Eberhard Zeidler

Lectures on Lyusternik-Schnirelman theory for indefinite nonlinear eigenvalue problems and its applications

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LECTURES ON LYUSTERNIK-SCHNIRELMAN THEORY FOR INDEFINITE NONLINEAR EIGENVALUE PROBLEMS AND ITS APPLICATIONS

Eberhard Zeidler

Introduction

The purpose of these lectures is to give an introduction to the Lyusternik-Schnirelman theory and its typical applications based on the ideas outlined in the papers of LYUSTERNIK (1930), KRASNOSEL'SKII (1956), VAINBERG (1956), BROWDER (1968), (1970a), (1970b), COFFMAN (1969), (1971), (1973), AMANN (1972), FUČÍK, NEČAS (1972a), FUČÍK, NEČAS, SOUČEK, SOUČEK (1973), RABINOWITZ (1973), (1974), ZEIDLER (1978).

The Lyusternik-Schnirelman theory is concerned with nonlinear eigenvalue problems in Banach spaces X of the type

$$Au = \lambda Bu, \quad u \in X, \quad \lambda \in \mathbb{R} \tag{1}$$

generalizing linear eigenvalue problems of the type

$$Au = \lambda u, \quad u \in X, \quad \lambda \in \mathbb{R}, \tag{2}$$

where A is a linear symmetric and completely continuous operator in a Hilbert space X .

AMANN (1972) has considered the problem (1) without definiteness restrictions upon A for the first time, and thus my lectures have been strongly influenced by his paper. In the indefinite case it is possible that there exists only a finite number of eigenvalues in (1), (2).

It is our goal to study the indefinite case extensively and to emphasize the connection between the results obtained for nonlinear and linear operators.

In Section 5 we shall formulate two general theorems strengthening the results of all the papers mentioned above (see Remarks 3, 4, 5 in Section 5). In Section 7 we shall restrict our main theorems to the case of linear operators. In this way we shall see that our results obtained for nonlinear operators are maximal in a certain sense. These lectures are organized as follows:

1. Notation
2. Some typical eigenvalue problems
   2.1. Nonlinear equations in $\mathbb{R}^N$
   2.2. Linear integral equations and the Hilbert-Schmidt theory
   2.3. Nonlinear integral equations
   2.4. Nonlinear elliptic partial differential equations
3. Courant's maximum-minimum principle
4. The genus of symmetric closed sets not containing the origin
5. The main theorems in infinite-dimensional Banach spaces
6. Sketched proofs of the main theorems
7. Restriction to the case of linear operators
8. An important special case of the main theorems concerning nonlinear operators
9. Applications to nonlinear elliptic partial differential equations
10. The main theorems in finite-dimensional Banach spaces
11. Applications to abstract Hammerstein equations
12. Applications to Hammerstein integral equations

    References

The contents of these lectures is closely related to Chapter 42 of the third volume of my "Lectures on Nonlinear Functional Analysis" (see ZEIDLER (1978)). Here we shall prove only the statements which are not contained in my book.

Furthermore, for the sake of technical simplicity we shall consider only simple but typical applications.

Remarks on the historical development of the Lyusternik-Schnirelman theory can be found in the papers of KRASNOSEL'SKII (1956), VAINBERG (1956), BROWDER (1970a), RABINOWITZ (1974).

Acknowledgments. I would like to express my gratitude to Prof. M. A. KRASNOSEL'SKII for telling me the sophisticated proof of Proposition 7 in Section 4.

Furthermore, I would like to thank my friends from the Organizing Committee for their invitation to this interesting and well-organized Spring School.

1. Notation

Let X be a Banach space. The space dual is denoted by X^*. We set $\langle x^*,x\rangle = x^*(x)$ for all $x \in X$, $x^* \in X^*$. The symbols $u_n \rightharpoonup u$ and $u_n \to u$ denote the weak and the strong convergence in X, respectively.

The set of all real or natural numbers is denoted by $\mathbb{R}$ or $\mathbb{N}$, respectively.

Let A be an operator from the Banach space X into the Banach space X^*. A is said to be completely continuous iff it is continuous and maps bounded sets into relatively compact sets. A is said to be strongly continuous iff $u_n \rightharpoonup u$ implies $Au_n \to Au$ $(n \to \infty)$.

A is said to be monotone iff $\langle Au - Av, u - v\rangle \geqq 0$ for all $u, v \in X$.

A is said to be uniformly monotone iff

$$\langle Au - Av, u - v\rangle \geqq c(\|u - v\|)\,\|u - v\| \quad \text{for all} \quad u, v \in X$$

where $c : [0, +\infty) \to [0, +\infty)$ is a real strictly monotone continuous function with $c(0) = 0$ and $c(t) \to +\infty$ as $t \to +\infty$.

A is said to be bounded iff A maps bounded sets into bounded sets. A is said to be a potential operator iff there exists a Gâteaux-differentiable real functional a on X such that $a'(u)=Au$ for all $u \in X$. The operator a is called the potential of A.

Figure 1 gives a survey on the connection between important operator properties. All the definitions and proofs can be found in Chapter 27 of ZEIDLER (1977).

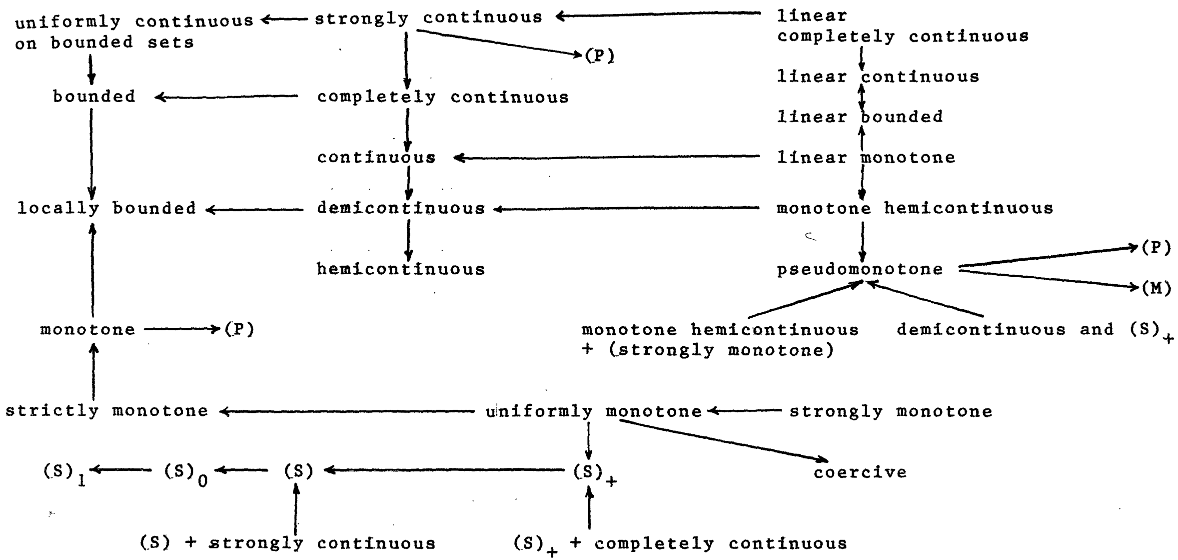

Fig. 1: Properties of nonlinear operators (+ means the sum of operators)

2. Some Typical Eigenvalue Problems

Let us consider four simple examples concerning

i) nonlinear equations in $\mathbb{R}^N$,

ii) linear integral equations,

iii) nonlinear integral equations,

iv) nonlinear elliptic partial differential equations.

2.1. Nonlinear equations in $\mathbb{R}^N$. We start with the real eigenvalue problem in $\mathbb{R}^N$

$$\frac{\partial g(x)}{\partial \xi_i} = \lambda \xi_i , \qquad i = 1,\dots,N , \tag{3}$$

where $x = (\xi_1,\dots,\xi_N) \in \mathbb{R}^N$, $\lambda \in \mathbb{R}$.

Proposition 1. (LYUSTERNIK (1930)).

Suppose $g : \mathbb{R}^N \to \mathbb{R}$ *has continuous first partial derivatives and is even.*

Then for each $r > 0$, *the eigenvalue problem* (3) *has at least* N *distinct pairs of eigenvectors* $(x, -x)$ *with* $||x|| = r$.

This basic result of the Lyusternik-Schnirelman theory is a special case of Theorem 3 in Section 10.

Let $A = (a_{ij})$ be a symmetric $N \times N$ - matrix. Set $g(x) = 2^{-1} \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j$. Then the equation (3) is equivalent to $Ax = \lambda x$, i.e. Proposition 1 generalizes the well-known fact that A has N linearly independent eigenvectors.

2.2. Linear integral equations and the Hilbert-Schmidt theory. Next we consider the linear integral equation

$$\int_G a(x,y)\, u(y)\, dy = \lambda u(x), \quad u \in L_2(G), \quad \lambda \in \mathbb{R} , \tag{4}$$

where G is an open bounded nonempty set in $\mathbb{R}^N$, $N \geq 1$.

The equation (4) is equivalent to the operator equation

$$Au = \lambda u , \qquad u \in X \equiv L_2(G) , \qquad \lambda \in \mathbb{R} . \tag{4'}$$

X is a real separable Hilbert space. Suppose that the real measurable function $a : G \times G \to \mathbb{R}$ is symmetric, i.e.

$$(5) \qquad a(x,y) = a(y,x) \quad \text{for all} \quad x,y \in G ,$$

and

$$(6) \qquad 0 < \int_{G\times G} a^2(x,y)\, dxdy < \infty .$$

Then the operator $A : X \to X$ is symmetric and completely continuous, $A \neq 0$.

The following main theorem of the Hilbert-Schmidt theory describes the solutions of the equations (4), (4').

Proposition 2. (cf. e.g. RIESZ-NAGY (1952), Chapter VI.)

Suppose :

X *is a real separable Hilbert space with a scalar product* $(\cdot\,|\,\cdot)$,

$A : X \to X$ *is a linear symmetric completely continuous operator,* $A \neq 0$, $\dim X = \infty$.

Then :

1) The equation

$$(7) \qquad Au = \lambda u , \quad u \in X , \quad \lambda \in \mathbb{R}$$

has at least one eigenvalue $\lambda \neq 0$.

2) Every eigenvalue $\lambda \neq 0$ *of* A *has a finite multiplicity.*

3) There exists an infinite sequence of eigensolutions (u_i,λ_i) *with* $(u_i|u_j) = \delta_{ij}$ *for* $i,j = 1,2,\ldots$ *and*

$$(8) \qquad u = \sum_{i=1}^{\infty} (u|u_i)\, u_i \qquad \text{for all} \quad u \in X .$$

If (u,λ), $u \neq 0$, $\lambda \in \mathbb{R}$ *is an arbitrary eigensolution of* (7), *then there exists a number* λ_i *with* $\lambda_i = \lambda$, *and in* (8) *it holds* $(u|u_j) = 0$ *for all* j *with* $\lambda_j \neq \lambda$.

