Hans Triebel

Recent developments in the theory of function spaces and linear regular differential equations

In: Svatopluk Fučík and Alois Kufner (eds.): Nonlinear Analysis, Function Spaces and Applications, Proceedings of a Spring School held in Horní Bradlo, 1978, [Vol 1]. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1979. Teubner Texte zur Mathematik. pp. 149--175.

Persistent URL: http://dml.cz/dmlcz/702409

RECENT DEVELOPMENTS IN THE THEORY OF FUNCTION SPACES AND LINEAR REGULAR ELLIPTIC DIFFERENTIAL EQUATIONS

Hans Triebel

1. Historical background

It is the opinion of the author that the history of the theory of function spaces can be divided in three periods. ("Function spaces" means always normed or quasi-normed spaces of functions and distributions.) The first period starts at the beginning and ends in the mid-thirties of our century. In this time the classical basic spaces L_p of p-integrable functions and C^m of m times differentiable functions were thoroughly investigated. (Here $m = 0,1,2,\dots$; we set $C^0 = C$.) The Hölder spaces C^s, where $0 < s \neq$ integer, and the Hardy spaces $\dot{H}_p$, where $0 < p < \infty$ (it is convenient for our purpose to mark these spaces with a dot, in contrast to the usual notation), belong to that period and anticipate the second period. Originally at that time the Hardy spaces were introduced in order to characterize the boundary values of analytic functions in a disk of the complex plane. The close connection of the Hardy spaces with the Fourier analysis was clear from the very beginning. The second period, the "constructive period", starts with S. L. SOBOLEV's papers (1935-1938), where the nowadays so-called Sobolev spaces W_p^m with $m=0,1,2,\dots$ were introduced. A new tool, the theory of distributions, was discovered and new techniques and results (e.g. imbedding theorems) were used successfully in order to investigate partial differential equations. The second period is characterized by more or less speculative constructions of a lot of new spaces with the help of explicit norms on the basis of the above classical spaces. This period had a culmination point at the end of the fifties and in the sixties, and flourishes also in our time. First of all we must note the direct des-

cendants of the above-mentioned classical spaces (including the Hölder spaces, the Hardy spaces, and the Sobolev spaces). These are the so-called isotropic spaces defined on the Euclidean n-space R_n (or on domains in R_n): (i) the Zygmund spaces $\mathcal{C}^s$, which are extensions of the Hölder spaces to indices s which are integers (A. ZYGMUND, 1945), (ii) the spaces W_p^s with $0 < s \neq$ integer (L. N. SLOBODECKIJ, N. ARONSZAJN, E. GAGLIARDO, 1955-1958), (iii) the Lebesgue (or Bessel-potential or Liouville) spaces H_p^s (N. ARONSZAJN, K. T. SMITH, A. P. CALDERÓN, 1961), (iv) the Besov (or Lipschitz) spaces $\Lambda_{p,q}^s$ (O. V. BESOV, 1959-1961; it should be emphasized that the special case $\Lambda_{p,\infty}^s$, sometimes denoted as Nikol'skij spaces, was discovered by S. M. NIKOL'SKIJ, 1951), (v) finally, the space BMO of functions of bounded mean oscillation (F. JOHN, L. NIRENBERG, 1961), which (as we know nowadays) is related to the real variable version of the n-dimensional Hardy spaces $\mathring{H}_p$ (E. M. STEIN, G. WEISS, 1960). Beside these spaces, which have become classical nowadays, many extensions, generalizations and modifications have been treated extensively : anisotropic spaces, weighted spaces, spaces with dominating derivatives, Lorentz spaces, Campanato-Morrey spaces, Orlicz spaces, Orlicz-Sobolev spaces etc. The third period, the "systematic period", starts in the sixties and is overlapping heavily with the contructive period. The inflation of spaces asks for simple and far-reaching methods which enable us to deal with function spaces (or at least with the hard core of the theory of function spaces, i.e. in our opinion, the theory of isotropic spaces in the sense of the above-mentioned spaces) from the point of view of few general principles.
The first remarkable success has been achieved in the framework of the abstract interpolation theory of Banach spaces and its applications to function spaces (E. GAGLIARDO, J. L. LIONS, E. MAGENES, J. PEETRE, A. P. CALDERÓN, 1960-1964). Starting with simple spaces (L_p-spaces, Hölder spaces, Sobolev spaces), one obtains on the basis

