Pierre-Louis Lions

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FULLY NONLINEAR ELLIPTIC EQUATIONS AND APPLICATIONS

P. L. Lions
Paris, France

Introduction

We want to present here some results on *fully nonlinear second order elliptic equations* i. e. equations of the following form:

$$F(D^2u,\ Du,\ u,\ x) = 0 \quad \text{in } \Omega \tag{1}$$

where Ω is an open set in $\mathbb{R}^N$, u is a real-valued function and F is a given nonlinearity continuous on $SL_N \times \mathbb{R}^N \times \mathbb{R} \times \Omega$ (where SL_N is the space of $N \times N$ symmetric matrices). The ellipticity of the equation is expressed by the condition

$$F(A,\ p,\ t,\ x) \geqq F(B,\ p,\ t,\ x) \quad \text{if } A \leqq B,\ A,\ B \in SL_N,\quad p \in \mathbb{R}^N,\ t \in \mathbb{R},\ x \in \Omega. \tag{2}$$

Of course when F is C^1, (2) is equivalent to

$$\left(\frac{\partial F}{\partial \xi_{ij}}(\xi,\ p,\ t,\ x)\right) \leqq 0 \quad \forall (\xi,\ p,\ t,\ x) \in SL_N \times \mathbb{R}^N \times \mathbb{R} \times \Omega. \tag{2'}$$

Obviously these equations may degenerate; in particular this class of equations contains the classical *first-order Hamilton-Jacobi equations*:

$$F(Du,\ u,\ x) = 0 \quad \text{in } \Omega. \tag{3}$$

In what follows we will first (Section I) present the notion of viscosity solutions of (1) or (3) introduced by M. G. Crandall and P. L. Lions [9], [10] - see also P. L. Lions [31], [32]; M. G. Crandall, L. C. Evans and P. L. Lions [8]; P. L. Lions [33]. Next (Section II) we indicate a few existence results for first order Hamilton-Jacobi equations (HJ equations for short). Finally in Section III we present existence and regularity results for Hamilton-Jacobi-Bellman equations and we will apply these results to the solutions of Monge-Ampère equations.

We will present many results on some special subclass of (1) namely the class of *Hamilton-Jacobi-Bellman equations* (HJB equations) i. e. equations like (1) where F is convex in $(D^2u,\ Du,\ u)$ or equivalently equations of the form

$$\sup_{\alpha \in I} \left[A^\alpha u - f^\alpha\right] = 0 \quad \text{in } \Omega \tag{4}$$

where I is a given set, $(A^\alpha)_{\alpha\in I}$ (resp. $(f^\alpha)_{\alpha\in I}$) is a family of second-order linear elliptic operator (possibly degenerate) (resp. a family of given functions). As we will see below these equations contain the famous *Monge-Ampère equations:*

(5) $\det(D^2u) = H(x, u, Du)$ in Ω, u convex on $\bar{\Omega}$.

Of course when we will discuss existence results for all these equations we will impose on u boundary conditions:we will restrict here our attention to Dirichlet type conditions.

To conclude this introduction, let us mention that HJ equations arise in many situations:calculus of variations, optimal control, differential games, geometrical optics ... In the same way HJ equations occur in optimal control, HJB equations occur in optimal stochastic control (see for example P. L. Lions [31], [32], N. V. Krylov [26]). Finally it is well-known that Monge-Ampère equations arise in differential geometry (see S. Y. Cheng and S. T. Yau [7], I. Bakelman [3], A. V. Pogorelov [47]).

I. On viscosity solutions of (1)

In this section we want to introduce a notion (and to present a few properties) of weak solutions of (1). We will begin by the case (totally degenerate) of (3).

I.1. Viscosity solutions of HJ equations

Before introducing the notion of solutions of (3) that we want to discuss, let us explain a few difficulties associated with (3):

(3) $$F(Du, u, x) = 0 \quad \text{in } \Omega.$$

Since (3) is a global nonlinear first-order problem, it is well-known that in general there does not exist classical solution u ($\in C^1$) of (3) - see Example 1 below for example. Then locally Lipschitz solutions u of (3) - that is functions $u \in W^{1,\infty}_{loc}(\Omega)$ satisfying (3) a. e. in Ω - were considered by many authors (A.Douglis [12];S. N. Kružkov [24],[25]; W. H. Fleming [18], [19]; A. Friedman [20]).It has been proved by these authors that under general assumptions there exist locally Lipschitz solutions of (3). Unfortunately as it will be seen in the following examples, with this notion of solutions one loses uniqueness, stability properties:

EXAMPLE 1. Consider the one-dimensional problem:

(6) $$|u'| - 1 = 0 \quad \text{in } (0, 1), \; u(0) = u(1) = 0.$$

Obviously there is no classical solution of (6). Now, if we look for Lipschitz solutions of (6), we see that for $n \geqq 1$ $u_n(x)$ defined by:

$$\begin{cases} u_n(x) = x - \frac{2j}{2^n} \text{ if } \frac{2j}{2^n} \leqq x \leqq \frac{2j+1}{2^n} & (0 \leqq j \leqq 2^{n-1} - 1), \\ u_n(x) = \frac{j}{2^{n-1}} - x \text{ if } \frac{2j-1}{2^n} \leqq x \leqq \frac{j}{2^{n-1}} & (1 \leqq j \leqq 2^{n-1}) \end{cases}$$

are Lipschitz, piecewise analytic functions on $(0, 1)$ and satisfy (6) except at a finite number of points. Thus, we see that (6) has infinitely many solutions and furthermore, remarking that $0 \leq u_n \leq \leq \frac{1}{2^n}$ in $(0,1)$, u_n converges uniformly on $(0, 1)$ to 0 which is *not* solution of (6).

EXAMPLE 2. Consider the following Cauchy problem:

$$(7) \qquad \frac{\partial u}{\partial t} + |Du|^{\alpha} = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \qquad u|_{t=0} = 0 \quad \text{on } \mathbb{R}^N$$

(where Du denotes the gradient in x and $\alpha > 0$).

Obviously $u \equiv 0$ is a solution of (7); but $u(x, t) = = \mathrm{Min}\,(|x| - t, 0)$ is Lipschitz, piecewise C^{∞} and satisfies (7) a.e.

This explains the need of finding a selection rule among (locally Lipschitz - for example) solutions of (3) such that we keep existence and we obtain uniqueness, stability results. Natural restrictions are also that any notion of solutions of (3) should contain solutions of (3) obtained via the vanishing viscosity procedure or optimal control - differential games problem. As we will see below, all these questions and requirements are answered by the following notion of viscosity solutions of HJ equations - in addition it can be proved that any notion of solutions satisfying these properties is contained in our notion. All the results presented below are taken from M. G. Crandall and P. L. Lions [10] (see also [8], [33]).

We first need to recall a few facts about sub- and superdifferentials of continuous functions: let $\phi \in C(\Omega)$; we define the subdifferential of ϕ at x $(\in \Omega)$, that we denote by $D^-\phi(x)$, and the superdifferential of ϕ at x, denoted by $D^+\phi(x)$, as follows:

$$D^+\phi(x) = \{p \in \mathbb{R}^N : \limsup_{\substack{y \to x \\ y \in \Omega}} [\phi(y) - \phi(x) - (p, y - x)]\,|y - x|^{-1} \leqq 0\},$$

$$D^-\phi(x) = \{p \in \mathbb{R}^N : \liminf_{\substack{y \to x \\ y \in \Omega}} [\phi(y) - \phi(x) - (p, y - x)]\,|y - x|^{-1} \geqq 0\}.$$

REMARKS.