2.3. Nonlinear integral equations. As another example let us consider the Hammerstein integral equation

(9) $$\int_G a(x,y)\, f(u(y))\, dy = \lambda u(x), \quad u \in L_2(G), \lambda \in \mathbb{R}, \ f \text{ odd}$$

and the corresponding linear equation

(10) $$\int_G a(x,y)\, u(y)\, dy = \lambda u(x), \quad u \in L_2(G), \quad \lambda \in \mathbb{R}.$$

In Section 12 we shall prove, roughly speaking, the following <u>result</u> : Suppose (5) and (6) are satisfied. Suppose that the linear integral equation (10) has only positive eigenvalues.

Then, under certain assumptions on f, the nonlinear integral equation (9) has an infinite number of distinct eigenvalues.

2.4. <u>Nonlinear elliptic partial differential equations.</u> For the sake of simplicity let us study the boundary value problem

(11) $$-\lambda \sum_{i=1}^{N} D_i(D_i u|D_i u|^{p-2}) = u|u|^{p-2}\phi(x) \quad \text{on} \quad G,$$
$$u = 0 \quad \text{on } \partial G$$

where $x = (\xi_1, \ldots, \xi_N)$, $D_i = \partial/\partial\xi_i$, $p \geq 2$. Let G be an open bounded nonempty set in $\mathbb{R}^N$, $N \geq 1$.

Suppose $\phi : \overline{G} \to \mathbb{R}$ is a continuous function with

(12) $$\min_{x \in \overline{G}} \phi(x) > 0 .$$

Definition 1. A function u *belonging to the Sobolev space* $X \equiv \overset{o}{W}{}^1_p(G)$ *is said to be a <u>generalized solution</u> of* (11) *iff*

(11') $$\lambda \tilde{b}(u,v) = \tilde{a}(u,v) \qquad \textit{for all} \quad v \in X,$$

where

$$\tilde{b}(u,v) = \int_G \sum_{i=1}^{N} D_i u|D_i u|^{p-2} D_i v\, dx ,$$

$$\tilde{a}(u,v) = \int_G \phi(x)\, u|u|^{p-2} v\, dx .$$

By integration by parts it is easily seen that every regular solution u of (11') is a solution of (11) as well. This justifies the term of generalized solution (see e. g. ZEIDLER (1977), p. 94).

Furthermore, it is not difficult to prove that there exist operators $A, B : X \to X^*$ with

$$\tilde{b}(u,v) = \langle Bu, v \rangle \ , \quad \tilde{a}(u,v) = \langle Au, v \rangle \quad \text{for all} \quad u, v \in X \ .$$

Therefore, the equation (11') is equivalent to

$$(11'') \qquad \lambda Bu = Au \ , \quad u \in X \ , \quad \lambda \in \mathbb{R} \ .$$

A, B are odd potential operators with potentials

$$b(u) = p^{-1} \int_G \sum_{i=1}^{N} |D_i u|^p \, dx \ , \quad a(u) = p^{-1} \int_G \phi(x) |u|^p \, dx \ ,$$

and all the hypotheses of the following Proposition 3 are satisfied with $B = B_1$, $B_2 = 0$ (see ZEIDLER (1978), p. 120).

Proposition 3.

Suppose :

(13) X *is a real reflexive separable Banach space,* $\dim X = \infty$.

(14) $A, B : X \to X^*$ *are odd potential operators with potentials* a, b;

$$a(0) = b(0) = 0 \ .$$

(15) $B = B_1 + B_2$, $B_i : X \to X^*$.

(16) B_1 *is bounded, continuous and uniformly monotone,* $B_1(0) = 0$.

(17) A, B_2 *are strongly continuous.*

(18) $\langle Au, u \rangle > 0$, $\langle B_2 u, u \rangle \geqq 0$ *for all* $u \neq 0$.

Let $\alpha > 0$ *be an arbitrary fixed real number.*

Then :

For each $m = 1, 2, \ldots$ *there exists an eigensolution* (u_m, λ_m) *of*

$$(19) \qquad \lambda Bu = Au \ , \quad b(u) = \alpha \quad (u \in X, \ \lambda \in \mathbb{R})$$

with $u_m \neq 0$, $\lambda_m > 0$ *and* $u_m \rightharpoonup 0$, $\lambda_m \to +0$ *as* $m \to \infty$.

Proposition 3 is a special case of Theorem 2 in Section 5 (see also Proposition 8 and Corollary 2 in Section 8).

If we suppose that the function ϕ has zeros on G, then the definiteness condition $\langle Au,u\rangle > 0$ if $u \neq 0$ is not satisfied. Nonetheless, it holds

(18')
$$\langle Au,u\rangle = 0 \Leftrightarrow a(u) = 0 .$$

This condition or the weaker condition

(18'')
$$Au = 0 \Rightarrow a(u) = 0$$

will play a crucial role in our main theorems (see Theorem 1, Theorem 2 in Section 5, and Section 9 for applications to partial differential equations).

3. Courant's Maximum-Minimum Principle

The Lyusternik-Schnirelman theory generalizes Courant's maximum-minimum principle. Therefore, let us formulate this principle in such a way that later the generalization will be obvious.

As in Proposition 2 (Hilbert-Schmidt theory) we shall make the following assumptions :

(i) X is a real separable infinite-dimensional Hilbert space with a scalar product $(.|.)$.

(ii) $A : X \to X$ is a linear symmetric completely continuous operator, $A \neq 0$.

Set
$$a(u) = 2^{-1}(Au|u) , \quad b(u) = 2^{-1}(u|u) .$$

Definition 2. Denote by S *the boundary of the unit ball, i.e.* $S = \{u \in X : ||u|| = 1\}$.

Denote by S_k *the boundary of an arbitrary* k-*dimensional unit ball in* X *, i.e.*
$$S_k = S \cap X_k , \quad X_k = k\text{-}dimensional\ linear\ subspace\ of\ X.$$

Let $\mathcal{L}_m$ *be the set of all* S_k *with* $k \geq m$, $m = 1, 2, \ldots$.

Define

(20)
$$\mathcal{L}_m^{\pm} = \{L \in \mathcal{L}_m : \pm a(u) > 0 \quad for\ all \quad u \in L\} .$$

Set

$$(21) \qquad \pm\lambda_m^{\pm} = \begin{cases} \sup\limits_{L\in\mathcal{L}_m^{\pm}} \min\limits_{u\in L} (\pm 2a(u)) \\ 0 \quad \text{if } \mathcal{L}_m^{\pm} = 0 \ . \end{cases}$$

Obviously, $\pm\lambda_1^{\pm} \geqq \pm\lambda_2^{\pm} \geqq \cdots \geqq 0$.

Proposition 4 (the maximum-minimum principle of COURANT (1920); *see also* FISCHER (1905), WEYL (1911)).

Suppose $\pm\lambda_m^{\pm} > 0$ (+ or −) . *Then* :

1) $\lambda = \lambda_m^{\pm}$ *is an eigenvalue of the operator* A . *All eigenvalues* $\lambda \neq 0$ *of* A *are obtained in this way.*

2) The multiplicity of λ *is equal to the number of indices* j *with* $\lambda_j^{\pm} = \lambda$.

3) There exist eigenvectors $u_1,\dots,u_m$ *of* A *with* $(u_i|u_j) = \delta_{ij}$ *such that*

$$\pm\lambda_m^{\pm} = \min_{u\in L_S} 2a(u)$$

where $L = S \cap \text{lin}\,\{u_1,\dots,u_m\} \in \mathcal{L}_m^{\pm}$.

REMARK 1. Our main theorems in Section 5 will generalize the maximum-minimum principle (21) to nonlinear operators A .

The basic idea due to LYUSTERNIK (1930) is to replace $\mathcal{L}_m$ by a larger class $\mathcal{K}_m \supseteq \mathcal{L}_m$. The sets $K \in \mathcal{K}_m$ are characterized by a topological invariant generalizing the dimension of spheres. LYUSTERNIK (1930) used the notion of category. Here we shall use the notion of genus (see Section 4).

P r o o f of Proposition 4. We choose eigensolutions (u_i, λ_i) of the operator A as in Proposition 2, i.e. $Au_i = \lambda_i u_i$ and

$$u = \sum_{i=1}^{\infty} (u|u_i)u_i \quad \text{for all } u \in X \ .$$

Hence

$$2a(u) = (Au|u) = \sum_{i=1}^{\infty} \lambda_i (u|u_i)^2, \qquad ||u||^2 = \sum_{i=1}^{\infty} (u|u_i)^2 \ .$$

(I) Suppose that A has at least r positive eigenvalues counted according to their multiplicity. Without any loss of generality we can assume that

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_r > 0 \ , \ \lambda_r \geqq \lambda_j \quad \text{if } r > j \ .$$

If $L \in \mathcal{L}_s^+$, i.e. $L = S \cap X_k$, $\dim X_k \geqq s$, and $s \leqq r$, then we can choose $u \in L$ such that

$$(u|u) = 1 \ , \quad (u|u_i) = 0 \ , \quad i = 1,\ldots,s-1 \ .$$

Hence

$$2a(u) \leqq \lambda_s \sum_{i=s}^{\infty} (u|u_i)^2 = \lambda_s ||u||^2 = \lambda_s \ , \quad \text{i.e.} \quad \lambda_s^+ \leqq \lambda_s \ .$$

Set $L_s = S \cap \text{lin}\,\{u_1,\ldots,u_s\} \in \mathcal{L}_s^+$ and observe that $(u|u_i) = 0$ if $u \in L_s$ and $i > s$, i.e.

$$1 = ||u||^2 = \sum_{i=1}^{s} (u|u_i)^2 \quad \text{for all} \quad u \in L_s \ .$$

Hence

$$2a(u) \geqq \lambda_s \sum_{i=1}^{s} (u|u_i)^2 = \lambda_s \quad \text{for all} \quad u \in L_s \ .$$

Since $2a(u_s) = \lambda_s$, we obtain

$$\lambda_s^+ \geqq \min_{u \in L_s} 2a(u) = \lambda_s \ ,$$

i.e. $\lambda_s^+ = \lambda_s$ if $s = 1,\ldots,r$.

(II) Let $\lambda_m^+ > 0$, i.e. $\mathcal{L}_m^+ \neq 0$. Our proof will be complete if we can show that there exist at least m positive eigenvalues counted according to their multiplicity.

Suppose there exist only r positive eigenvalues of A with $r < m$. Without any loss of generality we can assume that

$$\lambda_1 \geqq \ldots \geqq \lambda_r > 0 \quad \text{and} \quad \lambda_j \leqq \lambda_{r+1} \leqq 0 \quad \text{if} \quad j > r \ .$$

Let $L \in \mathcal{L}_m^+ \subseteq \mathcal{L}_{r+1}^+$. As in part (I) of our proof we can choose $u \in L$ with $2a(u) \leqq \lambda_{r+1} \leqq 0$, i.e. $\min_{u \in L} a(u) \leqq 0$. This is a contradiction to $\min_{u \in L} a(u) > 0$ for all $L \in \mathcal{L}_m^+$, q.e.d.