of general abstract procedures (interpolation methods) a lot of new spaces (e. g. H^s_p or $\Lambda^s_{p,q}$) in a systematic way. The disadvantage of this method is that one needs starting spaces. At the end of the sixties and the beginning of the seventies new methods (and essentially improved old methods) of the Fourier analysis and of the theory of the so-called maximal inequalities provided a new impetus in the theory of function spaces (J. PEETRE, E. M. STEIN, C. FEFFERMAN, 1967-1975). With the help of these far-reaching new powerful techniques one can deal with all the function spaces marked in Fig. 1 from a unified point of view and on the basis of very few general principles. All these spaces are contained in two scales $B^s_{p,q}$ and $F^s_{p,q}$ of spaces, which are defined in the framework of the methods of the Fourier analysis.

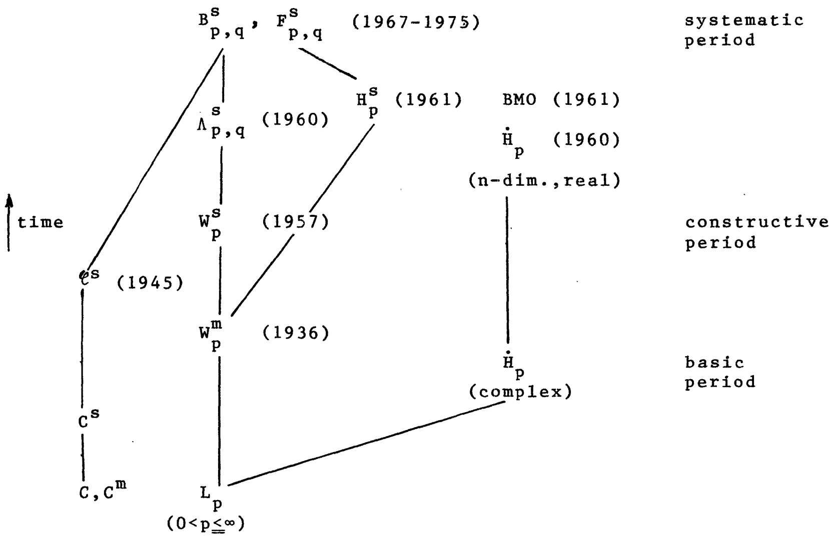

Fig. 1

The aim of this paper is to give an up-to-date description of some aspects of the theory of function spaces, where we restrict ourselves to isotropic unweighted spaces defined on R_n and on domains in R_n , in the framework of the Fourier analysis. Furthermore, we deal with some applications to regular elliptic differential operators. Section 2 contains a description of the "constructive spaces" (i.e. all the spaces mentioned in Fig.1 except of $B^s_{p,q}$ and $F^s_{p,q}$). In Section 3 we try to discuss in somewhat heuristical terms some ideas, principles and methods for spaces defined on R_n . The definition of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ and a description of some fundamental properties is given in Sections 4 and 5. Again in somewhat heuristical terms we discuss in Section 6 further principles which are necessary for a successful investigation of the corresponding spaces on domains. In Section 7 we describe some further properties and consider distinguished spaces. Finally, Section 8 deals with applications to regular elliptic differential operators.

The theory of the constructive spaces (and also of the basic spaces) has been treated extensively from several points of view in the books [1, 2, 4, 5, 7, 8, 9]. Systematic treatments of the spaces under consideration in the framework of the interpolation theory and the Fourier analysis may be found in the books [6, 9, 10, 11]. We do not quote the original papers and refer to the bibliography in the above books. In particular, this paper leans on [11]. However, there are also some new results, points of view and interpretations which (so we hope) justify this paper. On the other hand, for the sake of completeness, we take over some considerations of [11] in a modified and shortened way.

2. Spaces on R_n

The list of constructive spaces given below coincides essentially with the corresponding lists in [10], 2.5.1 and [11], 1.3. Let R_n be the n-dimensional real Euclidean space. Let

$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ be the usual abbreviation of derivatives,

$\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index, $\alpha_j \geq 0$ integers, $|\alpha| = \sum_{j=1}^{n} \alpha_j$.

Furthermore, all functions and distributions are complex.

