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THE INFLUENCE OF TOPOLOGY TO NON-LINEAR ANALYSIS

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1. Topological methods have been proved to be very important for studying fundamental non-linear problems in differential equations and in the calculus of variations. These methods have,now,been developed into mathematical theories (e.g.the fixed point theory,the degree theory of mappings,the Morse theory,and the Lusternik-Schnirelman theory) that play today a central role in pure and applied mathematics. *The purpose of this survey paper is to explain how some of the topological ideas become essential for the understanding of some research problems of non-linear analysis.*

2. Brouwer has proved the following theorem known as the Invariance of Domain Theorem (BIDTh) [cf.Brouwer [4],Brown [8],Nagata [12], Schwartz [16],Vick [17]].

<u>*THEOREM*</u>. *Let* U *and* V *be homeomorphic subsets of* S^n.*If* U *is open, then so is* V.

This theorem gives a remarkable topological property of S^n and more generally of n-dimensional topological manifolds. In fact one has the following property. If U and V are subsets of S^n and if $h:U\to V$ is a homeomorphism, then h maps interior points of U onto interior points of V and boundary points of U onto boundary points of V.

The theorem holds also for subsets of R^n and more generally in the following form:

<u>*THEOREM*</u>. *Let* M *and* N *be any two n-dimensional topological manifolds, and* U *and* V *subsets of* M *and* N *respectively such that* U *and* V *are homeomorphic. If* U *is an open subset of* M*, then* V *is also an open subset of* N.

BIDTh has played a significant role in the research work of Stoilow concerning Topological Analysis and to our opinion has led him to the concept of openness for a mapping in general. The Brouwer's papers of the years 1911-1913 [cf. [4],[5],[6]] in the development of topology presented the first notable synthesis of combinatorial-algebraic and set theoretic ideas and methods of topology.Out of this approach was discovered the so-called Vietoris homology theory in compact metric spaces,which was based in Brouwer's paper [5] and the first mathematical foundation of dimension theory contained in Brouwer's paper [4]. Uryson has posed the general problem of characterizing any n-dimensional sets in $R^m (n \leq m)$ by properties of their situation in R^m, Uryson had solved the problem only for closed subsets of the plane.

A complete solution of the problem for closed subsets of R^n is given by Aleksandrov's obstruction theorem. The study of the dimensional properties of subsets of Euclidean spaces led Aleksandrov to introduce a new dimensional invariant the metric dimension [cf. Aleksandrov and Fedorchuk [1]].

An interesting use of BIDTh is for the proof of Brouwer's Invariance of Dimension Theorem (BIDiTh): *if* $f:R^m \to R^n$ *is a homeomorphism, then* m=n. All the known proofs of the BIDiTh, as for the BIDTh use techniques of homology theory or of subdivisions of simplicial complexes. A stronger form of BIDiTh can be proved; if $f:R^m \to R^n$ is a continuous one-to-one and onto mapping, then m=n and f is a homeomorphism.

The Theorem on the Invariance of Domain is not true in general. For example [17, p.38], let $U=(\frac{1}{2},1]$ and $V=(0,\frac{1}{2}]$ be subsets of $[0,1]$. Let $h:U \to V$ be defined by $h(x)=x-\frac{1}{2}$. Then h is a homeomorphism, U is an open subset of $[0,1]$, but V is not. *The converse question* is still an interesting *open problem* and it can simply be stated as follows: *How does one characterize those spaces having the property that any subset homeomorphic with an open set is open?*

In an essentially intuitive direction BIDTh was considered in connection with Functional Analysis, which led, in particular, to the study and research of compact mappings [cf. Schwartz [16]]. The following gives an illustration of the application of BIDTh to Nonlinear Functional Analysis [cf. Rassias [13]].

__THEOREM__. Let $f:B \to B$ *be a continuous non-linear mapping, such that* $||f(x)-f(y)|| \geq ||x-y||$ *for all* $x,y \in B$ *with* $\dim B<\infty$, *where* B *is a Banach space,* $\dim B < \infty$. *Then* T *maps* B *onto* B.

P r o o f. *Claim that the range* $R[f]$ *of* f *is closed.*
Verification: Let $\{f(x_n)\}$ be a Cauchy sequence in B. Then by the expanding property of f we obtain $|| f(x_n)-f(x_m) || \geq || x_n-x_m||$ and therefore $\{x_n\}$ is a Cauchy sequence in B.

Thus $\{x_n\}$ converges to some element $x \in B$, i.e., $x_n \to x$ and by the continuity of f, it follows that $f(x)=\lim_{n\to\infty} f(x_n)$ which implies that $R[f]$ is closed as $f(x) \in B$.