1. $D^+\phi(x)$ (resp. $D^-\phi(x)$) is a closed, convex set in $\mathbb{R}^N$, possibly empty.
2. If ϕ is differentiable at $x \in \Omega$, then $D^+\phi(x) = D^-\phi(x) = \{D\phi(x)\}$.
3. If $D^+\phi(x)$ and $D^-\phi(x) \neq \emptyset$, then ϕ is differentiable at x.
4. $D^+\phi(x)$ (resp. $D^-\phi(x)$) is nonempty on a dense set in Ω.

We may now give the definition of viscosity solutions of (3):

DEFINITION. $u \in C(\Omega)$ *is said to be a viscosity solution of* (3) *if the following inequalities hold:*

(8) $\quad F(p, u(x), x) \leq 0 \qquad \forall\, p \in D^+u(x), \ \forall\, x \in \Omega$;

(9) $\quad F(p, u(x), x) \geq 0 \qquad \forall\, p \in D^-u(x), \ \forall\, x \in \Omega$.

REMARKS.

1. If $u \in C^1(\Omega)$ is a classical solution of (3), then obviously u is a viscosity solution of (3).
2. If u is a viscosity solution of (3) and if u is differentiable at $x_0 \in \Omega$, then we have $F(Du(x_0), u(x_0), x_0) = 0$.

   In particular if u is locally Lipschitz in Ω, then (3) holds a. e.

EXAMPLE. If we go back to the case of (6) - Example 1 - we check easily that u_1 is a viscosity solution of (6): since u_1 is C^1 except at $\frac{1}{2}$ and satisfies (6), we just have to consider the point $x = \frac{1}{2}$. At $x = \frac{1}{2}$ we easily obtain $D^-u_1(\frac{1}{2}) = \emptyset$, $D^+u_1(\frac{1}{2}) = [-1, +1]$ and (8) holds. On the other hand u_n for $n \geq 2$ is not a viscosity solution of (6): indeed taking $x = 1/2^{n-1}$, we have easily $D^-u_n(1/2^{n-1}) = [-1, +1]$, $D^+u_n(1/2^{n-1}) = \emptyset$ and (9) does not hold.

A more workable definition of viscosity solutions is given in the

PROPOSITION 1. *Let* $u \in C(\Omega)$; *then* u *is a viscosity solution of* (3) *if and only if for all* $\phi \in C^1(\Omega)$ *the following conditions hold:*

(8′) *at each local maximum point* x *of* $u - \phi$
$$F(D\phi(x), u(x), x) \leq 0 ,$$

(9′) *at each local minimum point* x *of* $u - \phi$
$$F(D\phi(x), u(x), x) \geq 0 .$$

REMARKS.

1. It is clear that there is a parallel between viscosity solutions and the use of distributions theory for (nonlinear) equations in

divergence form: the integration by parts is now replaced by some "differentiation by parts" performed inside the nonlinearity.

2. This form of the definition of viscosity solutions of (3) is somewhat related to the theory of accretive operators in L^∞ (see L. C. Evans [13]).
3. Let us also mention that the notion of viscosity solutions has some similarity with the so called entropy conditions for scalar conservation laws. This similarity is due to the relations between HJ equations and hyperbolic systems (see P. L. Lions [33] for more details).
4. Finally we want to point out that we could replace in the above proposition $\phi \in C^1(\Omega)$ by $\phi \in C^2(\Omega)$ or $\phi \in C^\infty(\Omega)$, and local maximum point by local strict, global, global strict maximum point ...

P r o o f of Proposition 1. If u is a viscosity solution and if $u - \phi$ has a local maximum at $x \in \Omega$, we have for $y \in \Omega$, y near x,

$$u(y) \leq u(x) + \phi(y) - \phi(x) = u(x) + (\nabla\phi(x), y - x) + o(|y-x|).$$

Thus $\nabla\phi(x) \in D^+u(x)$ and (8′)holds. Conversely if (8′) and (9′) hold and if $\xi \in D^+u(x)$ we have

$$u(y) \leq u(x) + (\xi, y - x) + o(|y - x|)$$

and it is an easy exercise in real analysis to find $\phi \in C^1(\Omega)$ such that

$$u(y) \leq \phi(y) \text{ in } \Omega, \quad u(x) = \phi(x), \quad D\phi(x) = \xi \quad .$$

An application of the above proposition is the following:

<u>*COROLLARY 1*</u> (*Stability*). *Let* $u_n \in C(\Omega)$ *be viscosity solution of the equation* $F_n(Du_n, u_n, x) = 0$ *in* Ω *, where* $F_n(p, t, x) \to F(p, t, x)$ *uniformly on compact sets of* $\mathbb{R}^N \times \mathbb{R} \times \Omega$. *We assume that* u_n *converges uniformly on compact subsets of* Ω *to some function* u. *Then* u *is a viscosity solution of* (3).

P r o o f . It is enough to consider a local strict maximum point x_0 of $u - \phi$ where $\phi \in C^1(\Omega)$. Then, for n large enough, $u_n - \phi$ has a local maximum point x_n and $x_n \to x_0$. By definition we have

$$F_n(D\phi(x_n), u_n(x_n), x_n) \leq 0$$

and we conclude easily sending n to ∞.

A related result is the following proposition showing that any limit function obtained via the vanishing viscosity method is really a viscosity solution: more precisely we take u_ε - the solution of

(10) $\quad - \varepsilon\Delta u_\varepsilon + F_\varepsilon(Du_\varepsilon, u_\varepsilon, x) = 0 \;$ in $\; \Omega, \; u_\varepsilon \in C^2(\Omega), \; \varepsilon > 0$.

COROLLARY 2. *Let* u_ε *be a solution of* (10) *and assume that* F_ε *converges uniformly on compact sets of* $\mathbb{R}^N \times \mathbb{R} \times \Omega$ *to some function* F. *We assume that* u_ε *converges uniformly on compact sets of* Ω *to some function* u . *Then* u *is a viscosity solution of* (3).

P r o o f. It suffices to consider a local strict maximum point x_0 of $u - \phi$ where $\phi \in C^2(\Omega)$. Then, for ε small enough, $u_\varepsilon - \phi$ has a local maximum at some point x_ε and $x_\varepsilon \to x$. Since $Du_\varepsilon(x_\varepsilon) = D\phi(x_\varepsilon)$ and $\Delta u_\varepsilon(x_\varepsilon) \leqq \Delta\phi(x_\varepsilon)$, we deduce from the equation (10) that

$$F_\varepsilon(D\phi(x_\varepsilon),\, u_\varepsilon(x_\varepsilon),\, x_\varepsilon) = \varepsilon\Delta u_\varepsilon(x_\varepsilon) \leqq \varepsilon\Delta\phi(x_\varepsilon)$$

and we conclude easily sending ε to 0.

REMARK. Let us also mention that the value function in deterministic optimal control or in deterministic differential games is always a viscosity solution of the associated Hamilton-Jacobi equation (often called the Bellman or the Isaacs equation in the engineering literature). We refer the interested reader to P. L. Lions [33], [32], P. L. Lions and M. Nisio [45].