(i) The spaces $C^m(R_n)$. If $C(R_n)$ is the set of all bounded uniformly continuous functions $f(x)$ on R_n and

$$||f||_{C(R_n)} = \sup_{x \in R_n} |f(x)|,$$

then, for $m = 0,1,2,\dots$,

$$C^m(R_n) = \{f \mid D^\alpha f \in C(R_n) \text{ for all } |\alpha| \leq m\},$$

$$||f||_{C^m(R_n)} = \sum_{|\alpha| \leq m} ||D^\alpha f||_{C(R_n)}.$$

Here $C^0(R_n) = C(R_n)$.

(ii) The spaces $L_p(R_n)$. If $0 < p < \infty$, then

$L_p(R_n) = \{f \mid f(x)$ Lebesgue-measurable on R_n,

$$||f||_{L_p(R_n)} = \left(\int_{R_n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \},$$

$L_\infty(R_n) = \{f \mid f(x)$ Lebesgue-measurable on R_n,

$$||f||_{L_\infty(R_n)} = \operatorname{ess\,sup}_{x \in R_n} |f(x)| < \infty\}.$$

(iii) The Hölder spaces $C^s(R_n)$. If s is a real number, then we put

$$s = [s] + \{s\}, \quad [s] \text{ integer}, \quad 0 \leq \{s\} < 1.$$

If $0 < s \neq$ integer, then

$$C^s(R_n) = \{f \mid f \in C^{[s]}(R_n),$$

$$||f||_{C^s(R_n)} = ||f||_{C^{[s]}(R_n)} + \sum_{|\alpha| \leq [s]} \sup_{\substack{x \neq y \\ x,y \in R_n}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{s\}}} < \infty\}.$$

of the Hardy spaces. We shall not give the definition of the classical Hardy spaces in the framework of the complex function theory, which may be found in [14], VII.7, cf. also [3, 7, 8].

Proposition 1. *(i) If* $0 < s \neq$ *integer, then* $C^s(R_n) = \mathcal{C}^s(R_n)$.
(ii) If $1 < p < \infty$ *and* $m = 0,1,2,\ldots$ *, then* $W^m_p(R_n) = H^m_p(R_n)$.
(iii) If $1 < p < \infty$ *and* $0 < s \neq$ *integer, then* $W^s_p(R_n) = \Lambda^s_{p,p}(R_n)$.

REMARK 2. These are well-known assertions, cf. e. g. [4, 5, 9]. Hence we may restrict our attention in the sequel to Zygmund spaces, Lebesgue spaces, Besov spaces, Hardy spaces, and BMO. (As we shall see, $L_\infty(R_n)$, $L_p(R_n)$ with $0 < p \leqq 1$, and $C^m(R_n)$ with $m=0,1,2,\ldots$ are not "good" spaces in the sense of our criterion below.)

3. Criterion and Principle

In [11], 1.1 and 1.2, we have given a rather extensive heuristical discussion of criteria, principles, and methods yielding the spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$. We recall briefly some arguments and refer for a more detailed version to [11], 1.1 and 1.2 . Furthermore, we want to modify here some of the considerations in [11].

We are concerned with quasi-Banach spaces $A(R_n)$ which are subspaces of $S'(R_n)$ (topological imbedding). A space A is said to be a quasi-Banach space if it has all the properties of a Banach space except for the triangle inequality which is replaced by

$$||a_1 + a_2||_A \leqq c(||a_1||_A + ||a_2||_A)$$

for all $a_1 \in A$ and all $a_2 \in A$ (here c is independent of a_1 and a_2). $L_p(R_n)$ with $0 < p < 1$ is an example of a quasi-Banach space which is not a Banach space (of course, Banach spaces are special quasi-Banach spaces). We are looking for spaces having "good" properties. Our point of view is that a space has good properties if it is useful in the theory of partial differential equations. If $A^2(R_n)$ denotes the space of all $f \in S(R_n)$ such that $D^\alpha f \in A(R_n)$

(iv) The Zygmund spaces $\mathscr{C}^s(R_n)$. If s is a real number, then we put

$$s = [s]^- + \{s\}^+ \ , \ [s]^- \text{ integer, } 0 < \{s\}^+ \leqq 1 \ .$$

Furthermore, if $h \in R_n$, then

$$(\Delta_h^\ell f)(x) = f(x+h) - f(x) \ , \quad \Delta_h^\ell = \Delta_h^\ell \ \Delta_h^{\ell-1} \quad \text{if } \ell = 2,3,\ldots$$