Claim that the $R[f]$ *is open.*
Verification: Let $f:B \to R[f]$. This is a homeomorphism and by the BIDTh, which is true if $\dim B<\infty$, the $R[f]$ is open, since B is open in B, and $R[f]$ is contained in B.

Because of the fact $R[f]$ is both open and closed it implies that

$R[f]=B$ and so f maps B onto B. Q.E.D.

REMARK. BIDTh *is not true for infinite dimensional Hilbert spaces* H. *Counterexample.* Let H be an infinite dimensional Hilbert space, and $S^{\infty}=\{x \in H: \|x\|=1\}$. It is a theorem that there exists a homeomorphism $f:H \to S^{\infty}$. However H is an open set in H and S^{∞} is closed in H. Thus BIDTh does not work for infinite dimensional Hilbert spaces.

P r o b l e m. *Find an infinite dimensional generalization of* BIDTh *under some relatively weak assumptions on the mapping. What is then the converse of* BIDTh?

C o n j e c t u r e. *Let* T *be a continuous mapping* $:H\to H$, H *is a Hilbert space, such that* $\|Tx-Ty\| \geq \|x-y\|$ *for all* $x,y \in H$, *and* $T(0)=0$. *Suppose* T *maps a neighborhood of the origin onto a neighborhood of the origin. Then* T *does not map* H *onto* H.

Remark. An answer to this conjecture will answer a question of L. Nirenberg [13].

C o n j e c t u r e. *There is no harmonic homeomorphism of the open unit ball* B *in* R^3 *onto* R^3, *i.e. there are no harmonic functions* f_1, f_2, f_3 *defined in* $B=\{\zeta=(\zeta_1,\zeta_2,\zeta_3):|\zeta|<1\}$, *such that the mapping* $\zeta \to (f_1,f_2,f_3)$ *is a homeomorphism of* B *onto all of* R^3.

3. We begin now with the definition of an isometry: Let X,Y be two metric spaces, d_1,d_2 the distances on X and Y. A bijection mapping $f:X\to Y$, of X onto Y, is defined to be an *isometry* if $d_2(f(x),f(y))= d_1(x,y)$ for all elements x,y of X. If $f:X\to Y$ is an isometry then the inverse mapping $f^{-1}:Y\to X$ is an isometry of Y onto X. Two metric spaces X and Y are defined to be *isometric* if there exists an isometry of X onto Y. It thus follows that an isometry is an isomorphism for the metric space structures. Some of the properties of an isometry are mentioned in [9] and in [10]. We now state in which sense an incomplete space can be fattened out to be complete: If (X,d_1) is an incomplete metric space, then there exist a complete metric space $\tilde{X}$ so that X is isometric to a dense subset of $\tilde{X}$. Mazur and Ulam [11] have proved that every isometry of a normed real vector space onto a normed real vector space is a linear mapping up to translation. Consider then the following condition (distance one preserving property), for $f:X\to Y$.

(DOPP) *Given* $x,y \in X$ *let* $d_1(x,y)=1$. *Then* $d_2(f(x),f(y))=1$.

P r o b l e m. *Let* $f:X\to Y$ *be a mapping (not necessarily continuous) satisfying condition (DOPP).*
Is $f:X\to f(X)\subseteq Y$ *an isometry?*

The problem still remains open even for the case where $X=R^n$ and $Y=R^m$ with $2\leq n<m$ (see for example [18, p.277]). Beadle [2] has covered

a number of cases for mappings $f:R^n \to R^m$ that preserve some distance. For these mappings it is clear that $n \leq m$ because R^m has equilateral n-simplices if and only if $n \leq m$. Beckman and Quarles [3] proved that f is an isometry if $1<n=m<\infty$. If $n<m$, f might not be an isometry if m is too large. It is not yet known if there is a distance preserving mapping $f:R^2 \to R^3$ which is not an isometry.

4. We are going now to state and prove a generalization of the Schauder's fixed point theorem, [14], [15].

THEOREM. *Assume* X *is a metric linear topological space where the metric* d *defined on* X *has been chosen so that balls are convex. Let* $A \subset X$ *be compact and convex and let* $f:A \to X$ *be a continuous mapping. Then there exists at least one point* $p \in A$ *such that* $d(f(p),p)=d(f(p),A)$ *where for a point* $x \in X$, $d(x,A)=\inf\{d(x,y):y \in A\}$.