I.2. Uniqueness results for viscosity solutions of HJ equations

We will treat only two simple cases and we refer the reader to [10] for more general results (and for complete proofs). The first case is

(11) $$H(Du) + \lambda u = f \quad \text{in} \quad \mathbb{R}^N$$

where $H \in C(\mathbb{R}^N)$, $\lambda > 0$ and f,g will be two functions in the space $BUC(\mathbb{R}^N) = \{u \in C_b(\mathbb{R}^N) : u$ is uniformly continuous on $\mathbb{R}^N\}$.

THEOREM 1. *Let* $u, v \in C_b(\mathbb{R}^N)$ *be viscosity solutions of* (11)*or of* (11) *where* f *is replaced by* g. *Then we have*

(12) $$\|(u - v)^+\|_\infty \leqq \frac{1}{\lambda} \|(f - g)^+\|_\infty .$$

REMARKS.

1. (12) implies the uniqueness of the solution u of (11): indeed if $f = g$ then $u \leqq v$ and by symmetry $u = v$.
2. In addition if $f \leqq g$ and u, v are the unique corresponding solutions then $u \leqq v$.

P r o o f of Theorem 1. We will give the proof only in the special case when $u, v \to 0$ as $|x| \to \infty$. In this case, we first show that if u, v are of class C^1, the proof of (12) is immediate. Indeed, let $\max(u - v) > 0$ and $x \in \arg\max(u - v)$. Obviously, since $u, v \in C^1$, it is $Du(x_0) = Dv(x_0)$ and we deduce from the equations that

$$\|(u - v)^+\|_\infty = (u - v)(x_0) = \frac{1}{\lambda}[f(x_0) - H(Du(x_0))] - \frac{1}{\lambda}[g(x_0) - H(Dv(x_0))] = \frac{1}{\lambda}(f - g)(x_0) \leq \frac{1}{\lambda}\|(f - g)^+\|_\infty .$$

Next, if we assume that $u, v \in C_0(\mathbb{R}^N)$, we may modify the above proof with the help of the following lemma on continuous functions:

<u>LEMMA 1.</u> *Let* $u, v \in C_0(\mathbb{R}^N)$ *be such that* $\max(u - v) > 0$. *Then, for* $\varepsilon > 0$, *there exist* $x_\varepsilon, y_\varepsilon, \xi_\varepsilon \in \mathbb{R}^N$ *such that* $x_\varepsilon \to x_0$, $y_\varepsilon \to x_0$ *where* $x_0 \in \arg\max(u - v)$ *and* $\xi_\varepsilon \in D^+u(x_\varepsilon) \cap D^-v(y_\varepsilon)$.

If we admit temporarily this lemma, we may use the definition of viscosity solutions to obtain

$$H(\xi_\varepsilon) + \lambda u(x_\varepsilon) \leq f(x_\varepsilon), \quad H(\xi_\varepsilon) + \lambda v(y_\varepsilon) \geq g(y_\varepsilon).$$

Thus
$$u(x_\varepsilon) - v(y_\varepsilon) \leq \frac{1}{\lambda}[f(x_\varepsilon) - g(y_\varepsilon)]$$
and letting $\varepsilon \to 0$ we conclude $\|(u - v)^+\|_\infty \leq \frac{1}{\lambda}\|(f - g)^+\|_\infty$.

P r o o f of Lemma 1. Let $M > \max(\|u\|_\infty, \|v\|_\infty)$ and let $\beta \in \mathcal{D}_+(\mathbb{R}^N)$ be such that $0 \leq \beta \leq 1$, $\operatorname{supp}\beta \subset B(0, 1)$, $\beta(0) = 1$. We denote by $\beta_\varepsilon(\xi) = \beta(\frac{\xi}{\varepsilon})$. Finally we introduce the functions

$$w_\varepsilon(x, y) = u(x) - v(y) + 3M\beta_\varepsilon(x - y) .$$

We claim that w_ε has a global maximum on $\mathbb{R}^N \times \mathbb{R}^N$ at some point $(x_\varepsilon, y_\varepsilon)$ such that $x_\varepsilon - y_\varepsilon \in \operatorname{supp}\beta_\varepsilon$ and $x_\varepsilon, y_\varepsilon$ remain in a bounded set in $\mathbb{R}^N$. Indeed, observe first that

$$\limsup_{|x|+|y|\to\infty} w_\varepsilon(x, y) \leq 3M \quad \text{uniformly with respect to } \varepsilon \in (0, 1] .$$

On the other hand if x_1 is such that $u(x_1) - v(x_1) > 0$, we obtain

$$\sup_{\mathbb{R}^N\times\mathbb{R}^N} w_\varepsilon(x, y) \geq w_\varepsilon(x_1, x_1) = u(x_1) - v(x_1) + 3M > 3M .$$

Finally, we prove our claim by remarking that if $x - y \notin \operatorname{supp}\beta_\varepsilon$, then $w_\varepsilon(x, y) \leq \varepsilon M$.

Since $w_\varepsilon(x, y_\varepsilon)$ has a global maximum at $x = x_\varepsilon$, we deduce $-3M\,D\beta_\varepsilon(x_\varepsilon - y_\varepsilon) \in D^+u(x_\varepsilon)$. In a similar way, since $-w_\varepsilon(x_\varepsilon, y)$ has a global minimum at $y = y_\varepsilon$, we deduce $-3M\nabla\beta_\varepsilon(x_\varepsilon - y_\varepsilon) \in D^-v(y_\varepsilon)$. To conclude we remark that if $x_{\varepsilon_n} \to x_0$ (with $\varepsilon_u \to 0$), then $y_{\varepsilon_n} \to x_0$ and

$$u(x_0) - v(x_0) + 3M = \lim_n \left[u(x_{\varepsilon_n}) - v(x_{\varepsilon_n}) + 3M\right] \geq$$

$$\geq \limsup_n w_{\varepsilon_n}(x_{\varepsilon_n} - y_{\varepsilon_n}) \geq \sup_x (u - v)(x) + 3M .$$

Another example of the uniqueness results proved in [10] is the following result concerning the Cauchy problem:

$$\frac{\partial u}{\partial t} + H(Du) + \lambda u = f \quad \text{in} \quad \mathbb{R}^N \times (0, T) \tag{13}$$

where $H \in C(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$, $f, g \in BUC(\mathbb{R}^N \times [0, T])$ and $T > 0$.

THEOREM 2. *Let* $u, v \in BUC(\mathbb{R}^N \times [0, T])$ *be viscosity solutions of* (13), *or of* (13) *with* f *replaced by* g . *Then we have for* $0 \leq t \leq T$

$$||(u - v)^+(t)||_\infty \leq e^{-\lambda t}||(u - v)^+(s)||_\infty + \int_0^t e^{-\lambda s}||(f-g)^+(s)||_\infty \, ds . \tag{14}$$

Finally let us mention that in M. G. Crandall and P. L. Lions [10] there are given various extensions of the uniqueness Theorems 1, 2 to general Hamiltonians $H(p, t, x)$ or $H(p, t, s, x)$. In [10] there is treated the case of HJ equations (3) in a domain Ω as well the uniqueness holds under the same assumptions as in $\mathbb{R}^N$ provided, for example, $u, v \in BUC(\overline{\Omega})$ and $u = v$ on $\delta\Omega$.

