then $(||Te_n||)_{n=1}^\infty \in \ell_p$ entails that $T \in S_p$ (give your own proof or else consult [12] or [47]). We use the analogous result for the natural "continuous" bases $(K_z)_{z \in D}$. One is then lead to prove the following statement: If $f \in B_p^{1/p,p}A(\underline{T})$, where $p > 1$, then the following integral is finite:

$$\iint_D \{ \iint_D \{ \frac{(K(z,\bar{z})^{1/2} K(w,\bar{w})^{1/2} |f(z) - f(w)|}{K(z,\bar{w})} \}^2 d\Pi(z) \}^{p/2} d\Pi(w)$$

where $K(z,\bar{w}) = (1 - z\bar{w})^{-(\alpha+2)}$ is the reproducing kernel in $A^{\alpha,2}(D)$ (see §2) and $d\Pi(z) = d\sigma(z)/(1 - |z|^2)^2$ is the Poincaré measure. (Incidentally, this leads to a new (?) characterization of Besov spaces too.) The hard thing is $p \leq 2$ and this is this case which really is needed. Actually, in the strict sense the statement is true only for $\alpha \geq 0$ and in the case $0 < \alpha < 1$ a modification of the argument is needed (another related perhaps less natural "basis" has to be used).

Non-analytic symbols. In connection with the operators H_b^α there is no *a-priori* need of requiring the symbol b to be analytic. Below we indicate briefly what happens if we drop the latter assumption.

It is convenient (cf. §2) to formulate the results in terms of the commutator

$$C_b^\alpha = [M_b, P^\alpha] ,$$

where M_b is multiplication with b , $M_b f = bf$, which in the decomposition $L^2(D,\mu_\alpha) = A^{\alpha,2}(D) \oplus A^{\alpha,2}(D)^\perp$ is given by the block matrix

$$\begin{pmatrix} 0 & -H_b^{\alpha *} \\ H_{\bar{b}}^{\alpha} & 0 \end{pmatrix}.$$

Here it is interesting to look already at the S_2-theory. (Indeed, some of what we say can also be developed in the more general setting of §4.) More precisely, let F^{α} be the class of symbols b such that C_b^{α} is in S_2, i.e.

$$\iiiint_{D\times D} |K(z,\bar{w})|^2 |b(z) - b(w)|^2 \, d\mu_{\alpha}(z) \, d\mu_{\alpha}(w) < \infty ,$$

with $K(z,\bar{w})$ as before and $d\mu_{\alpha}(z) = (1 - |z|^2)^{\alpha} \, d\sigma(z)$. It is plain that the Möbius group acts via isometries on F^{α} but this action is *not* reducible. One irreducible factor F_1 consists of analytic symbols and is identified with the Dirichlet space $\mathrm{Dir} \equiv B_2^{1,2}A(\underline{T})$. Another one F_2 consists of anti-analytic symbols (= the conjugates of the elements in F_1) and is, consequently, identified with $\overline{\mathrm{Dir}}$ A third invariant factor F_3 (but not an irreducible one; it is a *continuous* sum of irreducible representations) is obtained by taking the closure of $C_0^{\infty}(D)$ in F^{α}. One can show that $F^{\alpha} = F_1 \oplus F_2 \oplus F_3$ (orthogonal sum). Thus there are three types of symbols:

1) analytic symbols,
2) anti-analytic symbols,
3) symbols "vanishing on the boundary".

Actually, one can show that $F_3 = L^2(D, d\Pi)$, where again $d\Pi$ is the Poincaré measure, but only upto equivalence of norm. This is done by invoking some more group theory. Let $\tilde{\Delta}$ be the Möbius invariant Laplacian

$$\tilde{\Delta} = (1 - |z|^2)^2 \Delta ,$$

where Δ is the ordinary Euclidean Laplacean. Then by general principles it is clear that

$$||f||_{F^{\alpha}} = ||h(\tilde{\Delta})f||_{L^2(D,d\Pi)} \qquad (f \in C_0^{\infty}(D))$$

for some positive Borel function h. Using the *spherical (Fourier) transform* (see e.g. [13]) one can write down h explicitly (in terms of, once more, Euler's gamma function Γ) and this shows that h indeed is bounded and bounded from below.

Let us now turn to L^p-theory. The natural L^p space to be considered (cf. §4) is Λ_{α}^{p} (*ad hoc*-notation) defined by the condition

$$\iint_{D\times D} |f(z)|^2 K(z,\bar{z})^{1-p/2} \, d\mu_{\alpha}(z) < \infty .$$

One then readily shows that C_b^α is a bounded map from Λ_α^1 onto itself (or from Λ_α^∞ into itself) iff the symbol b satisfies the condition

$$\sup_{w \in \Delta} \iint_D |b(z) - b(w)| \frac{K(z,\bar{w})}{K(z,\bar{z})^{1/2} K(w,\bar{w})^{1/2}} d\mu_\alpha(z) < \infty .$$

By (Schur) interpolation then in the same hypothesis C_b^α is bounded on Λ_α^p for all $1 < p < \infty$.