If $s > 0$, then

$$\mathscr{C}^s(R_n) = \{f \mid f \in C^{[s]^-}(R_n)\},$$

$$||f||_{\mathscr{C}^s(R_n)} = ||f||_{C^{[s]^-}(R_n)} + \sum_{|\alpha|=[s]^-} \sup_{0 \neq h \in R_n} |h|^{-\{s\}^+} ||\Delta_h^2 D^\alpha f||_{C(R_n)} < \infty\}.$$

(v) The Sobolev spaces $W_p^m(R_n)$. If $1 < p < \infty$ and $m = 0,1,2,\ldots,$ then

$$W_p^m(R_n) = \{f \mid f \in S'(R_n), \quad ||f||_{W_p^m(R_n)} = \sum_{|\alpha| \leqq m} ||D^\alpha f||_{L_p(R_n)} < \infty \},$$

where $S'(R_n)$ denotes the set of all tempered distributions on R_n. $W_p^0(R_n) = L_p(R_n)$.

(vi) The Slobodeckij spaces $W_p^s(R_n)$. If $1 < p < \infty$ and $0 < s \neq$ integer, then

$$W_p^s(R_n) = \{f \mid f \in W_p^{[s]}(R_n), \ ||f||_{W_p^s(R_n)} =$$

$$= ||f||_{W_p^{[s]}(R_n)} + \sum_{|\alpha|=[s]} \left(\int_{R_n \times R_n} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}} dxdy \right)^{\frac{1}{p}} < \infty \} .$$

(vii) The Besov spaces (= Lipschitz spaces) $\Lambda_{p,q}^s(R_n)$. If $s > 0$, $1 < p < \infty$ and $1 \leqq q < \infty$, then

$$\Lambda_{p,q}^s(R_n) = \{f \mid f \in W_p^{[s]^-}(R_n), \ ||f||_{\Lambda_{p,q}^s(R_n)} = ||f||_{W_p^{[s]^-}(R_n)} +$$

$$\sum_{|\alpha|=[s]^-} \left(\int_{R_n} |h|^{-\{s\}^+ q} ||\Delta_h^2 D^\alpha f||_{L_p(R_n)}^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < \infty \}$$

and if $s > 0$ and $1 < p < \infty$, then

$$\Lambda^s_{p,\infty}(R_n) = \{f \mid f \in W_p^{[s]^-}(R_n), \quad ||f||_{\Lambda^s_{p,\infty}(R_n)} = ||f||_{W_p^{[s]^-}(R_n)}$$

$$+ \sum_{|\alpha|=[s]^-} \sup_{0 \neq h \in R_n} |h|^{-\{s\}^+} ||\Delta_h^2 D^\alpha f||_{L_p(R_n)} < \infty \} .$$

(viii) <u>The Lebesgue spaces (= Bessel potential spaces = Liouville spaces) $H^s_p(R_n)$</u>. If s is real and $1 < p < \infty$, then

$$H^s_p(R_n) = \{f \mid f \in S'(R_n),$$
$$||f||_{H^s_p(R_n)} = ||F^{-1}[(1+|\xi|^2)^{\frac{s}{2}} Ff||_{L_p(R_n)} < \infty \} .$$

Here F and F^{-1} denote the Fourier transform and its inverse transform on $S'(R_n)$, respectively.

(ix) <u>The Hardy spaces $\dot{H}_p(R_n)$</u>. If $\phi(x)$ is an infinitely differentiable function on R_n with compact support and $\phi(0) = 1$, and if $0 < p < \infty$, then

$$\dot{H}_p(R_n) = \{f \mid f \in S'(R_n), \ ||f||_{\dot{H}_p(R_n)} =$$
$$= ||\sup_{t>0} |(F^{-1}[\phi(t.)Ff(.)])(x)| \ ||_{L_p(R_n)} < \infty \} .$$

(x) <u>The space BMO</u>. If $f(x)$ is a locally Lebesgue-integrable function on R_n and if Q is a cube in R_n, then

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx$$

is the mean value of $f(x)$ with respect to Q.

$\mathrm{BMO} = \{f \mid f(x)$ locally Lebesgue-integrable in R_n,

$$||f||_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty \} ,$$

(the supremum is taken over all finite cubes in R_n).