REMARKS. Let us mention a few applications of these uniqueness results (and uniqueness proofs): in M. G. Crandall and P. L. Lions [11] they are used in order to prove the convergence of finite difference schemes and error estimates for the numerical approximation of HJ equations. These results are also used in P. L. Lions and M. Nisio [45], P. L. Lions [34] to obtain abstract characterizations of HJ semi-groups. Finally these considerations are used in P. L. Lions, G. Papanicolaou and S. R. S. Varadhan [46] in the study of asymptotic problems in HJ equations - in particular homogeneization of HJ equations or of associated problems in the Calculus of Variations.

I.3. On viscosity solutions of fully nonlinear elliptic equations

We now turn to the general equations (1). We first need to define some kind of second-order derivatives of continuous functions. Let $\phi \in C(\Omega)$; we consider the following sets:

$$D^+_{21}\phi(x) = \Big\{(A, p) \in SL_N \times \mathbb{R}^N : \limsup_{\substack{y \to x \\ y \in \Omega}} \left[\phi(y) - \phi(x) - (p, y - x) - \tfrac{1}{2}(A(y - x), y - x)\right] |y - x|^{-2} \leq 0\Big\} ,$$

$$D^-_{21}\phi(x) = \{(A,p) \in SL_N \times \mathbb{R}^N : \liminf_{\substack{y\to x \\ y\in\Omega}} [\phi(y) - \phi(x) - (p, y - x) - \frac{1}{2}(A(y - x), y - x)]|y - x|^{-2} \geq 0\} .$$

REMARKS.

1. $D^+_{21}\phi(x)$ (resp. $D^-_{21}\phi(x)$) is a closed, convex set in $SL_N \times \mathbb{R}^N$, possibly empty.
2. $D^+_{21}\phi(x)$ (resp. $D^-_{21}\phi(x)$) is non-empty on a dense set in Ω.
3. If $(A,p) \in D^+_{21}\phi(x)$ (or $D^-_{21}\phi(x)$) then $(B,p) \in D^+_{21}\phi(x)$ (or $D^-_{21}\phi(x)$ respectively) for all $B \in SL_N$ such that $B \geq A$ (or $B \leq A$ respectively).
4. If ϕ satisfies $\phi(y) = \phi(x) + (p, y - x) + \frac{1}{2}(A(y - x), y - x) + o(|y - x|^2)$ for some $(A,p) \in SL_N \times \mathbb{R}^N$ then

$$D^+_{21}\phi(x) = \{(B,p) \text{ with } B \in SL_N \text{ and } B \geq A\} ,$$

$$D^-_{21}\phi(x) = \{(B,p) \text{ with } B \in SL_N \text{ and } B \leq A\} .$$

We may now define viscosity solutions of (1):

DEFINITION. Let $u \in C(\Omega)$ *. The function* u *is said to be a viscosity solution of* (1) *if the following inequalities hold:*

(15) $\quad F(A,p,u(x),x) \leq 0 \quad \forall (A,p) \in D^+_{21}u(x) , \quad \forall x \in \Omega ,$

(16) $\quad F(A,p,u(x),x) \geq 0 \quad \forall (A,p) \in D^-_{21}u(x) , \quad \forall x \in \Omega .$

REMARKS.

1. It is clear that if $u \in C^2(\Omega)$ is a classical solution of (1) then u is a viscosity solution of (1) (recall that F satisfies (2)).
2. On the other hand if u is a viscosity solution of (1) and if u is differentiable near a point $x_0 \in \Omega$ and if u is twice differentiable at x_0 then

$$F(D^2u(x_0),Du(x_0),u(x_0),x_0) = 0 .$$

In particular, if $u \in W^{2,p}_{loc}(\Omega)$ for some $p > N$, then u satisfies (1) a.e.

A sharper result than Remark 1 above is given by the

PROPOSITION 2. Let $u \in W^{2,N}_{loc}(\Omega)$ *be a solution of the equation*

$$F(D^2u,Du,u,x) = 0 \quad a.e.\ in\ \Omega .$$

Then u *is a viscosity solution of* (1).

REMARK. This result is quite sharp as it is shown by the following example: Take $u(x) = -|x|$. Obviously $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ for $p < N$ and $D^2u \in M^N(\mathbb{R}^N)$ (Marcinkiewicz space). And it is clear that u is not a viscosity solution of the following equation (actually HJ equation):

$$1 - |Du| = 0 \quad \text{in } \mathbb{R}^N .$$

(Indeed, $F(\xi,p,t,x) = 1 - |p|$, $D^+_{21}u(0) = \{(A,p) \in SL_N \times \mathbb{R}^N : A \geq 0 ,\ |p| \leq 1\}$.)

Proposition 2 is proved in P. L. Lions [35]: it involves an extension of Bony's maximum principle [4] and is somewhat related to Alexandrov study on maximum principle [1].

Exactly as in Section I.1, we derive the following results:

PROPOSITION 3. Let $u \in C(\Omega)$. *The function* u *is a viscosity solution of* (1) *if and only if the following conditions hold for all* $\phi \in C^2(\Omega)$:

(15′) $F(D^2\phi(x),D\phi(x),u(x),x) \leq 0$ *at each local maximum point* x *of* $u - \phi$;

(16′) $F(D^2\phi(x),D\phi(x),u(x),x) \geq 0$ *at each local minimum point* x *of* $u - \phi$.

PROPOSITION 4. Let $u_n \in C(\Omega)$ *be viscosity solutions of the equation*

$$F_n(D^2u_n,Du_n,u_n,x) = 0 \quad in\ \Omega .$$

Let us assume that (F_n) *satisfy* (2) *and converges uniformly on compact sets of* $SL_N \times \mathbb{R}^N \times \mathbb{R} \times \Omega$ *to some function* F. *Finally we assume that* u_n *converges uniformly on compact sets of* Ω *to some function* u. *Then* u *is a viscosity solution of* (1).

REMARKS.

1. If F does not depend on $A \in SL_N$ (i.e. if (1) reduces to (3))

then viscosity solutions of (1) are viscosity solutions of (3).

2. In Proposition 3, we may replace $\phi \in C^2(\Omega)$ by $\phi \in C^\infty(\Omega)$ and local extremum by local strict extremum (or global, or global strict,...).

3. If one considers the optimal cost function of optimal stochastic control problems (or the value function of stochastic differential games) then it is a viscosity solution of the associated HJB equation (or Isaac's equation) - for more details see P. L. Lions [31], [32].

4. The combination of the preceding remark and Propositions 2, 4 yields immediately the stability results proved in N. V. Krylov [22], [28] by sophisticated probabilistic methods.

Of course, the remaining question concerns the uniqueness of viscosity solutions of (1) and this is an open question except essentially two cases: 1) when $N \leq 2$ and the equation is uniformly elliptic; 2) when F is convex with respect to $A \in SL_N$ (i.e. for HJB equations). To be more specific, let us consider the following simple case:

$$F(D^2u,Du) + \lambda u = f(x) \quad \text{in } \mathbb{R}^N \tag{17}$$

where $\lambda > 0$, $f \in BUC(\mathbb{R}^N)$ and the function $F \in W_{loc}(SL_N \times \mathbb{R}^N)$ - for example - satisfies the condition: $\forall R < \infty \quad \exists \nu_R \geq 0$

$$\left(\frac{\partial F}{\partial \xi_{ij}}(\xi,p)\right) \geq \nu_R I_N \text{ for a.e. } (\xi,p) \in SL_N \times \mathbb{R}^N , \quad ||\xi|| + |p| \leq R . \tag{18}$$

The following result holds:

THEOREM 3. Let $u, v \in C_b(\mathbb{R}^N)$ *be viscosity solutions of* (17), *or of* (17) *with* f *replaced by* g. *Let one of the following conditions hold:*

$$N \leq 2 , \quad F' \in W^{1,\infty}(SL_N \times \mathbb{R}^N) , \quad \nu_R \geq \nu > 0 \quad \forall R < \infty ; \tag{19}$$

$$F \textit{ is convex in } \xi , \quad \nu_R \geq \nu > 0 \quad \forall R < \infty \textit{ and } \frac{\partial^2 F}{\partial \xi \partial p} \in L^\infty(SL_N \times \mathbb{R}^N) ; \tag{20}$$

$$F \textit{ is convex in } (\xi,p) . \tag{21}$$

Then we have

$$||(u - v)^+||_\infty \leq \frac{1}{\lambda} ||(f - g)^+||_\infty . \tag{12}$$

REMARKS.