4. The Genus of Symmetric Closed Sets Not Containing the Origin

Definition 3. Let X *be a real Banach space. A subset* $M \subseteq X$ *is called symmetric iff* $u \in M \Rightarrow -u \in M$.

A symmetric closed set $M \subseteq X - \{0\}$ *is said to have* <u>*genus*</u> n, *notation* $\gamma(M) = n$, *iff there exists*

(22) *an odd continuous map* $f : M \to \mathbb{R}^n - \{0\}$

and n *is the smallest natural number with this property.*

If there is no such natural number n, *we set* $\gamma(M) = +\infty$. *For the empty set* $\emptyset$ *we define* $\gamma(\emptyset) = 0$.

The following Proposition describes a crucial property of the genus.

Proposition 5.

Let $S = \{u \in X : ||u|| = 1\}$ *be the unit sphere in a real Banach space* X.

Then $\gamma(S) = \dim X$.

The proof is given, for example, in ZEIDLER(1978), p. 102. This proof is an easy consequence of Borsuk's antipodal theorem (see e.g. ZEIDLER (1976)).

REMARK 2. The definition of genus given here is that used by COFFMAN (1969). It is equivalent to an earlier definition given by KRASNOSEL'SKII (1952), (1956). This equivalence has been proved by RABINOWITZ (1973). The genus appears also in CONNER, FLOYD (1960), where it is called the coindex.

In Lyusternik's category approach to nonlinear eigenvalue problems (see LYUSTERNIK (1930), (1934), (1947)) an important role is played by the fact that real k-dimensional projective spaces P^k, obtained by identifying the antipodal points of a k-dimensional unit sphere, have the category $k + 1$ with respect to P^n. The proof of this deep topological result is due to SCHNIRELMAN (1930) (see also SCHWARTZ (1969), BROWDER (1970 a)).

It was Krasnosel'skii's idea to simplify proofs of the main results of the Lyusternik-Schnirelman theory by using the notion of genus. For example, the proof of Proposition 5 is extremely simpler than the proof of Schnirelman's theorem concerning the category of projective spaces. Furthermore, there is no need to pass to projective spaces when using the genus.

Now let us summarize some further properties of the genus.

Proposition 6. *Let* X *be a real Banach space. Suppose* M, M_1 *are symmetric closed subsets of* $X - \{0\}$.

Then :

1) $M_1 \subseteq M_2 \Rightarrow \gamma(M_1) \leq \gamma(M_2)$.

2) *If* $F : M_1 \to M_2$ *is a continuous odd map, then* $\gamma(M_1) \leq \gamma(M_2)$. *Furthermore, if* F *is an odd homeomorphism from* M_1 *onto* M_2, *then* $\gamma(M_1) = \gamma(M_2)$.

3) $\gamma(M_1 \cup M_2 \cup \ldots \cup M_k) \leq \gamma(M_1) + \ldots + \gamma(M_k)$, $1 \leq k < \infty$.

4) $\gamma(M_1) < \infty \Rightarrow \gamma(\overline{M_2 - M_1}) \geq \gamma(M_2) - \gamma(M_1)$.

5) M *is a compact set* $\Rightarrow \gamma(M) < \infty$.

6) *If* M_1 *is a compact set, then there exists an open symmetric set* U *such that* $M_1 \subseteq U$ *and* $\gamma(M_1) = \gamma(\overline{U})$.

7) $\gamma(M) \leq \dim X$.

8) *If* M *is a finite nonempty set, then* $\gamma(M) = 1$.

9) *Let* $X_1 \subseteq X$ *be an m-dimensional subspace with* $1 \leq m < \infty$. *Suppose* $P : X \to X_1$ *is a linear continuous projector onto* X_1.

Then : $\gamma(M) > m \Rightarrow M \cap (I - P)(X) \neq \emptyset$.

10) *If* $M_1 \cap M_2 = \emptyset$, *then* $\gamma(M_1 \cup M_2) = \max(\gamma(M_1), \gamma(M_2))$.

P r o o f . The proofs of 1) ... 9) are given, for example, in ZEIDLER (1978), p. 102 .

Let us prove 10). Since $\gamma(\emptyset) = 0$, the case $M_1 = \emptyset$ or $M_2 = \emptyset$ is trivial. Suppose $M_1, M_2 \neq \emptyset$. Then 1) implies

$$\max(\gamma(M_1), \gamma(M_2)) \leq \gamma(M_1 \cup M_2).$$

If $\gamma(M_1) = \infty$ or $\gamma(M_2) = \infty$, then 10) is proved.

Now, suppose $\gamma(M_i) = n_i$. Then there exist continuous odd maps

$$f : M_i \to \mathbb{R}^{n_i} - \{0\} , \; i = 1,2 .$$

Define

$$f(u) = f_i(u) \quad \text{if} \quad u \in M .$$

Hence

$$f : M_1 \cup M_2 \to \mathbb{R}^m - \{0\} , \quad m = \max (n_1, n_2),$$

i.e. $\gamma(M_1 \cup M_2) \leqq \max\{\gamma(M_1), \gamma(M_2)\}$, q.e.d.

The following Proposition 7 seems to be new.

Proposition 7 *(Krasnosel'skii).*

Let X *be a real Banach space with* $\dim X = \infty$ *. Set* $S = \{u \in X : ||u|| = 1\}$ *. Suppose* $M \subseteq S$ *is a compact set.*

Then, for every $m \in \mathbb{N}$ *, there exists a compact symmetric subset* $K_m \subseteq S - M$ *with* $\gamma(K_m) \geqq m$.

P r o o f . If X is a Hilbert space, Proposition 7 follows easily from orthogonal decomposition arguments (see ZEIDLER (1978), p. 113). If X is an arbitrary Banach space, then the proof is more sophisticated. The following proof based on a selection theorem of MICHAEL (1956) is due to KRASNOSEL'SKII (oral communication during his stay in Leipzig, December 1977). Figure 2 describes the main idea of the proof.

Step 1. A selection theorem of MICHAEL (1956).

Suppose :

i) T is a metric space, X is a real Banach space.

ii) There exists a lower semi-continuous map $\phi : T \to 2^X$, i.e. if $\tau \in T$, $u \in \phi(\tau)$, and $U(u) \subseteq X$ is a neighbourhood of u , then there exists a neighbourhood $V(\tau) \subseteq T$ of τ such that

$$\phi(\tau') \cap U(u) \neq \emptyset \quad \text{for all} \quad \tau' \in V(\tau) .$$

iii) For all $\tau \in T$, $\phi(\tau)$ is a nonempty closed convex subset of X .

Then there exists a continuous function $f : T \to X$ with $f(\tau) \in \phi(\tau)$ for all $\tau \in T$.

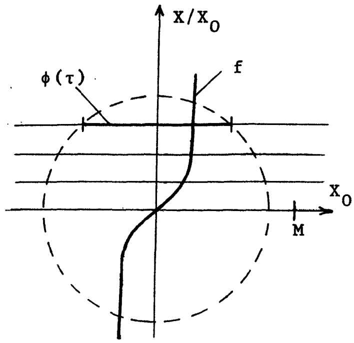

Fig. 2

Step 2. Since M is compact, there exists a linear finite-dimensional subspace X_0 of X such that $\mathrm{dist}(u,X_0) < \frac{1}{2}$ for all $u \in M$.

Step 3. Set $T = X/X_0$. The elements τ of the factor space X/X_0 are the sets $\tau = u_0 + X_0$. T is a Banach space under the norm

$$||\tau||_T = \inf\{||u|| : u \in \tau\} .$$

Hence

$$\begin{aligned}\inf\{||u_1 - u_2||_X : u_i \in \tau_i\} &= \inf\{||v|| : v \in \tau_1 - \tau_2\} \\ &= ||\tau_1 - \tau_2||_T .\end{aligned}$$

Step 4. Define $\phi : T \to 2^X$ by

$$\phi(\tau) = \{u \in \tau : ||u||_X \leqq (4/3)||\tau||\} .$$

For all $\tau \in T$, $\phi(\tau)$ is a nonempty closed and convex set.

We assert that ϕ is lower semi-continuous. Suppose this is not true. Then there exist elements $\tau \in T$, $u \in \phi(\tau)$, a neighbourhood $U(u)$ of u and a sequence (τ_n) such that

$$\tau_n \to \tau \text{ in } T \text{ as } n \to \infty \text{ and } \phi(\tau_n) \cap U(u) = \emptyset \text{ for all } n \in \mathbb{N} .$$

Choose a small number $\eta > 0$ and an element $v \in \tau$ with $v \in U(u)$ and $||v|| \leqq ((4/3) - \eta)||\tau||$. Furthermore, there exists a sequence of elements $u_n \in \tau_n$ such that

$$||v - u_n|| \leqq (4/3)||\tau - \tau_n||_T \text{ for all } n \in N .$$

Now, from $u_n \to v$, $||\tau_n|| \to ||\tau||$ $(n \to \infty)$ we obtain $||u_n|| \leqq (4/3)||\tau_n||$ if $n \geqq n_0$, i.e. $u_n \in \phi(\tau_n) \cap U(u)$ if $n \geqq n_1$

This contradicts $\phi(\tau_n) \cap U(u) = \emptyset$ for all $n \in \mathbb{N}$.

Step 5. The Michael selection theorem in Step 1 implies that there exists a continuous map $f_0 : T \equiv X/X_0 \to X$ such that $f_0(\tau) \in \phi(\tau) \subseteq \tau$ for all $\tau \in T$.

Define $f(\tau) = (f_0(\tau) - f_0(-\tau))/2$. Then $f : T \to X$ is a continuous odd map with $f(\tau) \in \tau$ for all $\tau \in T$.

f is also a homeomorphism. This follows from

$$||\tau_1 - \tau_2||_T \leq ||f(\tau_1) - f(\tau_2)||_X .$$

Furthermore, the construction of $f(\tau)$ yields

$$||\tau|| \leq ||f(\tau)|| \leq (4/3)||\tau|| .$$

Step 6. Define $g(\tau) = f(\tau)/||f(\tau)||$.

Set $S' = \{\tau \in T \equiv X/X_0 : ||\tau|| = 1\}$. Then $g : S' \to S$ is a continuous odd map.

Since $\dim X = \infty$, $\dim X_0 < \infty$, we have $\dim X/X_0 = \infty$. For every $m \in \mathbb{N}$, there exists an $(m-1)$-dimensional unit sphere $S_m \subseteq S'$, i.e. $\gamma(S_m) = m$. Hence $\gamma(g(S_m)) \geq m$ (see Proposition 5 and Proposition 6,2)).