<u>REMARK 1.</u> Comments may be found in [11], 1.3. The above definition of $\dot{H}_p(R_n)$ which is due to C. FEFFERMAN and E. M. STEIN ([3], 1972, p. 183) corresponds to the n-dimensional real variable version

for all α with $|\alpha| \leqq 2$, then it would be desirable for the elliptic differential operator $-\Delta + E$ (here E is the identity) to yield a one-to-one mapping from $A^2(R_n)$ onto $A(R_n)$. We discussed this situation in some detail in [11], 1.1. As an almost direct consequence we obtained the following assertion.

Criterion. *The space* $A(R_n)$ *should satisfy the following weak Michlin-Hörmander multiplier property: If* $m(x)$ *is a complex-valued infinitely differentiable function on* R_n *such that*

$$\sup_{\xi \in R_n} (1 + |\xi|)^{|\beta|} |D^\beta m(\xi)| < \infty$$

for all multi-indices β *, then there exists a number* $c \geqq 0$ *such that*

$$||F^{-1}[m(.)(Ff)(.)]||_{A(R_n)} \leqq c||f||_{A(R_n)}$$

for all $f \in A(R_n)$.

Proposition 2. *(i) The spaces* $\mathcal{C}^s(R_n)$ *(with* $s > 0$*),* $H_p^s(R_n)$ *(with* $-\infty < s < \infty$ *and* $1 < p < \infty$ *),* $\Lambda_{p,q}^s(R_n)$ *(with* $s > 0$, $1 < p < \infty$, *and* $1 \leqq q \leqq \infty$ *),* $\dot{H}_p(R_n)$ *(with* $0 < p < \infty$*) and* BMO *satisfy the Criterion.*

(ii) The spaces $L_1(R_n)$, $L_\infty(R_n)$ *and* $C^m(R_n)$ *with* $m = 1,2,\dots$) *do not satisfy the Criterion.*

REMARK 3. The spaces $L_p(R_n)$ with $0 < p < 1$ cannot be treated as subspaces of $S(R_n)$. Hence, by Proposition 1, covers all spaces described above which are subspaces of $S'(R_n)$. Again we refer to [11].

After the criterion has been established, we are looking for general principles. We skatch some ideas, which can be found at least partly in more detail in [11], 1.2. However, here are also some essential modifications in comparison with [11]. The weight $(1+|\xi|^2)^{\frac{s}{2}}$ in the definition of the Lebesque spaces $H_p^s(R_n)$, cf. 2 (viii), expresses the differentiability properties of the functions or distributions belonging to $H^s_p(R_n)$. If one uses the Hilbert space version of Proposition 2 or a theorem of Littlewood-Paley type for $L_p(R_n)$, $1 < p < \infty$, then one can prove the following assertion (for a detailed proof we refer to [9], p. 177-78). If $\{\phi_j(x)\}_{j=0}^{\infty} \subset S(R_n)$ is a smooth resolution of unity, i.e.

$$\operatorname{supp} \phi_o \subset \{y \mid |y| \leq 2\} , \tag{1}$$

$$\operatorname{supp} \phi_j \subset \{y \mid 2^{j-1} \leq |y| \leq 2^{j+1}\} , \quad j = 1,2,3,\dots , \tag{2}$$

for any multi-index γ there exists a constant c_γ such that

$$|D^\gamma \phi_j(x)| \leq c_\gamma 2^{-j|\gamma|} \tag{3}$$

and for all $x \in R_n$ we have $\sum\limits_{j=0}^{\infty} \phi_j(x) = 1$, then

$$H^s_p(R_n) = \{f \mid f \in S'(R_n), \ \|f\|^*_{H^s_p(R_n)} =$$

$$= \|(\sum_{j=0}^{\infty} |2^{js}F^{-1}[\phi_j \ Ff](.)|^2)^{\frac{1}{2}}\|_{L_p(R_n)} < \infty\} \, .$$