P r o o f. If this is not the case, then there are no fixed points for f and therefore for every point $p \in A$, $d(f(p),p)>0$. Define a mapping $T:A \to X$ by

$$T(p)=A \cap B(f(p),d(f(p),p)) \text{ for each } p \in A,$$

where

$$B(f(p),d(f(p),p))$$

is the *open ball* in X with center f(p) and radius d(f(p),p). If there exists a point $p \in A$ such that $T(p)=\emptyset$ then p is the required point satisfying $d(f(p),p)=d(f(p),A)$. Assuming the contrary, then for each $p \in A$, T(p) is a non-empty and convex set. Assume x is a point in T(A) and p in $T^{-1}(x)$ where $T^{-1}(x)=\{p \in A: x \in T(p)\}$. Let $\varepsilon>0$ be such that $d(f(p),x)=d(f(p),p)-\varepsilon$ and assume $\delta>0$ is such that f maps the open ball $B(p,\delta)$ into $B(f(p),\frac{\varepsilon}{4})$.

Consider δ_1 to be the minimum between δ and $\frac{\varepsilon}{4}$ and assume p' is a point inside $B(p,\delta_1)$. Then

$$\begin{aligned} d(f(p'),x) &\leq d(f(p),x)+\frac{\varepsilon}{4}=d(f(p),p)-\varepsilon+\frac{\varepsilon}{4}=d(f(p),p)-\frac{3\varepsilon}{4} \\ &\leq d(f(p),f(p'))+d(f(p'),p')+d(p',p)-\frac{3\varepsilon}{4} \\ &\leq d(f(p'),p')+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}-\frac{3\varepsilon}{4}=d(f(p'),p')-\frac{\varepsilon}{4} \\ &< d(f(p'),p') \end{aligned}$$

because $d(f(p),f(p'))<\frac{\varepsilon}{4}$ and $d(p',p)<\frac{\varepsilon}{4}$. Therefore $d(f(p'),x)<d(f(p'),p')$ and so p′ is a point in $T^{-1}(x)$, as defined above, which means $B(p,\delta_1)$ is contained in $T^{-1}(x)$ and so $T^{-1}(x)$ is an open set. Thus there exists some point $p \in A$ so that $p \in T(p)$ according to a theorem by F.Browder [7; Theorem 1, p.285], but by the definition of T(p) we obtain a contradiction which implies that there exists a point $p \in A$ such that $d(f(p),p)=d(f(p),A)$. Q.E.D.

REMARK. If $f:A \to A \subset X$ then Theorem 1 implies that there exists a point $p \in A$ such that $f(p)=p$, because $d(f(p),A)=0$.

It can be proved that *there exists a metric linear topological space* X *where the metric* d *defined on* X *has been chosen so that balls are convex, but* X *is not a normed space*. In fact, let X be any topological vector space whose topology is given by a countable family of norms $\|\cdot\|_n$ but not by any single norm. Define $\rho(x,y)=\sup_n \frac{1}{2^n} \arctan(\|x-y\|_n)$. Then ρ is a metric on X which defines the topology. Given $\varepsilon>0$ and $y \in X$, we show that $\{x \in X: \rho(x,y)<\varepsilon\}$ is convex. It is true that

$$\{x \in X: \rho(x,y)<\varepsilon\}=\bigcap_n \{x \in X: \arctan(\|x-y\|_n)<2^n\varepsilon\}.$$

If $2^n\varepsilon<\frac{\pi}{2}$ then

$$\{x \in X: \arctan(\|x-y\|_n)<2^n\varepsilon\}=\{x \in X: \|x-y\|_n<\tan(2^n\varepsilon)\}$$

which is convex; otherwise

$$\{x \in X: \arctan(\|x-y\|_n)<2^n\varepsilon\}$$

is the whole space which is also convex. Combining the two cases we obtain that

$$\{x \in X: \rho(x,y)<\varepsilon\}$$

is an intersection of convex sets and is therefore convex. It is also true here that the metric ρ induces the same topology on X as the given one.

CONCLUSION. Schauder's fixed point theorem shows that any compact convex non-empty subset Y *of a normed space has the fixed point property.* Therefore our Theorem generalizes the Schauder's fixed point theorem to metric linear topological spaces X where the metric d defined on X has been chosen so that balls are convex.

P r o b l e m. *Let* $X \subset R^2$ *be a non-empty, compact and starshaped subset of* R^2 *and let* $f:X \to X$ *be a continuous mapping such that* $f(\partial X) \subset X$. *Does there exist a point* $q \in X$ *such that* $f(q)=q$?

P r o b l e m. *Let* X *be an open convex subset of a topological vector space over the real field and* $f:X \to X$ *a mapping such that the closure of* $f(X)$ *is a compact subset of* X. *Does there exist a point* $q \in X$ *such that* $f(q)=q$?

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