We claim $g(S_m) \cap M \neq \emptyset$. Indeed,

$$\inf\{||v - u||_X : v \in \tau ,\ u \in X_0\} = ||\tau||$$

and $g(\tau) \in \tau/||f(\tau)||$. Hence

$$\operatorname{dist}(g(\tau), X_0) = ||\tau||/||f(\tau)|| \geq 3/4$$

for all $\tau : ||\tau|| = 1$. Now, $g(S_m) \cap M = \emptyset$ follows from $\operatorname{dist}(u, X_0) < \frac{1}{2}$ for all $u \in M$.

Thus we have constructed symmetric compact sets $K_m \equiv g(S_m) \subseteq S$ with $K_m \cap M = \emptyset$ and $\gamma(K_m) \geq m$, q.e.d.

5. The Main Theorems in Infinite-Dimensional Banach Spaces

We turn now to the nonlinear eigenvalue problem

$$(23) \qquad Au = \lambda Bu , \quad b(u) = \alpha \quad (u \in X,\ \lambda \in \mathbb{R})$$

where $\alpha > 0$ is a fixed real number. The condition $b(u) = \alpha$ normalizes the eigenvector u.

If (u,λ) is an eigensolution of (23) with $\langle Bu,u\rangle \neq 0$, then $\lambda = \langle Au,u\rangle / \langle Bu,u\rangle$.

Problem (23) generalizes the linear eigenvalue problem

$$Au = \lambda u, \quad b(u) = \alpha \quad (u \in X, \ \lambda \in \mathbb{R}) \tag{23'}$$

studied in Section 2.2 and Section 3, where X is a real separable infinite-dimensional Hilbert space, $A : X \to X^* \equiv X$ is a linear symmetric completely continuous operator $A \neq 0$, $B = I$ (identity) and

$$a(u) = 2^{-1}\langle Au,u\rangle \equiv 2^{-1}(Au|u), \quad b(u) = 2^{-1}(u|u) .$$

($(.|.)$ is the scalar product in X. We identify $X^* \equiv X$, i.e. $(u|v) = \langle u,v\rangle$.) In this special linear case all the hypotheses of the following two theorems are satisfied.

Theorem 1 *(Eigensolutions of the equation (23)).*

Suppose that the following conditions hold :

(24) X *is a real reflexive separable Banach space,* $\dim X = \infty$.

(25) $A,B : X \to X^*$ *are continuous odd potential operators with potentials* a, b; $a(0) = b(0) = 0$.

(26) A *is strongly continuous.*

(27) $Au = 0 \Rightarrow a(u) = 0$.

(28) B *is uniformly continuous on bounded sets of* X .

(29) B *satisfies the condition*

$(S)_1$: $u_n \rightharpoonup u$, $Bu_n \to v \Rightarrow u_n \to u$ $(n \to \infty)$.

(30) $u \neq 0 \Rightarrow \langle Bu,u\rangle > 0$.

(31) *The level set* $N_\alpha = \{u \in X : b(u) = \alpha\}$ *is bounded* *(e.g.* $b(u) \to +\infty$ *as* $||u|| \to \infty$) .

(32) $\inf_{u \in N_\alpha} \langle Bu,u\rangle > 0$.

(33) *For each* $u \neq 0$ *there exists a real number* $r(u) > 0$ *such that* $b(r(u)u) = \alpha$ *(i.e. each ray through the origin inter-*

sects N_α *; see Fig. 3).*

(34) $a \not\equiv 0$ *on* N_α .

Then, under all these assumptions, the following statements are true :

1) The level set N_α *is homeomorphic to the unit sphere. There exist real numbers* c,d *such that* $0 < c \leq ||u|| \leq d$ *on* N_α .

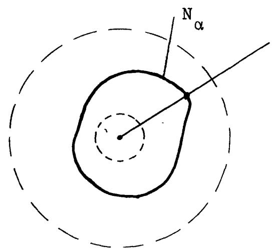

Fig. 3

2) The critical levels $\beta_m, \beta_m^\pm$.

Define, for all $m \in N$

(35)
$$\beta_m^\pm = \sup_{K \in \mathcal{K}_m} \min_{u \in K} |2a(u)| ,$$

($35_\pm$)
$$\pm\beta_m^\pm = \begin{cases} \sup_{K \in \mathcal{K}_m^\pm} \min_{u \in K} (\pm 2a(u)) , \\ 0 \quad \text{if } \mathcal{K}_m^\pm = \emptyset , \end{cases}$$

where

$$\mathcal{K}_m = \{K \subseteq N_\alpha : K \text{ compact, symmetric, } \gamma(K) \geq m\}$$

$$\mathcal{K}_m^\pm = \{K \in \mathcal{K}_m : \pm a(u) > 0 \text{ on } K\} .$$

Then $\mathcal{K}_m \neq \emptyset$ *for all* $m \in \mathbb{N}$ *and*

$$\beta_1 \geq \beta_2 \geq \ldots \geq 0, \quad \beta_1 > 0 ,$$
$$\pm\beta_1^\pm \geq \pm\beta_2^\pm \geq \ldots \geq 0 , \quad \pm\beta_i \leq \beta_i .$$

Furthermore, $\beta_m, \beta_m^\pm \to 0$ *as* $m \to \infty$.

3) Lyusternik's maximum-minimum principle generalizing Courant's maximum-minimum principle.

a) If $\beta_m > 0$ *, then the equation* (23) *has an eigensolution*

(36)
$$u_m \neq 0, \quad \lambda_m \neq 0, \quad |2a(u_m)| = \beta_m .$$

b) If $\pm\beta_m^\pm > 0$ *(+ or −), then the equation* (23) *has an eigensolution*

($36_\pm$)
$$u_m^\pm \neq 0, \ \lambda_m^\pm \neq 0, \quad 2a(u_m^\pm) = \beta_m^\pm .$$

If A *is homogeneous, i.e.* $Atu = t^\rho Au$ *for all* $t > 0, u \in X$ *and a fixed* $\rho \geq 0$ *, then*

$$a(u) \equiv \int_0^1 <Atu,u>\, dt = (1+\rho)^{-1}<Au,u> \quad , \quad i.e.$$

$$\pm\lambda_m^{\pm} = \pm(1+\rho)\beta_m^{\pm}/2<Bu_m,u_m> > 0 .$$

4) <u>The global multiplicities</u> $\chi(a,N_\alpha)$, $\chi_\pm(a,N_\alpha)$. *Observe that*

$\beta_m > 0 \Leftrightarrow$ *there exists* $K \in \mathcal{R}_m$ *with* $a(u) \neq 0$ *on* K ,

$\pm\beta_m^{\pm} > 0 \Leftrightarrow$ *there exists* $K \in \mathcal{R}_m$ *with* $\pm a(u) > 0$ *on* K .

Define

$$\chi = \sup \{m : \beta_m > 0\} \quad ,$$

$$\chi_\pm = \begin{cases} \sup \{m : \pm\beta_m^{\pm} > 0\} \\ 0 \quad \text{if} \quad \beta_1^{\pm} = 0 . \end{cases}$$

Then :

a) $\chi = \max\{\chi_+,\chi_-\} \geqq 1$.

b) If the set $\{u \in N_\alpha : a(u) = 0\}$ *is compact, then* $\chi = \chi_+ = \infty$ *or* $\chi = \chi_- = \infty$.

c) If $a(u) \neq 0$ *on* N_α *(e.g.* $a(u) = 0 \Leftrightarrow u = 0$ *), then* $\chi = \chi_+ = \infty$, $\chi_- = 0$ *or* $\chi = \chi_- = \infty$, $\chi_+ = 0$.

d) If X_0 *is a linear subspace of* X *and* $a(u) \neq 0$ *on* $N_\alpha \cap X_0$ *, then* $\chi \geqq \dim X_0$.

If $\pm a(u) > 0$ *on* $N_\alpha \cap X_0$ *(+ or -), then* $\chi_\pm \geqq \dim X_0$.

5) <u>Existence of an infinite number of distinct eigenvectors on</u> N_α .

a) If $\chi = \infty$ *then, for all* $m \in \mathbb{N}$ *, the equation* (23) *has an eigensolution* $(u_m, \lambda_m) : u_m \in N_\alpha, \lambda_m \neq 0, |2a(u_m)| = \beta_m$.

b) If $\chi_\pm = \infty$ *(+ or -) then, for all* $m \in \mathbb{N}$ *, the equation* (23) *has an eigensolution* $(u_m^{\pm}, \lambda_m^{\pm}) : u_m^{\pm} \in N_\alpha$, $\lambda^{\pm}{}_m \neq 0$, $2a(u_m^{\pm}) = \beta_m^{\pm}$.

Since β_m, $\beta_m^{\pm} \to 0$ *as* $m \to \infty$*, all the sequences* (u_m), $(u_m^{\pm})$ *contain an infinite number of distinct eigenvectors on* N_α .

6) <u>Existence of an infinite number of distinct eigenvalues</u>.

Suppose $a(u) = 0 \Rightarrow <Au,u> = 0$.

Let $(\tilde{u}_m, \tilde{\lambda}_m)$ *be an arbitrary sequence of eigensolutions of the equation* (23) *with* $a(\tilde{u}_m) \to 0$ *as* $m \to \infty$. *Then* $\tilde{\lambda}_m \to 0$ *as* $m \to \infty$.

This together with the fact that β_m, $\beta_m^{\pm} \to 0$ *as* $m \to \infty$ *implies* λ_m, $\lambda_m^{\pm} \to 0$ *as* $m \to \infty$ *for the sequences* (u_m, λ_m), $(u_m^{\pm}, \lambda_m^{\pm})$ *in 5a), 5b). This means that if* $\chi \equiv \max(\chi_+, \chi_-) = \infty$, *then the equation* (23) *has an infinite number of distinct eigenvalues.*

7) <u>*Weak convergence of the eigenvectors*</u>.

Suppose $a(u) = 0 \Longleftrightarrow u = 0$.

Then $\chi = \infty$. *Furthermore, let* $(\tilde{u}_m)$ *be an arbitrary sequence on* N_α *with* $a(\tilde{u}_m) \to 0$; *then* $\tilde{u}_m \rightharpoonup 0$, $\langle A\tilde{u}_m, \tilde{u}_m\rangle \to 0$ *as* $m \to \infty$.

This together with the fact that β_m, $\beta_m^{\pm} \to 0$ *as* $m \to \infty$ *implies* u_m, $u_m^{\pm} \rightharpoonup 0$, $\lambda_m, \lambda_m^{\pm} \to 0$ *as* $m \to \infty$ *for the sequences* (u_m, λ_m), $(u_m^{\pm}, \lambda_m^{\pm})$ *in 5a), 5b).*

8) <u>*Existence of at least one eigenvalue if* A, B *are not necessarily odd*</u>.