1. We could treat more general nonlinearities F, as well, but no final result without severe restriction like F convex in ξ has yet been proved. Another example of uniqueness result is given in Section III.
2. The proof of the above result is based in each case on existence results: the case when (21) holds relies on results given in Section III while the cases when (19) and (20) hold depend on the fact that if $f \in W^{2,\infty}(\mathbb{R}^N)$ in each case there exists $u \in C_b^2(\mathbb{R}^N)$, the solution of the equation

$$F(D^2u,Du) + \lambda u = f \quad \text{in} \quad \mathbb{R}^N .$$

II. Existence results for viscosity solutions

II.1. Existence results for HJ equations

In P. L. Lions [33], [43] there are discussed existence results. We will give here only two easy examples and we refer to [33] for more general results including boundary value problems.
First we consider the case of (11):

$$(11) \qquad H(Du) + \lambda u = f \quad \text{in} \quad \mathbb{R}^N$$

where $H \in C(\mathbb{R}^N)$, $\lambda > 0$, $f \in BUC(\mathbb{R}^N)$.

PROPOSITION 5. (i) *There exists a unique viscosity solution* u *of* (11) *in* $BUC(\mathbb{R}^N)$.

(ii) *For all* $h > 0$ *the estimate*

$$\sup_{|x-y|\leq h} |u(x) - u(y)| \leq \frac{1}{\lambda} \sup_{|x-y|\leq h} |f(x) - f(y)|$$

holds.
In particular if $f \in C^{0,\alpha}(\mathbb{R}^N)$ $(0 < \alpha \leq 1)$ *then* $u \in C^{0,\alpha}(\mathbb{R}^N)$.

P r o o f . Consider the vanishing viscosity method for the equation (11):

$$(22) \qquad -\varepsilon\Delta u + H_\varepsilon(Du_\varepsilon) + \lambda u_\varepsilon = f_\varepsilon \quad \text{in} \quad \mathbb{R}^N$$

where $H_\varepsilon \in C^\infty(\mathbb{R}^N)$, $H_\varepsilon \to H$ uniformly on bounded sets and $f \in C_b^\infty(\mathbb{R}^N)$, $f_\varepsilon \to f$ in $L^\infty(\mathbb{R}^N)$.
It is quite easy to show that, if $\varepsilon > 0$, there exists a unique

solution u_ε of (22) and $u_\varepsilon \in C_b^\infty(\mathbb{R}^N)$. Applying the maximum principle, we find easily that

$$||u_\varepsilon||_\infty \leqq \frac{1}{\lambda} ||f_\varepsilon - H_\varepsilon(0)||_\infty \leqq C \quad \text{(independent of } \varepsilon \text{)}.$$

Now, if we assume $f \in W^{1,\infty}(\mathbb{R}^N)$, we may take f_ε such that f_ε is bounded in $C_b^1(\mathbb{R}^N)$. Differentiating the equation (22) and applying again the maximum principle, this yields:

$$||Du_\varepsilon||_\infty \leqq \frac{1}{\lambda} ||Df_\varepsilon||_\infty \leqq C \quad \text{(independent of } \varepsilon \text{)}.$$

Therefore u_ε (or a subsequence u_{ε_n}) converges uniformly to some function u $(\in W^{1,\infty}(\mathbb{R}^N))$ which, by Corollary 2, is a (and thus the) viscosity solution of (11). Therefore for $f \in W^{1,\infty}(\mathbb{R}^N)$ there exists a unique viscosity solution $u(f)$ $(\in W^{1,\infty}(\mathbb{R}^N))$; in addition, in view of Theorem 1 we have: $\forall f,\ g \in W^{1,\infty}(\mathbb{R}^N)$

$$||u(f) - u(g)||_\infty \leqq \frac{1}{\lambda} ||f - g||_\infty . \tag{23}$$

By the density and Corollary 1, we deduce the existence (and uniqueness) of a viscosity solution $u(f)$ for any $f \in BUC(\mathbb{R}^N)$; in addition, (23) holds for $f,\ g \in BUC(\mathbb{R}^N)$. Finally, part (ii) of Propos. 5 follows easily from the fact that the map $f \mapsto u(f)$ commutes with translations of $\mathbb{R}^N$.

In a similar way one obtains the following result concerning the problem

$$\frac{\partial u}{\partial t} + H(Du) + \lambda u = f \quad \text{in} \quad \mathbb{R}^N \times (0,T) \ , \quad u|_{t=0} = u_0 \tag{13}$$

where $H \in C(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$, $f \in BUC(\mathbb{R}^N \times [0,T])$, $u_0 \in BUC(\mathbb{R}^N)$.

<u>*PROPOSITION 6.*</u> (i) *There exists a unique viscosity solution* u *of* (13) *in* $BUC(\mathbb{R}^N \times [0,T])$.

(ii) *For all* $h > 0$ *, we have:*

$$\forall t \in [0,T] \quad \sup_{|x-y| \leqq h} |u(x,t) - u(y,t)| \leqq$$

$$\leqq e^{-\lambda t} \sup_{|x-y| \leqq h} |u_0(x) - u_0(y)| + $$
$$+ \int_0^t e^{-\lambda s} \sup_{|x-y| \leqq h} |f(x,s) - f(y,s)| ds .$$

In particular, if $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, $f(\cdot,t)$ is bounded in $W^{1,\infty}(\mathbb{R}^N)$ uniformly for $t \in [0,T]$, then $u \in W^{1,\infty}(\mathbb{R}^N \times (0,T))$.

II.2. Existence results for equation (1)

Just to give an example, we mention the following existence result: Let $F \in C(SL_N \times \mathbb{R}^N)$ and satisfy (2). Then it holds

PROPOSITION 7. Let $f \in BUC(\mathbb{R}^N)$ *and let* $\lambda > 0$ *. Then there exists* $u \in BUC(\mathbb{R}^N)$ *viscosity solution of*

(17) $$F(D^2u,Du) + \lambda u = f \quad \text{in } \mathbb{R}^N .$$

REMARK. This proposition can be easily proved by stochastic considerations or by the use of results due to R. Jensen and P. L. Lions [23]. By similar methods, one can give very general existence results for equations (1) (with boundary conditions) - see in particular Section III. But as long as no satisfactory uniqueness result exists, this seems to be without great interest.