Finally, applying Russo's theorem [46] (the "Hausdorff-Young" theorem for integral operators; cf. [27]) one can get an S_p-result in one direction at least for $2 < p < \infty$.

6. Manifolds

In this Section we take up a theme treated already in [27], App. 1, and then again briefly mentioned in [39], Sec. 11. In all the previous discussion we have exclusively been dealing with situations with lots of symmetries (invariance for a large class of automorphisms). These again may be viewed as "flat" (or "isotropic") cases on which more general "curved" (or "non-isotropic") object may be modelled (in a sense familiar from Differential Geometry).

For instance, it is conceivable that much of theory of Hankel forms over a weighted Bergman space, even in the case of several variables (we are thinking of the theory in [1], in the first place), can be extended to the case of general strictly pseudo-convex domains, in as much as the unit ball in $\underline{C}^d$ may be viewed as a "flat" model for a general strictly convex domain (the sphere is of course not flat in the naive sense). This is connected with to what extent one can define Besov spaces for a strongly pseudo-convex domain. One has to take care of the complex geometry of the boundary. More precisely, in dealing with "boundary values" of analytic functions one has to impose smoothness (differentiability) in complex tangential directions only (cf. [29], [18]).

Let us now instead consider a real variable extension of the theory of Hankel and Toeplitz operators (essentially we repeat only in some greater detail what was said already in [39]). One can associate a theory of Hankel and Toeplitz operators with any injective elliptic P.D.O. For simplicity we consider only the following model case: The Laplace-Beltrami operator Δ on a Riemannian manifold Ω with boundary $S = \partial\Omega$, $d = \dim S$. Then there is a natural Hilbert space associated, namely $L^2 = L^2(S)$ consisting of square integrable vector fields "along" S (section of the tangent bundle $T\Omega$ of Ω restricted to S). Let $H^2 = H^2(S)$ be the subspace of L^2 obtained by taking restrictions of $f = \text{grad } u$ to S for solutions u of the equation $\Delta u = 0$. (Then f is a harmonic field; if Ω is not simply connected one should perhaps work with the harmonic forms themselves.) Informally, we may think of H^2 as the space of Cauchy data and it is the Cauchy data we should consider also in more general situations. (Notice that the above situation with several complex variables then also becomes formally a special case, in as much as the Cauchy-Riemann equations written in real form may be viewed as an injective (overdetermined) elliptic system.) Let P be the orthogonal projection of L^2 onto H^2 . By a Hankel-Toeplitz operator we mean an operator of the form $Tf = PVf$, where V denotes multiplication by a "matrix-valued" function v , i.e. strictly speaking, a section of the bundle of linear endomorphism of the bundle $T\Omega \mid S$; in terms of local coordinates: if f_i are the components of f then Vf has components of the form $\sum v_{ik} f_k$. (More generally, we could let V be a Ψ.D.O. of degree 0 .)

EXAMPLE. $\Omega = D$ = unit disk, $S = \underline{T}$ = unit circle. In this case we identify f with the analytic function $\partial u/\partial x - i\partial u/\partial y$. Now the formula for T takes the form $Tf = P(af + b\bar{f})$. (Notice that in our treatment L^2 and H^2 are *a priori* taken as $\underline{R}$-vector spaces.) This formula comprises thus in one stroke both what is classically considered as a Toeplitz operator ($b = 0$) and a Hankel operator ($a = 0$). In [27] we proved (with some effort) a theorem to the effect that (in the general case) T is bounded in L^2 iff both its Toeplitz and its Hankel part are bounded, that is, by Nehari's theorem (§1) and its unnamed Toeplitz counterpart, iff $a \in L^\infty$ and $b \in$ PBMO . (Later Svante Janson pointed out to me that this trivially follows by just separating real and imaginary parts in the obvious way.)

Let us return to our general (model) case. In general a Hankel--Toeplitz operator is a Ψ.D.O. of degree 0 in the technical sense. However, it may "accidentally" happen that it has lower degree. When this is the case we say that we have a pure Hankel operator. This is very nice, because than T may be replaced by a commutator. Indeed, let $p(x,\xi)$ be the (principal) symbol of the projection P (this is a "matrix-valued" function on the cotangent bundle T^*S of S). The symbol of T is then (in general) $v(x)p(x,\xi)$. But if T is pure Hankel we must have $v(x)p(x,\xi) \equiv 0$; in other words, there are relations between the components v_{ik} of v. Write $T = PV = VP + [P,V]$. Here the first term may be written as VQ where Q is a Ψ.D.O. of order -1. It can easily (?) be dealt with directly. What remains is thus the commutator $[P,V]$ and then the Janson-Wolff theory (commutators of Calderón-Zygmund operator; see §2) is at our disposal, as the latter is essentially of local character. Assuming that S is compact one can also derive global results, at least in one direction. Especially, one finds that $T \in S_p$, where $d < p < \infty$, iff v belongs to a suitable Besov space $B_p^{d/p,p}(S)$ (of sections of the bundle in question). As there are relations between the components v_{ik} we get however no conclusions in the other direction. For instance, it is not clear if there ever exist (pure) Hankel operators in our sense which have finite rank (analogue of Kronecker's theorem; see §1), except of course in the classical case (see the above example).

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