If there exists an element $u_0^{\pm} \in N_\alpha$ *(+ or −) with* $\pm a(u_0^{\pm}) > 0$, *then the equation (23) has an eigensolution*

$$u^{\pm} \neq 0 , \lambda^{\pm} \neq 0 , \pm a(u^{\pm}) = \max_{u \in N_\alpha} \pm a(u) .$$

(Here we do not suppose that A , B *are odd.)*

<u>*Corollary 1*</u> *(multiplicity of the critical levels* $\beta_m, \beta_m^{\pm}$*)* .

Under the assumptions made in Theorem 1 it holds :

a_1*)* *If* $\beta_m = \beta_{m+1} = \cdots\cdots = \beta_{m+p-1} > 0$, $p \geq 1$, *then* $\gamma(\{u \in N_\alpha : u$ *eigenvector in* (23), $|2a(u)| = \beta_m\}) \geq p$.

a_2*)* *The equation* $Au = \lambda Bu$, $b(u) = \alpha$ *(*$\alpha > 0$ *fixed) has at least* $\chi = \max(\chi_+, \chi_-)$ *distinct pairs of eigenvectors* $(u, -u)$ *with nonzero eigenvalues obtained by the maximum-minimum principle* (35) .

b_1*)* *If* $\pm \beta_m^{\pm} = \pm\beta_{m+1}^{\pm} = \ldots = \pm\beta_{m+p-1}^{\pm} > 0$, $p \geq 1$, *(+ or −) , then* $\gamma(\{u^{\pm} \in N_\alpha : u^{\pm}$ *eigenvector in* (23), $2a(u^{\pm}) = \beta_m^{\pm}\}) \geq p$.

b_2*)* *The equation* $Au = \lambda Bu$, $b(u) = \alpha(\alpha > 0$ *fixed) has at least* $\chi_+ + \chi_-$ *distinct pairs of eigenvectors* $(u, -u)$ *with nonzero eigen-*

values obtained by the maximum-minimum principle $(35_\pm)$.

The purpose of the next theorem is to weaken the continuity assumptions upon B.

Theorem 2.

Let all the assumptions made in Theorem 1 hold except of the following changes :

(28') *Replace* (28) (B *is uniformly continuous on bounded sets) by the weaker assumption :* B *is bounded.*

(29') *Replace* (29) (B *satisfies the condition* $(S)_1$) *by the stronger assumption :*

$$(S)_0 : u_n \rightharpoonup u,\ Bu_n \rightharpoonup v,\ \langle Bu_n, u_n\rangle \to \langle v,u\rangle \Rightarrow u_n \to u \quad (n \to \infty)$$

(27') *Replace* (27) $(Au = 0 \Rightarrow a(u) = 0)$ *by the stronger assumption :*

$$a(u) = 0 \Leftrightarrow \langle Au, u\rangle = 0 .$$

Then all the statements of Theorem 1 are true.

An important special case of Theorems 1, 2 will be considered in Section 8.

REMARK 3. Theorem 1, points 1), 3a), 5a) and Corollary 1, a_1) with respect to the critical levels

$$\beta_m = \sup_{K\in \mathcal{R}_m} \min_{u\in K} |2a(u)|$$

have been proved by AMANN (1972). On Amann's paper X is supposed to be a uniformly convex Banach space.

Since $\chi = \max(\chi_+, \chi_-)$ by Theorem 1, 4a), Corollary 1, b_2) gives in general more eigenvectors than Corollary 1, a_2). This is the reason why we have introduced the critical levels $\beta_m^\pm$.

In Corollary 1' of Section 7 we shall show that the multiplicity result in Corollary 1, b_2) is a straightforward generalization of the corresponding results for linear operators.

REMARK 4. Under the additional definiteness assumption

(37) $$Au = 0 \Longleftrightarrow u = 0, \quad a(u) > 0 \text{ if } u \neq 0 ,$$

Theorem 1, points 1), 2), 3), 5), 7) have been proved by FUČÍK, NEČAS (1972a), for Banach spaces equipped with the so-called usual structure.

In the case (37) it holds $\beta_m = \beta_m^+$, $\beta_m^- = 0$, $\chi_+ = \infty$. DANCER (1976) has shown that every real reflexive separable Banach space has the usual structure.

Furthermore, under the assumption (37), Corollary 1,a_1) has been proved also by FUČÍK, NEČAS (1972a), the notion of genus $\gamma(.)$ being replaced by ord(.) (order of a set). However, ord(.) gives not so good multiplicity results as $\gamma(.)$. Suppose, for example, that there exist only two nonzero critical levels $\beta_1 = \beta_2$, i.e. $\chi = 2$. Then the paper of FUČÍK, NEČAS implies

$$\text{ord}\ (\{u \in N_\alpha : Au = \lambda Bu\}) \geqq 2 .$$

On the other hand, $\text{ord}\ \{u_1, -u_1\} = 2$. Therefore we cannot conclude that there exist at least the distinct pairs of eigenvectors. In contrast to this, $\gamma(M) \geqq 2$ implies that M contains an infinite number of distinct pairs $(u, -u)$.

REMARK 5. Theorem 2 is closely related to general theorems due to BROWDER (1968), (1970a), (1970b). However, in these papers it is assumed that

$$(38) \qquad \langle Au, u\rangle = 0 \Longleftrightarrow u = 0$$

Since $X - \{0\}$ is connected, it follows immediately from (38) that $\langle Au,u\rangle$ has the same sign for all $u \neq 0$. Hence

$$a(u) \equiv \int_0^1 \langle Atu,u\rangle \, dt = 0 \Longleftrightarrow u = 0 .$$

Theorem 2 removes the condition (38). In case of a linear operator A the condition (38) means $Au = 0 \Longleftrightarrow u = 0$ (see Theorem 1', 7')).

6. Sketched Proofs of the Main Theorems

6.1. Proof of Theorem 1, 1), 4), 5), 6), 7).

Proof of 1) Set $\phi(u) = r(u)u$. It is easily seen that ϕ is an odd homeomorphism from the unit sphere S onto the level set N_α (see e.g. ZEIDLER (1978), p. 108).

Proof of 4a) Obviously, $\chi_+, \chi_- \leqq \chi$. Now, suppose $\beta_m > 0$. This is equivalent to the existence of a symmetric compact set $K \in \mathcal{K}_m$ with $a(u) \neq 0$ on K. Define

$$K^\pm = \{u \in K : \pm a(u) > 0\} .$$

$K^\pm$ is symmetric, compact, $K^+ \cap K^- = \emptyset$, and by Proposition 6,10)

$$\gamma(K) = \max(\gamma(K^+), \gamma(K^-)), \quad \gamma(K) \geqq m .$$

Hence $K^+ \in \mathcal{K}_m^+$ or $K^- \in \mathcal{K}_m^-$, i.e. $\beta_m^+ > 0$ or $-\beta_m^- > 0$.

Thus $\max(\chi_+, \chi_-) \geqq \chi$.

Proof of 4b) Set $N = \{u \in N_\alpha : a(u) = 0\}$. Suppose Φ is an odd homeomorphism from the unit sphere S in X onto N_α.

By the hypothesis N is compact, i.e. $\phi^{-1}(N)$ is compact in S. Proposition 7 yields that for each $m \in N$ there exists a compact symmetric subset $K_m \subseteq S - \phi^{-1}(N)$ with $\gamma(K_m) \geqq m$, i.e. $\phi(K_m)$ is a compact symmetric subset of N_α with $\phi(K_m) \cap N = \emptyset$ and $\gamma(\phi(K_m)) \geqq m$. Hence $\min_{u \in \phi(K_m)} |2a(u)| > 0$, i.e. $\beta_m > 0$. Thus $\chi = \infty$.

Proof of 4c) N_α is connected. Hence $a(N_\alpha)$ is connected, too. Thus $a(u) \neq 0$ on N_α implies that $a(u)$ has the same sign for all $u \in N_\alpha$. Suppose $a(u) > 0$ on N_α. Then $\mathcal{K}_1^- = \emptyset$, i.e. $\chi_- = 0$.

It follows from 4a), 4b) that $\chi = \chi_+ = \infty$.

Proof of 4d) Suppose $a(u) \neq 0$ on $K = N_\alpha \cap X_0$. Let $\dim X_0 < \infty$. K is compact symmetric set. $\phi^{-1}(K)$ is a sphere in X_0. By Proposition 5, $\gamma(\phi^{-1}(K)) = \dim X_0$. Hence $\gamma(K) = \dim X_0$ by Proposition 6,2). Thus $\min_{u \in K} |2a(u)| > 0$, i.e. $\beta_m > 0$, $m = \dim X_0$.

Therefore $\chi \geqq m$. Similarly we obtain the other assertions in 4d).

Proof of 5) Compare Theorem 1,3) and the definition on $\chi, \chi_{\pm}$ in Theorem 1,4).

Proof of 6) Suppose $A\tilde{u}_m = \tilde{\lambda}_m B\tilde{u}_m$, $\tilde{u}_m \in N_\alpha$ for all $m \in \mathbb{N}$ and $a(\tilde{u}_m) \to 0$ as $m \to \infty$. Let $(\tilde{\lambda}_{m'})$ be an arbitrary subsequence of $(\tilde{\lambda}_m)$. Since $(\tilde{u}_{m'})$ is bounded, we can choose a subsequence $(\tilde{u}_{m''})$ with $\tilde{u}_{m''} \rightharpoonup u$ as $m'' \to \infty$, i.e. $a(u) = 0$. Hence $\langle Au,u\rangle = 0$, i.e. $\langle A\tilde{u}_{m''}, \tilde{u}_{m''}\rangle \to \langle Au,u\rangle = 0$. Therefore

$$\tilde{\lambda}_{m''} = \langle A\tilde{u}_{m''},\tilde{u}_{m''}\rangle \;/\; \langle B\tilde{u}_{m''}, \tilde{u}_{m''}\rangle \to 0 \quad \text{as} \quad m \to \infty .$$

Thus we have shown that the sequence $(\tilde{\lambda}_m)$ has only one accumulation point, i.e. $\tilde{\lambda}_m \to 0$ as $m \to \infty$.

Proof of 7) Proceed similarly as in the proof of 6) .

6.2. Sketched proofs of Theorem 1, 2), 3), 8) and Corollary 1.

All the assertions of Theorem 1, 2), 3), 8) and Corollary 1 have been proved in ZEIDLER (1978), p. 112 with respect to the critical levels β_m . The corresponding proofs for $\beta_m^{\pm}$ work similarly.

The proofs in ZEIDLER (1978) combine various ideas taken from the papers of AMANN (1972), FUČÍK, NEČAS (1972a), FUČÍK, NEČAS, SOUČEK, SOUČEK (1973) and DANCER (1976).