III. On Hamilton-Jacobi-Bellman equations

In this section, we restrict our attention to equations (1) in the case when F is convex in (D^2u,Du,u), that is when (1) reduces to the HJB equation

(4) $$\sup_\alpha \left[A^\alpha u - f^\alpha\right] = 0 \quad \text{in } \Omega$$

and we will impose for example Dirichlet boundary conditions

(24) $$u = 0 \quad \text{on } \partial\Omega .$$

In all what follows, we assume that Ω is a regular domain in $\mathbb{R}^N$ and we denote by $u(x)$ the unit outward normal to $\partial\Omega$ at x.
In the following subsections III.1-2, we present and show in a particular case an existence and a regularity result for (4). Finally in Section III.3 we will consider Monge-Ampère equations.

III.1. An existence and regularity result

First, we need some notations and assumptions: I will be a separable metric space; for $\alpha \in I$, A^α is given by

$$A^\alpha = - a^\alpha_{ij}(x)\partial_{ij} + b^\alpha_i(x)\partial_i + c^\alpha(x) , \quad a^\alpha_{ij} = \sigma^\alpha_{ik}\sigma^\alpha_{jk}$$

where σ^α_{ij}, b^α_i, c^α, f^α remain in a bounded set of $W^{2,\infty}(\Omega)$ as α describes I and are continuous in α (for $x \in \overline{\Omega}$) and where

$$(25) \qquad \lambda = \inf_{(\alpha,x) \in I\times\bar{\Omega}} c^\alpha(x) > 0 .$$

Let us give a typical example of existence, uniqueness and regularity results:

THEOREM 4. *Let us assume*

$$(26) \qquad \exists \nu > 0 \quad \forall (x,\alpha) \in \partial\Omega \times I \quad a^\alpha_{ij}(x) n_i(x) n_j(x) \geq \nu .$$

(i) Then there exists a unique viscosity solution u of (4) - (24) in $BUC(\bar{\Omega})$. In addition $u \in C^{0,\theta}(\bar{\Omega})$ for some $\theta \in (0,1]$ depending only on ν , σ^α , b^α .

(ii) There exists λ_0 depending only on $||D^\beta\phi||_{L^\infty(\Omega)}$ for $|\beta| = 1,2$, $\phi = \sigma^\alpha$, b^α ($\lambda_0 = 0$ if σ^α , b^α do not depend on x) such that if $\lambda > \lambda_0$, we have

$$(27) \qquad u \in W^{1,\infty}(\Omega) , \quad \sup_{\alpha \in I} ||A^\alpha u||_\infty < \infty ;$$

$$(28) \qquad u \text{ is semi-concave on } \Omega : \exists C > 0 , \ \forall \xi \in \mathbb{R}^N \ |\xi| = 1 , \ \partial^2_\xi u \leq C \text{ in } \mathcal{D}'(\Omega) ;$$

$$(29) \qquad \sup_\alpha [A^\alpha u - f^\alpha] = 0 \quad \text{a.e. in } \Omega .$$

(iii) If $\lambda > \lambda_0$ and if for some open set $\omega \subset \Omega$ there exist $\nu > 0$, $p \in \{1,\dots,N\}$ such that for all $x \in \omega$, we can find $n \geq 1$, $\alpha_1,\dots,\alpha_n \in I$ and $\theta_1,\dots,\theta_n \in (0,1)$ such that

$$(30) \qquad \sum_i \theta_i = 1 , \quad \sum_i \theta_i a^{\alpha_i}_{k\ell}(x)\xi_k\xi_\ell \geq \nu \sum^p \xi_j^2 \quad \forall \xi \in \mathbb{R}^N ;$$

then $\partial_{ij} u \in L^\infty(\Omega)$ for $1 \leq i,j \leq p$.

(iv) If for some open set $\omega \subset \Omega$ there exists $\nu > 0$ such that

$$(31) \qquad \forall x \in \omega \quad \forall \alpha \in I \quad (a^\alpha_{ij}(x)) \geq \nu I_N$$

then $u \in C^{2,\beta}(\omega)$ (for some $\beta \in (0,1)$ depending on ω , ν , $\sup_\alpha ||a^\alpha||_\infty$).

THEOREM 5. *Assume* (26).

(i) *If* $w \in W^{1,\infty}(\Omega)$ *satisfies*

$$(32) \qquad A^\alpha w \leq f^\alpha \ \textit{in} \ \mathcal{D}'(\Omega) , \quad w \leq 0 \ \textit{on} \ \partial\Omega ,$$

then $w \leq u$.

(ii) *If* w *satisfies* (27), (29) *with* $w = 0$ *on* $\partial\Omega$ *and*

(33) $$\exists C > 0 \quad \Delta w \leq C \quad in \quad \mathcal{D}'(\Omega) ,$$

then $w = u$.

REMARKS.

1. Since we want u to be 0 on $\partial\Omega$, some assumption like (26) has to be assumed (more general assumptions can be found in P. L. Lions [31], [32] where cases when we want u to be 0 only on some part of $\partial\Omega$ are also considered).
2. If $\Omega = \mathbb{R}^N$, the condition (26) becomes vacuous.
3. It is possible to show on easy examples that the regularity involved in Theorem 4 is optimal. In particular the assumption $\lambda > > \lambda_0$ is in general necessary in order to have (27) (see Genis and Krylov [22] for an example). In the uniformly elliptic case, however, this condition is not necessary (see below). Part (iv) of Theorem 4 is an easy consequence of the regularity results due to L. C. Evans [15], [14].
4. All these results adapt easily to the case of Cauchy problems:

(34) $$\frac{\partial u}{\partial t} + \sup_{\alpha} [A^\alpha u - f^\alpha] = 0 \quad in \quad \Omega \times (0,T) ,$$
$$u|_{t=0} = u_0 \quad in \quad \bar{\Omega} ,$$

(24′) $$u = 0 \quad on \quad \partial\Omega \times [0,T] .$$

In this case λ may be taken arbitrarily in $\mathbb{R}$.

5. These results extend some particular results obtained by N. V. Krylov [29], [30], [26]; P. L. Lions and J. L. Menaldi [44]. The case $\Omega = \mathbb{R}^N$ is treated in P. L. Lions [36] by purely probabilistic methods and the general case (announced in [37], [31]) is treated in [32] by combinations of p. d. e. and probabilistic arguments.

In what follows, we will only treat the following

COROLLARY 3. *Assume that* Ω *is a bounded smooth domain,* $\lambda \geq 0$ *and*

(31′) $$\exists \nu > 0 \quad \forall (x,\alpha) \in \Omega \times I \quad (a_{ij}(x)) \geq \nu I_N .$$

Then there exists a unique solution $u \in W^{2,\infty}(\Omega)$ *of* (4) - (24).

REMARKS.

1. Corollary 3 is proved in P. L. Lions [38], L. C. Evans and P. L. Lions [17] by p. d. e. techniques sketched below. Previously, very

particular results were obtained by N. V. Krylov [29], H. Brèzis and L. C. Evans [5], L. C. Evans and A. Friedman [16], P. L. Lions and J. L. Menaldi [44].

2. The exact range of λ such that (4) - (24) is solvable is investigated in P. L. Lions [42]. This involves a notion of demi-eigenvalues for the operator associated with the HJB equation.

III.2. Sketch of the proof of Corollary 3

The proof consists first in approximating the problem and getting uniform a priori estimates in $W^{2,\infty}(\Omega)$ and next in passing to the limit. The passage to the limit is an exercise on the notion of viscosity solutions and thus we will skip it. Various approximations of (4) - (24) are possible: see L. C. Evans and A. Friedman [16], R. Sensen and P. L. Lions [23]. We will follow the one introduced in [16]: taking $I = \{1,\dots,m\}$ to simplify: one considers a system, called the penalized system

$$(35)\quad \begin{cases} A^1u_\varepsilon^1 + \beta_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2) = f^1 \text{ in } \Omega \,, \quad u_\varepsilon^1 = 0 \text{ on } \partial\Omega \,, \\ A^2u_\varepsilon^2 + \beta_\varepsilon(u_\varepsilon^2 - u_\varepsilon^3) = f^2 \text{ in } \Omega \,, \quad u_\varepsilon^2 = 0 \text{ on } \partial\Omega \,, \\ \vdots \\ A^mu_\varepsilon^m + \beta_\varepsilon(u_\varepsilon^m - u_\varepsilon^1) = f^m \text{ in } \Omega \,, \quad u_\varepsilon^m = 0 \text{ on } \partial\Omega \end{cases}$$

where $\beta_\varepsilon(t) = \frac{1}{\varepsilon}\beta(t)$ and $\beta(t) \in C^\infty(\mathbb{R})$, $\beta(t) = 0$ if $t \leq 0$, $\beta'(t) > 0$ if $t > 0$ and $\beta''(t) \geq 0$ on $\mathbb{R}$.

It is easy to show that there exists a unique solution $(u_\varepsilon^1,\dots$ $\dots,u_\varepsilon^m)$ of (35) in $(C^2(\bar\Omega))^m$. Of course by an obvious approximation of the coefficients, one may assume $u_\varepsilon^i \in C^\infty(\Omega)$. Now if we prove $||u_\varepsilon^i||_{W^{2,\infty}(\Omega)} \leq C$ (independent of ε) then it is easy to show first that $u_\varepsilon^i \underset{\varepsilon\to 0}{\longrightarrow} u$ (independent of i) (or subsequences of u_ε^i), and u is a viscosity solution of (4). Obviously $u \in W^{2,\infty}(\Omega)$ and thus (4) holds a.e.

The a priori estimates are obtained in three steps: first, one proves a priori estimates in $W^{1,\infty}(\Omega)$ by the use of standard barrier functions and maximum principle. Then, if we set $M_\varepsilon = \sup_{i,x} ||D^2u_\varepsilon^i(x)||$ by a convenient adaptation of a device due to Kohn and Nirenberg - for more details see P. L. Lions [38] - one shows by a sophisticated argument of barrier functions that

$$\forall x \in \partial\Omega \quad \forall i \in \{1,\dots,m\} \quad |D^2u_\varepsilon^i(x)| \leq c + cM_\varepsilon^{1/2}$$

where c is independent of x , i and ε .

Let us now show how we conclude - firstly in a simple case namely when a^i , b^i , c^i do not depend on x : Let ξ be a unit vector in $\mathbb{R}^N$, we differentiate twice (35) with respect to ξ and we obtain

$$A^i(\partial_\xi^2 u_\varepsilon^i) + \beta'(u_\varepsilon^i - u_\varepsilon^{i+1})(\partial_\xi^2 u_\varepsilon^i - \partial_\xi^2 u_\varepsilon^{i+1}) +$$

$$+ \beta''(u_\varepsilon^i - u_\varepsilon^{i+1})(\partial_\xi u_\varepsilon^i - \partial_\xi u_\varepsilon^{i+1})^2 = \partial_\xi^2 f^i$$

and since β is convex we deduce

$$(36) \qquad A^i(\partial_\xi^2 u_\varepsilon^i) + \beta'(u_\varepsilon^i - u_\varepsilon^{i+1})(\partial_\xi^2 u_\varepsilon^i - \partial_\xi^2 u_\varepsilon^{i+1}) \leq \partial_\xi^2 f^i \leq c .$$

Next let (i_0,x_0) be a maximum point of $\sup\limits_{i,x} \partial_\xi^2 u^i(x)$, without loss of generality we may assume that $i_0 = 1$. If $x_0 \in \partial\Omega$, we deduce

$$\partial_\xi^2 u_\varepsilon^i(x) \leq c + cM_\varepsilon^{1/2} \quad \forall x \in \bar{\Omega} , \quad \forall i ;$$

on the other hand if $x_0 \in \Omega$, then applying maximum principle in (36) (for $i = 1$) at the point x_0 , we obtain $\partial_\xi^2 u_\varepsilon^1(x_0) \leq c$. Therefore in all cases, we proved:

$$\forall \xi \in \mathbb{R}^N , \quad |\xi| = 1 \quad \forall x \in \bar{\Omega} \quad \forall i \quad \partial_\xi^2 u_\varepsilon^i(x) \leq c + cM_\varepsilon^{1/2} .$$

Observing that: $A^i u_\varepsilon^i \leq c$ we deduce easily

$$|D^2 u_\varepsilon^i(x)| \leq c + cM_\varepsilon^{1/2}$$

and thus M_ε is bounded.

In the general case, the dependence of the coefficients on x creates difficulties (supplementary terms appear in (36)) which are solved in [38], [17] in the following way. Let x_0 be such that $|D^2 u_\varepsilon^{i_0}(x_0)| = M_\varepsilon$ for some i_0 . If $x_0 \in \partial\Omega$, we conclude easily; if $x_0 \in \Omega$, we may assume without loss of generality that $D^2 u_\varepsilon^{i_0}(x_0)$ is diagonal and we denote $\alpha_k = a_{kk}^{i_0}(x_0)$. Finally we introduce

$$w^i(x) = |D^2 u_\varepsilon^i(x)|^2 + 2N^2 M_\varepsilon \sum_k \alpha_k \partial_k^2 u_\varepsilon^i(x) + \mu |Du_\varepsilon^i|^2$$

where $\mu > 0$ will be determined later on and where we normalize the (a^i) such that $(a^i(x)) \geq I_N$ for $x \in \bar{\Omega}$, $i \in \{1,\dots,m\}$. Then making similar computations as those performed above with $\partial_\xi^2 u_\varepsilon^i$, differentiating twice the equations (35) and using maximum principle on the inequalities satisfied by $A^i w^i$, one obtains the following inequality: if μ is large enough, then

$$\max_{i,x} w^i(x) \leqq \frac{1}{2} M_\varepsilon^2 + C \,. \tag{37}$$

Thus in particular

$$M_\varepsilon^2 \leqq w^{i_0}(x_0) - 2N^2 M_\varepsilon \sum_k \alpha_k \partial_k^2 u_\varepsilon^{i_0}(x_0) + C \leqq$$

$$\leqq 2N^2 M_\varepsilon \{A^{i_0} u_\varepsilon^{i_0}(x_0)\} + \frac{1}{2} M_\varepsilon^2 + CM_\varepsilon + C \leqq$$

$$\leqq \frac{1}{2} M_\varepsilon^2 + CM_\varepsilon + C$$

and we conclude.