A sketched proof of Theorem 1, 2). The proof for $\beta_m \to 0$ as $m \to \infty$ in ZEIDLER (1978) is based on the following

Lemma 1 (DANCER). *Let* X *be a reflexive separable infinite-dimensional Banach space.*

Then, for each $n \in \mathbb{N}$ *, there exist continuous odd operators* $P_n : X \to X$ *with finite dimensional ranges such that*

$$u_n \rightharpoonup u \Rightarrow P_n u_n \rightharpoonup u \quad (n \to \infty) .$$

This lemma shows that every reflexive separable Banach space has the usual structure in the sense of FUČÍK, NEČAS (1972a). Hence we obtain by a similar argument as in FUČÍK, NEČAS (1972a) that

(39) $$\beta_m \to 0 \quad \text{as} \quad m \to \infty \quad .$$

Now, the convergence $\beta_m^{\pm} \to 0$ as $m \to \infty$ follows from $\pm\, \beta_m^{\pm} \leqq \beta_m$.

<u>A sketched proof of Theorem 1,3)</u>. The proof of this crucial assertion given in ZEIDLER (1978), p. 112 is based on the following Lemmas.

<u>Lemma 2</u> (TROYANSKI (1971); *an equivalent norm on* X).*In every reflexive Banach space* X *we can introduce an equivalent norm* $||\cdot||_1$ *such that* X , X^* *are locally uniformly convex.*

This implies that the duality map $J : X^* \to X^{**} \equiv X$ is continuous with respect to $||\cdot||_1$ (see e.g. ZEIDLER (1978), p. 146).

<u>Lemma 3</u> (*curves on the level set* N_α). *Set*

$$Du = Au - (\langle Au, u\rangle / \langle Bu,u\rangle)\ Bu \ ,$$

$$Eu = JDu - (\langle Bu,JDu\rangle / \langle Bu,u\rangle)\ u \ .$$

Define curves on N_α *by* $g(t, u) = r(u + tEu)\ (u + tEu)$.

Then the maps $g, g_t : [-t, t_o] \times N_\alpha \to N_\alpha$ *are bounded, continuous, and* $t \mapsto g(t, u)$, $t \mapsto g_t(t, u)$ *are equicontinuous on* $[-t_o, t_o]$ *with respect to all* $u \in N_\alpha$.

Furthermore, $g(0, u) = u$, $\langle Au, g_t(0, u)\rangle = ||Du||^2$.

<u>REMARK 6.</u> This lemma has been proved by AMANN (1972). Since we do not suppose that X is uniformly convex, we cannot prove, as in Amann's paper, that g, g_t are uniformly continuous on $[-t_o, t_o] \times N_\alpha$ (see Remark 7).

<u>Lemma 4</u> (*the critical sets* L_σ, $L_\sigma^{\pm}$).

Set $L_\sigma = \{u \in N_\alpha : |\ |2a(u)| - \beta_m| \leqq \sigma \ ,\ ||Du||^2 \leqq \sigma\}$.

Suppose $\beta_m > 0$.

Then for each $\sigma > 0$ *there exists* $\mu(\sigma) > 0$ *such that*

$$\min_{u\in K} |2a(u)| > \beta_m - \mu(\sigma) \Rightarrow L_\sigma \cap K \neq \emptyset$$

for all $K \in \mathcal{R}_m$.

2) Set $L_\sigma^\pm = \{u \in N_\alpha : |2a(u) - \beta_m^\pm| \leq \sigma , ||Du||^2 \leq \sigma \}$.

Suppose $\pm\beta_m^\pm > 0$ *(+ or −) .*

Then for each $\sigma > 0$ *there exists* $\mu^\pm(\sigma) > 0$ *such that*

$$\min_{u\in K} (\pm 2a(u)) > \pm\beta_m^\pm - \mu^\pm(\sigma) \Rightarrow L_\sigma^\pm \cap K \neq \emptyset$$

for all $K \in \mathcal{R}_m^\pm$.

REMARK 7. This crucial lemma is related to the Main Lemma in the paper of FUČÍK, NEČAS (1972a).

The proof of Lemma 4, 1) is based on the careful deformation argument along the curves $t \mapsto g(t, u)$, due to AMANN (1972). The situation described in Remark 6 complicates the proof.

Lemma 1, 2) follows by a similar argument.

Lemma 5 *(local Palais-Smale condition;* AMANN(1972)).*Let* (u_n) *be a sequence on* N_α . *Suppose* $Du_n \to 0$, $a(u_n) \to \beta$, $\beta \neq 0$ $(n \to \infty)$.

Then there exists a convergent subsequence $(u_{n'})$ *with* $u_{n'} \to u$ *as* $n \to \infty$ *and* $Du = 0$ *, i.e.* u *is an eigenvector of* (23).

Now, Theorem 1, 3) is an easy consequence of Lemma 4 and Lemma 5.

Sketched proof of Theorem 1,8). This assertion follows by a slight modification of Lemma 4 and by Lemma 5 (see ZEIDLER (1978), p. 114).

Sketched proof of Corollary 1. The assertion a_1), b_1) follow from Lemma 4, Lemma 5 and Proposition 6, 6) by an argument due to LYUSTERNIK (1930) (see ZEIDLER (1978), p. 114).

Now a_2) and b_2) in Corollary 1 are easy consequences of a_1), b_1). Observe that $\gamma(M) \geq 2$ implies that M is an infinite set.

6.3. A sketched proof of Theorem 2.

The proof is based on a Galerkin procedure due to BROWDER (1968), (1970a), (1970b). In ZEIDLER (1978), p. 116 it is shown that this procedure converges also if we replace the definiteness condition (38) ($\langle Au, u\rangle = 0 \Leftrightarrow u = 0$) by weaker condition $\langle Au, u\rangle = 0 \Leftrightarrow a(u) = 0$.

It is important that we can prove β_m, $\beta_m^{\pm} \to 0$ as $m \to \infty$ without using the uniform continuity of B on bounded sets.

7. Restriction to the Linear Case

To check the quality of the statements for nonlinear operators made in Theorem 1 and Corollary 1 let us consider the special case of linear operators. The following Theorem 1' and Corollary 1' show that our results in Section 5 are maximal in a certain sense (Theorem 1, X) corresponds to Theorem 1', x')).

For fixed $\alpha > 0$, consider the equation

$$(40) \qquad Au = \lambda u , \quad b(u) = \alpha \quad (u \in X, \lambda \in \mathbb{R}).$$

Define $N(A) = \{u \in X : Au = 0\}$.

Theorem 1'

Suppose :

i) X *is a real separable infinite-dimensional Hilbert space with a scalar product* $(\cdot|\cdot)$.

ii) $A : X \to X$ *is a linear completely continuous symmetric operator,* $A \neq 0$.

iii) Set $B = I$ *(identity),* $a(u) = 2^{-1}(Au|u)$, $b(u) = 2^{-1}(u|u)$.

Then :

0) All the hypotheses made in Theorem 1 and Theorem 2 are satisfied.

2'), 3'). Let $\lambda_m^{\pm}$ *be defined as in* (21) *by Courant's maxi-*

mum-minimum principle. Then $\beta_m^{\pm} = 2\alpha\lambda_m^{\pm}$.

The set of all $\lambda_m^{\pm} \neq 0$ *is equal to the set of all nonzero eigenvalues of* A *counted according to their multiplicity (see Proposition 4).*

4') χ_+ and χ_- are equal respectively to the number of all positive and negative eigenvalues of A *counted according to their multiplicity.*

a') $\chi \equiv \max(\chi_+, \chi_-) \geqq 1$.

b') $N \equiv \{u \in N_\alpha : a(u) = 0\}$ *is compact*

$$\Leftrightarrow \begin{cases} \dim N(A) < \infty, & \chi_+ = \infty, \quad \chi_- = 0 \\ \textit{or } \dim N(A) < \infty, & \chi_- = \infty, \quad \chi_+ = 0. \end{cases}$$

If N *is compact, then* $N = N_\alpha \cap N(A)$.

c') See 7').

6') $\langle Au, u\rangle = 0 \Leftrightarrow a(u) = 0$; $\lambda_m^{\pm} \to 0$ *as* $m \to \infty$.

7') The following conditions are equivalent :

a) $a(u) \neq 0$ *on* N_α,

b) $a(u) = 0 \Leftrightarrow u = 0$,

c) $N(A) = \{0\}$, $\chi_+ = \infty$, $\chi_- = 0$

or $N(A) = \{0\}$, $\chi_- = \infty$, $\chi_+ = 0$,

d) $\chi = \infty$, *and if* $(\tilde{u}_m)$ *is an arbitrary sequence on* N_α *with* $a(\tilde{u}_m) \to 0$ *as* $m \to \infty$, *then* $\tilde{u}_m \rightharpoonup 0$ *and* $\langle Au_m, u_m\rangle \to 0$ *as* $m \to \infty$.

<u>*Corollary 1'*</u>. *b_1') Let* $p \geqq 1$ *be the multiplicity of* $\pm\lambda_m^{\pm} > 0$ *(+ or −). Then* $\pm\beta_m^{\pm} = \pm\beta_{m+1}^{\pm} = \ldots = \pm\beta_{m+p-1}^{\pm} > 0$ *and* $\gamma(\{u \in N_\alpha : u^{\pm}$ *eigenvector in* (40), $2a(u^{\pm}) = \beta_m^{\pm}\}) = p$.

b_2') The equation (40) *has at least* $\chi_+ + \chi_-$ *distinct pairs of eigenvectors* (u, −u) *belonging to nonzero eigenvalues.*

There exist operators A *such that the equation* (40) *has exactly* $\chi_+ + \chi_-$ *distinct pairs of eigenvectors* (u, −u) *belonging to nonzero eigenvalues.*

Proof of 0). This is easy to check.

Proof of 2'), 3'). Without any loss of generality we can assume $\alpha = \frac{1}{2}$, i.e. $N_\alpha = \{u \in X : ||u|| = 1\}$.

(I) First we shall show : If $\pm\beta_m^{\pm} > 0$ (+ or −), then $\beta_m^{\pm}$ is an eigenvalue of A .

Indeed, Theorem 1, 3) says that there exists an eigensolution $Au_m^{\pm} = \lambda_m u_m^{\pm}$, $b(u_m^{\pm}) = \frac{1}{2}$ with $2a(u_m^{\pm}) = \beta_m^{\pm}$, i.e.

$$2a(u_m^{\pm}) = (Au_m^{\pm} | u_m^{\pm}) = \lambda_m . \text{ Hence } \beta_m^{\pm} = \lambda_m .$$

(II) Consider $\mathscr{L}_m^{\pm}$ introduced in (20). By Proposition 5, $\mathscr{L}_m^{\pm} \subseteq \mathscr{K}_m^{\pm}$, i.e. $\pm\lambda_m^{\pm} \leqq \pm\beta_m^{\pm}$ for all $m \in \mathbb{N}$.

(III) Let $\beta_1^+ > 0$. Then $\lambda_1^+ \geqq \beta_1^+$ by (I) and Proposition 4, 1). Thus $\lambda_1^+ = \beta_1^+$ by (II).