III.3. Applications to the Monge-Ampère equations

We now turn to the solution of the classical Monge-Ampère equations:

$$\det(D^2 u) = H(x,u,Du) \quad \text{in } \Omega \,, \quad u \text{ is convex in } \bar{\Omega} \,, \quad u = 0 \text{ on } \partial\Omega \,, \tag{38}$$

where Ω is a bounded, convex domain in $\mathbb{R}^N$ and $H \in C^\infty(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$. We shall assume that

$$\forall R < \infty \quad \exists\, \alpha_R > 0 \quad \forall (x,t,p) \in \bar{\Omega}^R \times [-R,+R] \times \bar{B}_R \qquad H(x,t,p) \geqq \alpha_R > 0 \,, \tag{39}$$

where $\bar{\Omega}^R = \{x \in \bar{\Omega} : \operatorname{dist}(x,\partial\Omega) \geqq \frac{1}{R}\}$ and

$$\underline{u} \in C(\bar{\Omega}) \,, \text{ convex in } \bar{\Omega} \,, \text{ satisfying} \quad \det(D^2\underline{u}) \geqq H(x,\underline{u},D\underline{u}) \quad \text{in } \Omega \,. \tag{40}$$

The precise meaning of the inequality (40) is to be understood in the sense of A. D. Alexandrov [2] (see also A. V. Pogorelov [47], S. Y. Cheng and S. T. Yau [7]). Then we have

THEOREM 6. Under assumptions (39) *and* (40) *there exists a solution* u *of* (38) *in* $C^\infty(\Omega) \cap C(\bar{\Omega})$, $u \geqq \underline{u}$ *in* Ω .

REMARKS.

1. If $\frac{\partial H}{\partial t}(x,t,p) \geqq 0$ for $(x,t,p) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, then u is unique.
2. This result is proved in P. L. Lions [39], [40] (and some version of it was announced in P. L. Lions [41]), where more general results are proved (requiring less regularity on H); in particular the case of non-homogeneous boundary conditions is treated in [39], [40].

3. It is easy to show that if H satisfies the condition

(41) $$\limsup_{|p|\to\infty} H(x,t,p)|p|^{-N} < \infty \quad \text{uniformly in } (x,t) \in \bar{\Omega} \times \mathbb{R}$$

and if Ω is strictly convex then (40) holds, i.e. a subsolution $\underline{u}$ exists. It is of course the case when H depends only on x and thus we see that we recover as a very special case the result of S. Y. Cheng and S. T. Yau [7] which is proved in [7] by completing the method of A. V. Pogorelov [47] based on geometrical arguments involving first the solution of the Minkowski problem. On the other hand, our proof is a direct p.d.e. proof and applies to general non-linearities $H(x,t,p)$ while the method of Pogorelov-Cheng-Yau does not seem to cover this case.

4. A major open question is the regularity of u up to the boundary.

5. Let us consider, as an example, a particular case: $H(x,t,p) = H(x)(1+|p|^2)^{\alpha}$ where $\alpha \geqq 0$. We will also assume, to simplify, that Ω is strictly convex and $H > 0$ in Ω. Then (see Remark 3 above):

(i) if $\alpha \leqq \frac{N}{2}$, there exists a unique solution of (38) in $C^{\infty}(\Omega) \cap C(\bar{\Omega})$. This obviously contains the case $\alpha = 0$ which gives the solution of the Minkowski problem.

(ii) On the other hand, if $\alpha > \frac{N}{2}$ and if u solves (38), then necessarily

$$\int_{\Omega} H(x)dx = \int_{\Omega} \det(D^2u)(1+|Du|^2)^{-\alpha}dx =$$
$$= \int_{Du(\Omega)} \frac{dp}{(1+|p|^2)^{\alpha}} \leqq \int_{\mathbb{R}_N} \frac{dp}{(1+|p|^2)^{\alpha}} = c_{\alpha} .$$

Therefore the condition

(42) $$\int_{\Omega} H(x)dx \leqq c_{\alpha}$$

is a necessary condition for the existence of u (or $\underline{u}$).
On the other hand, it has been proved by I. Bakelman [3] that $\underline{u}$ exists if

(43) $$\exists \lambda \geqq 0 , \quad \alpha \leqq \frac{N+1+\lambda}{2}, \quad \int_{\Omega} H(x)dx < c_{\alpha} ,$$
$$H(x) \leqq C \operatorname{dist}(x,\partial\Omega)^{\lambda}.$$

Therefore if (43) holds, there exists a solution $u \in C^{\infty}(\Omega) \cap C(\bar{\Omega})$ of (38). In particular we see that if $\alpha \leqq \frac{N+1}{2}$ we may take $\lambda = 0$

and (42) is necessary and "almost" sufficient for the existence of u . A case of interest for differential geometry is the case $\alpha = \frac{N+2}{2}$ - in this case solving (38) amounts to build a convex hypersurface with prescribed Gauss curvature $H(x)$. In this case we see that (42) is a necessary condition and that if : $\int_\Omega H(x)dx < C_\alpha$, $H(x) \leq$ $\leq C \operatorname{dist}(x,\partial\Omega)$ then there exists a unique solution $u \in C^\infty(\Omega) \cap$ $\cap\ C(\bar{\Omega})$.

Let us mention the main lines of the proof of Theorem 6: The main difficulty of (38) lies with $\partial\Omega$ since us a priori estimates for u up to $\partial\Omega$ are known. This is why we approximate (38) by problems in $\mathbb{R}^N$ of the following form:

$$(38\text{-}\varepsilon)\quad \begin{cases} \det(D^2u_\varepsilon - \frac{1}{\varepsilon}\,pu_\varepsilon I_N) = H(x,u_\varepsilon,Du_\varepsilon) \quad \text{in } \mathbb{R}^N , \\ (D^2u_\varepsilon - \frac{1}{\varepsilon}\,pu_\varepsilon I_N) \geq 0 \quad \text{in } \mathbb{R}^N , \quad u_\varepsilon \in C_b^\infty(\mathbb{R}^N) \end{cases}$$

where $p \in C_b^\infty(\mathbb{R}^N)$, $p \equiv 0$ in $\bar{\Omega}$, $p > 0$ on $\mathbb{R}^N - \bar{\Omega}$.

The idea of the proof is to solve firstly (38-ε), to let $\varepsilon \to 0$ and to make sure that $u_\varepsilon \to 0$ on $\mathbb{R}^N - \bar{\Omega}$ by the use of appropriate barrier functions and finally to apply the general a priori estimates due to A. V. Pogorelov [47], E. Calabi [6].

To conclude, let us explain the relations between Monge-Ampère equations and HJB equations: This is explained by the following algebraic lemma observed by B. Gareau [21] and N. V. Krylov [30].

<u>LEMMA 2.</u> *Let* A *be a symmetric* N×N-*matrix.*

(i) *If* $A \geq 0$ *then* $(\det A)^{1/N} = \inf\{\operatorname{Tr}(AB) : B \in SL_N , B > 0 , \det B = 1/N^N\}$.

(ii) *If* $A > 0$, $B = \frac{1}{N}(\det A)^{-1/N} A^{-1}$ *is a minimum in the above infimum.*

(iii) *If* $\inf\{\operatorname{Tr}(AB) : B \in SL_N , B > 0 , \det B = 1/N^N\} > -\infty$, *then* $A \geq 0$.

This shows that (38-ε) is equivalent to:

$$\sup_{B \in V} \{- b_{ij}\partial_{ij}u^\varepsilon + \frac{1}{\varepsilon}\, pu^\varepsilon \operatorname{Tr}(B)\} + \left(H(x,u_\varepsilon,Du_\varepsilon)\right)^{1/N} = 0 \quad \text{in } \mathbb{R}^N$$

where $V = \{B \in SL_N : B > 0 , \det B = 1/N^N\}$.

In particular, if H depends only on x, the resolution of (38-ε) is then an immediate consequence of Theorem 4.

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