(IV) Now, let us prove $\lambda_2^+ = \beta_2^+$. Indeed, $\lambda_2^+ \leqq \beta_2^+ \leqq \beta_1^+ = \lambda_1^+$ by (II), (III). If $\lambda_2^+ = \lambda_1^+$, then $\beta_2^+ = \lambda_2^+$.

Next suppose $\lambda_2^+ < \lambda_1^+$. Proposition 4, 1) and (I) yield either $\beta_2^+ = \lambda_1^+$ or $\beta_2^+ = \lambda_2^+$.

Let $\beta_2^+ = \lambda_1^+$. Then $\beta_1^+ = \beta_2^+ > 0$. Corollary 1 implies

$$\gamma(\{u : Au = \lambda_1^+ u, ||u|| = 1\} \geqq 2 .$$

Proposition 5 shows that the multiplicity of λ_1^+ is at least $p = 2$. However, from $\lambda_1^+ < \lambda_2^+$ and Proposition 4,2) we conclude that the multiplicity of λ_1^+ is equal to $p = 1$. This is a contradiction.

(V) By induction we obtain 2'), 3') .

Proof of 4'). See 2').

Proof of 4'b'). $\Rightarrow$: Let N be compact. Then $\dim N(A) < \infty$, since $N(A) \cap N_\alpha \subseteq N$. Hence $\chi_+ = \infty$ or $\chi_- = \infty$ by 4').

Let $\chi_+ = \infty$. Prove $\chi_- = 0$.

Suppose $\chi_+ = \infty$, $\chi_- > 0$. Choose on orthonormal sequence of eigenvectors $u_1^-, u_1^+, u_2^+, \ldots$ belonging to the eigenvalues $\lambda_1^- < 0$, $\lambda_1^+ \geqq \lambda_2^+ \geqq \ldots > 0$, respectively. Set

$$u_m = \sqrt{2\alpha\,\lambda_m^+}\,\Big(\frac{u_m^+}{\sqrt{\lambda_m^+}} + \frac{u_1^-}{\sqrt{|\lambda_1^-|}}\Big)\Big(1 + \frac{\lambda_m^+}{|\lambda_1^-|}\Big)^{-\frac{1}{2}}, \quad m = 1,2,\ldots .$$

Observe that $(u_1^-|u_m^+) = 0$, $||u_m||^2 = 2\alpha$, $a(u_m) = 0$, i.e. $u_m \in N$ for all $m \in \mathbb{N}$. N is compact by hypothesis. Thus (u_m) contains a convergent subsequence $u_{m'} \to u$ as $m' \to \infty$. Since $\lambda_m^+ \to 0$ it follows that $u_{m'}^+ \to u/\sqrt{2\alpha}$ as $m' \to \infty$. This contradicts $||u_m^+ - u_n^+||^2 = 2$ if $m \neq n$.

$\Leftarrow$: Let $\dim N(A) < \infty$, $\chi_+ = \infty$; $\chi_- = 0$. Then $\lambda_1^- = 0$, i.e. $a(u) \geqq 0$ for all $u \in X$. Hence $N = \{u \in N_\alpha : Au = 0\}$, i.e. $N = N_\alpha \cap N(A)$. Thus N is compact.

<u>Proof of 7')</u>. a) $\Longleftrightarrow$ b) $\Longleftrightarrow$ c). See 4'b').

a) $\Longrightarrow$ d). See Theorem 1, 7).

d) $\Longrightarrow$ a). Let $\chi = \infty$, i.e. $\chi_+ = \infty$ or $\chi_- = \infty$. Suppose $\chi_+ = \infty$. According to c) we have to prove $N(A) = \{0\}$ and $\chi_- = 0$.

(VI) First assume $N(A) \neq \{0\}$. i.e. there exists an element $\tilde{u}$ with $A\tilde{u} = 0$, $||\tilde{u}|| = 1$. Choose an orthonormal system of eigenvectors $u_1^+, u_2^+, \ldots$ belonging to the eigenvalues $\lambda_1^+, \lambda_2^+, \ldots,$ respectively. Bessel's inequality yields

$$\sum_{i=1}^{\infty} (u|u_m^+)^2 \leqq ||u||^2 \text{ , i.e. } u_m^+ \rightharpoonup 0 \text{ as } m \to \infty.$$

Set $\tilde{u}_m = \sqrt{\alpha}\,(\tilde{u} + u_m^+)$ if $m \geqq 2$. Observe that $(\tilde{u}|u_m^+) = 0$ for all $m \geqq 2$, i.e. $\tilde{u}_m \in N_\alpha$ and $2a(\tilde{u}_m) = 2\alpha a(u_m^+) = \alpha\lambda_m^+ \to 0$ as $m \to \infty$. By the hypothesis d) we obtain $\tilde{u}_m \rightharpoonup 0$. This contradicts $\tilde{u}_m \rightharpoonup \sqrt{\alpha}\tilde{u}$ as $m \to \infty$.

(VII) Secondly, assume $\chi_- > 0$, i.e. $\lambda_1^- < 0$. Set

$$\tilde{u} = \Big(\frac{u_1^+}{\sqrt{\lambda_1^+}} + \frac{u_1^-}{\sqrt{|\lambda_1^-|}}\Big)\Big(\frac{1}{\lambda_1^+} + \frac{1}{|\lambda_1^-|}\Big)^{-\frac{1}{2}}$$

and proceed as in (VI).

Proof of Corollary 1'b_1'). The set of all eigenvectors on N_α with $\alpha = \frac{1}{2}$ belonging to the eigenvalue $\pm \lambda_m^{\pm} > 0$ is equal to the unit sphere S_p in the p-dimensional eigenspace. Now, b_1') follows from Proposition 5.

Proof of Corollary 1' b_2'). Compare Theorem 1', 2').

If A has only simple nonzero eigenvalues and $Au = 0 \Leftrightarrow u = 0$, then A has exactly $\chi_+ + \chi_-$ pairs of eigenvectors (u, -u), q.e.d.

8. An Important Special Case of the Main Theorems

In this Section we shall restrict our main theorems to a special situation. This will be useful for applications to partial equadifferential equations in the next Section.

Proposition 8.

Suppose:

(41) X *is a real reflexive separable infinite-dimensional Banach space.*

(42) $A, B : X \to X^*$ *are potential operators with potentials* a, b *and* $a(0) = b(0) = 0$.

(43) A *is strongly continuous.*

(44) $\langle Au, u\rangle = 0 \Leftrightarrow a(u) = 0$.

(45) $a \not\equiv 0$ *on* $N_\alpha = \{u \in X : b(u) = \alpha\}$ *($\alpha > 0$ fixed).*

(46) $B = B_1 + B_2$, $B_i : X \to X^*$.

(47) B_1 *is bounded, continuous, uniformly monotone and* $B_1(0) = 0$.

(48) B_2 *is strongly continuous and* $\langle B_2u, u\rangle \geq 0$ *for all* $u \in X$.

Then:

1) The equation

(49) $$Au = \lambda Bu, \quad b(u) = \alpha$$

has an eigensolution $u \neq 0$, $\lambda \neq 0$.

2) Suppose that A, B *are odd. In this case it holds:*

a) If $\chi = \infty$ *, then for every* $m \in \mathbb{N}$ *there exists an eigen-solution* (u_m, λ_m) *of* (49) *with* $u_m \neq 0$, $\lambda_m \neq 0$ *and* $\lambda_m \to 0$ *as* $m \to \infty$ *, i.e. there exists an infinite number of distinct eigenvectors and eigenvalues.*

(If the set $\{u \in N_\alpha : a(u) = 0\}$ *is compact, or if there exists a linear infinite-dimensional subspace* $X_0 \subseteq X$ *and* $a(u) \neq 0$ *on* $X_0 \cap N_\alpha$ *, then* $\chi = \infty$ *.)*

b) If $a(u) = 0 \Longleftrightarrow u = 0$ *(e.g.* $\langle Au, u\rangle > 0$ *if* $u \neq 0$) *, then* $\chi = \infty$ *and* $u_m \rightharpoonup 0$ *as* $m \to \infty$ *in a).*

c) If B_1 *is uniformly continuous on bounded sets, then the equation* (49) *has at least* $\chi_+ + \chi_-$ *distinct pairs of eigenvectors* $(u, -u)$ *belonging to nonzero eigenvalues.*

(If there exists a linear subspace $X_0 \subseteq X$ *with* $\pm a(u) > 0$ *on* $N_\alpha \cap X_0$ *(+ or −), then* $\chi_\pm \geqq \dim X_0$ *.)*

<u>*Corollary 2*</u>. *Let* $A(0) = 0$ *. The condition* (44)*, i.e.* $\langle Au, u\rangle = 0 \Longleftrightarrow a(u) = 0$*, is satisfied if one of the following conditions holds :*

(44a) The real function $t \mapsto \langle Atu, u\rangle$ *is monotone on* $[0,1]$ *for all* $u \in X$ *(e.g.* A *is monotone).*

(44b) The real function $t \mapsto a(tu)$ *is convex on* $[0,1]$ *for all* $u \in X$ *(e.g.* a *is convex).*

(44c) $\langle Au, u\rangle > 0$ *if* $u \neq 0$ *.*

(44d) A *is homogeneous, i. e.* $Atu = t^\rho u$ *for all* $u \in X$, $t > 0$ *and fixed* $\rho \geqq 0$ *.*

<u>Proof of Corollary 2</u>. Set $\phi(t) = \langle Atu, u\rangle$. Then $\phi(0) = 0$. Now, consider

$$a(u) = \int_0^1 \phi(t)\, dt \quad \text{and} \quad \frac{da(tu)}{dt} = \phi(t) .$$

Proof of Proposition 8. Using

$$\langle B_1u - B_1v, u - v\rangle \geqq c(||u - v||)\,||u - v|| \quad \text{for all} \quad u, v \in X$$

and the relations mentioned in Figure 1 we see easily that all the hypotheses of Theorem 2 are satisfied (see ZEIDLER (1978), p. 106).

Now, the proof follows from Theorem 2 and Corollary 1.

9. Application to Nonlinear Elliptic Equations

Proposition 8 will be now applied to nonlinear elliptic equations. For technical convenience we shall consider only a simple example related to Section 2.4.

Consider the boundary value problem

$$(50) \qquad -\lambda\Big(\sum_{i=1}^{N} D_i(D_iu|D_iu|^{p-2}) + f'(u)\Big) = g'(u)\phi(x) \quad \text{on} \quad G ,$$
$$u = 0 \quad \text{on} \quad \partial G ,$$

where G is an open bounded nonempty set in $\mathbb{R}^N$, $N \geqq 1$, and $x = (\xi_1, \ldots, \xi_N)$, $D_i = \partial/\partial\xi_i$, $p \geqq 2$.

Suppose $f, g \in C^1(\mathbb{R})$ with the growth conditions

$$|f(u)| , \quad |g(u)| \leqq c + d|u|^p ,$$
$$|f'(u)|, \quad |g'(u)| \leqq c + d|u|^{p-1}$$

for all $u \in \mathbb{R}$ where c, d are fixed positive constants.

Definition 4. A function u *belonging to the Sobolev space* $X \equiv \overset{o}{W}{}^1_p(G)$ *is said to be a generalized solution of* (50) *iff*

$$(50') \qquad \lambda\tilde{b}(u, v) = \tilde{a}(u, v) \quad \textit{for all} \quad v \in X \quad \textit{and} \quad b(u) = \alpha ,$$

where $\tilde{b} = \tilde{b}_1 + \tilde{b}_2$, $b = b_1 + b_2$ *and*

$$\tilde{b}_1(u, v) = \int_G \sum_{i=1}^{N} D_iu|D_iu|^{p-2} D_iv \, dx ,$$

$$\tilde{b}_2(u, v) = \int_G f'(u)v \, dx, \qquad \tilde{a}(u, v) = \int_G \phi(x)\, g'(u)v \, dx ,$$

$$b_1(u) = p^{-1}\int_G \sum_{i=1}^N |D_i u|^p \, dx, \quad b_2(u) = \int_G f(u) \, dx, \quad a(u) = \int_G \phi(x) \, g(u) \, dx \, .$$

REMARK 8. (50') is obtained from (50) by multiplying (50) by $v \in C_0^\infty(G)$ and integrating by parts.

Proposition 9. Suppose $\phi \in C(\overline{G})$ *and* $\phi(x) \geqq 0$ *for all* $x \in \overline{G}$, $\phi \not\equiv 0$ *on* $\overline{G}$. *Assume*

(51) $\quad f'(u)u \geqq 0$ *for all* $u \in \mathbb{R}$,

(52) $\quad g'(u)u > 0$ *if* $u \neq 0$, $g(0) = 0$.

Let $\alpha > 0$ *be an arbitrary fixed number.*

Then :

1) The equation (50') *has an eigensolution* $u \neq 0$, $\lambda > 0$.

2) Suppose f, ġ *are even. Then for all* $m \in \mathbb{N}$, *the equation* (50') *has an eigensolution* (u_m, λ_m) *with* $u_m \neq 0$, $\lambda_m > 0$ *and* $\lambda_m \to 0$ *as* $m \to \infty$.

If $\phi(x) > 0$ *on* G , *then* $u_m \rightharpoonup 0$ *in* X *as* $m \to \infty$.

P r o o f . (I) It is not difficult to show that there exist operators $A, B_1, B_2 : X \to X^*$ with $\langle Au, v\rangle = \tilde{a}(u, v)$, $\langle B_i u, v\rangle = \tilde{b}_i(u, v)$ for all $u, v \in X$.

A, B_i are potential operators with the corresponding potentials a, b_i , respectively. A, B_2 are strongly continuous, B_1 is continuous, bounded and uniformly monotone (see ZEIDLER (1978), p.120)

(II) From (52) it follows that $g(u) > 0$ if $u \neq 0$. Hence

$$\langle Au, u\rangle = 0 \Longleftrightarrow a(u) = 0 \, .$$

(III) Set $K = \{x \in \overline{G} : \phi(x) = 0\}$. Since $\phi \not\equiv 0$ on $\overline{G}$, we have $G - K \neq \emptyset$. Let X_0 be the set of all $u \in C_0^\infty(G)$ with $\operatorname{supp} u(x) \subset G - K$. Obviously, X_0 is an infinite-dimensional linear subspace of X and $a(u) > 0$ for all $u \in X_0 - \{0\}$.

Now Proposition 9 is a consequence of Proposition 8, q.e.d.

REMARK 9. If we combine Theorems 1, 2 with the general results proved by BROWDER (1970b) concerning the properties of operators induced by general classes of quasilinear elliptic differential equations of 2m-th order, then it is possible to generalize Proposition 9 rigorously.

10. The Main Theorem in Finite-Dimensional Banach Spaces

For fixed $\alpha > 0$ consider the nonlinear eigenvalue problem

$$Au = \lambda Bu, \quad b(u) = \alpha \qquad (u \in X, \lambda \in \mathbb{R}). \tag{53}$$

Set $N_\alpha = \{u \in X : b(u) = \alpha\}$.

Theorem 3.

Suppose :

i) X *is a real finite-dimensional Banach space.*

ii) $A, B : X \to X^*$ *are continuous potential operators with potentials* a, b *respectively;* $a(0) = b(0) = 0$.

iii) $\langle Bu, u\rangle > 0$ *if* $u \neq 0$.

iv) For every $u \neq 0$ *there exists a real number* $r(u) > 0$ *such that* $b(r(u)u) = \alpha$.

Then :

1) The equation (53) *has an eigensolution* $u \neq 0, \lambda \in \mathbb{R}$.

2) If A, B *are odd, then* (53) *has at least* $\dim X$ *distinct pairs of eigenvectors* $(u, -u)$.

Corollary 3. If $Av \neq 0$ *for all* $v \in N_\alpha$*, then all the eigenvalues of* (53) *are different from zero.*

Corollary 4 (multiplicity). Suppose that all the hypotheses of Theorem 3 are satisfied. Suppose A, B *are odd. Define critical levels*

$$\tilde{\beta}_m = \sup_{K \in \mathcal{K}_m} \min_{u \in K} a(u), \qquad m = 1, \ldots, \dim X,$$

where $\mathcal{K}_m$ *is defined as in Theorem 1.*

Then :

a) $\mathcal{N}_m \neq \emptyset$ *for all* $m = 1, \ldots, \dim X$.

b) If $\tilde{\beta}_m = \tilde{\beta}_{m+1} = \ldots = \tilde{\beta}_{m+p-1}$, $p \geq 1$, *then*

$\gamma(\{u \in N_\alpha : u$ *is eigenvector in* (53), $a(u) = \tilde{\beta}_m\}) \geq p$.

P r o o f . See e.g. ZEIDLER (1978), p. 115.

11. Application to Abstract Hammerstein Equations

Now we shall apply Theorem 1, Theorem 3 to the eigenvalue problem

(54) $$KF(u) = \lambda u \qquad (u \in X, \lambda \in \mathbb{R}) ,$$

$$\langle u, w \rangle_X = \alpha > 0 \quad \text{for all} \quad w \in K^{-1}(u) \quad .$$

Theorem 4.

Suppose :

i) X *is a real reflexive separable Banach space.*

ii) $K : X \to X^*$ *is a linear completely continuous operator with* $\langle Kv, v\rangle \geq 0$, $\langle Kv, w\rangle = \langle Kw, v\rangle$ *for all* $v, w \in X$.

iii) $F : X^* \to X$ *is a continuous potential operator with a potential* ϕ .

iv) $\phi(0) = F(0) = 0$; $\phi(u) \neq 0$, $KF(u) \neq 0$ *if* $u \neq 0$.

Then :

1) For all $\alpha > 0$, *the equation* (54) *has at least one eigensolution* $u \neq 0$, $\lambda \neq 0$.

2) Suppose F *is odd.*

a) Then for all $\alpha > 0$, *the equation* (54) *has at least* $\dim K(X)$ *distinct pairs of eigenvectors* $(u, -u)$ *belonging to nonzero eigenvalues.*

b) If $\dim K(X) = \infty$ *then, for all* $\alpha > 0$, *the equation* (54) *has an infinite number of distinct eigenvalues* λ_m *with* $\lambda_m \to 0$ *as* $m \to \infty$.

REMARK 10. Under stronger assumptions this result is contained in VAINBERG (1956) and COFFMAN (1971).

Theorem 4,2a) is a special case of a **more** general result due to AMANN (1972).

A sketched proof of Theorem 4. A proof of Theorem 4 is given in ZEIDLER (1978), p. 121. The main idea of the proof due to AMANN (1972) is

i) to factorize $K = S^*S$ ($S : X \to H$, H a Hilbert space) by a general factorization theorem due to BROWDER, GUPTA (1969) (see also ZEIDLER (1977), p. 107);

ii) to replace (54) by the equivalent problem

$$SF(S^*v) = \lambda v \tag{54'}$$

in the Hilbert space H ($v = S^{*-1}u$);

iii) to apply the Lyusternik-Schnirelman theory to (54').

If we set $A = SFS^*$, then A is a potential operator with a potential $a(u) = \phi(S^*u)$ and $a(u) = 0 \Longleftrightarrow Au = 0 \Longleftrightarrow u = 0$.

Theorem 1, points 8), 5), 6) lead to Theorem 4, points 1), 2a), 2b), respectively. In the case $\dim K(X) < \infty$ one has to use Theorem 3.

12. Application to Hammerstein Integral Equations

For technical convenience we shall apply Theorem 4 only to a simple example. Consider an integral equation

$$\lambda u(x) = \int_G k(x, y)\, f(u(y))\, dy \qquad (u \in X, \quad \lambda \in \mathbb{R}) \tag{55}$$

where G is an open bounded nonempty set in $\mathbb{R}^N$, $N \geq 1$. Set $X = X^* = L_2(G)$.

The corresponding linear integral equation reads

$$\lambda u(x) = \int_G k(x, y)\, u(y)\, dy \qquad (u \in X, \quad \lambda \in \mathbb{R}). \tag{56}$$

Proposition 9.

Suppose :

i) $k(.,.)$ *is a real measurable function on* $G \times G$ *with* $k(x, y) = k(y, x)$ *for all* $x, y \in G$ *and*

$$0 < \int_{G \times G} k^2(x, y)\, dxdy < \infty .$$

ii) f *is a real continuous function on* $\mathbb{R}$ *with* $|f(u)| \leqq c + d|u|$ *for all* $u \in \mathbb{R}$, c, d *are positive constants and* $\pm f(u) > 0$ *if* $\pm u > 0$.

iii) The linear integral equation (56) *has only positive eigenvalues.*

Then :

1) For every $\alpha > 0$ *, the equation* (55) *has an eigensolution* $u \in X$, $\lambda \neq 0$ *with*

$$\int_G u(x)\, w(x)\, dx = \alpha \qquad \textit{for all} \quad w \in K^{-1}u . \tag{60}$$

2) Suppose f *is odd. The for every* $\alpha > 0$ *, the equation* (55 *has an infinite number of eigensolutions* (u_m, λ_m) *with* (60) *,* $\lambda_m \neq 0$ *and* $\lambda_m \to 0$ *as* $m \to \infty$ *, i.e. there exists an infinite number of distinct eigenvalues.*

P r o o f . We write (56) as $\lambda u = Ku$ and (55) as $\lambda u = KFu$, $u \in X$. From iii) we obtain $\langle Kv, v\rangle > 0$ if $v \neq 0$. F is a potential operator with a potential

$$\phi(u) = \int_G \left(\int_0^{u(x)} f(v)\, dv \right) dx .$$

Now Proposition 9 is a consequence of Theorem 4 with $\dim K(X) = \infty$ (see also ZEIDLER (1978), p. 69), q.e.d.

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