Here $-\infty < s < \infty$ and $1 < p < \infty$. In other words, the original weight $(1 + |\xi|^2)^{\frac{s}{2}}$, expressing the differentiability properties of the elements of $H^s_p(R_n)$ is decomposed into weights $2^{js}\phi_j(\xi)$ and an ℓ_2-norm comes in. One can give a more suggestive argument (at least for $p = 2$) if one uses spectral analysis in Hilbert spaces, cf. [11], 1.2. In any case the above result shows clearly our intention :
A decomposition of differentiability properties of distributions with the help of smooth resolutions of unity, and a measurement of the decomposed distribution in appropriate spaces, e.g. in the above case $L_p(R_n,\ell_2)$ with $1 < p < \infty$. If $p = 2$ then, of course, $L_p(R_n,\ell_2)$ can be replaced by $\ell_2(L_p(R_n))$. In any case, decompositions of the above type, sequence spaces, and L_p-spaces are involved. It seems reasonable to fix the above decomposition of differentiability properties and to measure these decompositions in the more general spaces $L_p(R_n,\ell_q)$ and $\ell_q(L_p(R_n))$ and to investigate the spaces obtained in this way. As we shall see such an attempt is successful

for $0 < p \leqq \infty$ and $0 < q \leqq \infty$ (except $p = \infty$ in the case $L_p(R_n, \ell_q)$). At the first glance it seems to be somewhat surprising that also values p with $p < 1$ are admissible. However, by the Paley-Wiener-Schwartz theorem and the Plancherel-Pólya-Nikol'skij inequality, $F^{-1}[\phi_j\ Ff] \in L_p(R_n) \cap S'(R_n)$ is an entire analytic function and belongs to all spaces $L_r(R_n)$ with $p \leqq r \leqq \infty$ (cf. [10], Chapter 1). In order to overcome the difficulties with the spaces related to $L_\infty(R_n, \ell_q)$ we modify the above approach slightly. Let again $1 < p < \infty$ and $-\infty < s < \infty$. If one uses again the Hilbert space version of the multiplier theorem contained in Proposition 2, one can prove that

$$H_p^s(R_n) = \{f \mid \exists \{f_j\}_{j=0}^\infty \subset L_p(R_n) \text{ such that} \tag{4}$$
$$\left\| \left(\sum_{j=0}^\infty |2^{js} f_j(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{L_p(R_n)} < \infty \text{ and } f \underset{S'}{=} \sum_{j=0}^\infty F^{-1}[\phi_j\ Ff_j]\} .$$

Furthermore,

$$\inf \left\| \left(\sum_{j=0}^\infty |2^{js} f_j(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{L_p(R_n)} ,$$

where the infimum is taken over all admissible systems $\{f_j\}$, is an equivalent norm in $H_p^s(R_n)$. Of course, this is also a description of spaces via a decomposition method. One can ask for generalizations, where $L_p(R_n, \ell_2)$ is replaced by $L_p(R_n, \ell_q)$ or $\ell_q(L_p(R_n))$. The disadvantage is that p must be restricted to $p \geqq 1$. This seemingly more complicated approach has a famous counterpart, the charactrization of BMO with the help of $L_\infty(R_n)$-functions, cf. [3], p. 145. We summarize our considerations as follows.

Principle. *The space* $A(R_n)$ *should be treated by measuring the decomposition* $\{F^{-1}[\phi_j Ff]\}_{j=0}^\infty$ *of the differentiability properties of* $f \in S'(R_n)$ *(or the above-described modification with the help of appropriate quasi-norms of type* $L_p(R_n, \ell_q)$ *or* $\ell_q(L_p(R_n))$.

4. The Spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$

Let $\mathcal{A}^c$ be the set of all systems $\phi = \{\phi_j(x)\}_{j=0}^{\infty} \subset S(R_n)$ with (1), (2) and (3). Let $\mathcal{A}^0$ be the set of all systems $\phi \in \mathcal{A}^c$ with the additional property $\sum_{j=0}^{\infty} \phi_j(x) = 1$ for all $x \in R_n$ (smooth resolution of unity). If $0 < p \leq \infty$ and $0 < q \leq \infty$, then we put

$$||a_j||_{\ell_q(L_p(R_n))} = \left(\sum_{j=0}^{\infty} ||a_j||^q_{L_p(R_n)}\right)^{\frac{1}{q}} , \tag{5}$$

$$||a_j||_{L_p(R_n,\ell_q)} = \left|\left|\left(\sum_{j=0}^{\infty} |a_j(.)|^q\right)^{\frac{1}{q}}\right|\right|_{L_p(R_n)} \tag{6}$$

with the usual modification for $q = \infty$.

Definition 1. *Let* $-\infty < s < \infty$ *and* $0 < q \leq \infty$.

(*i*) *If* $0 < p \leq \infty$, *then*

$$B^s_{p,q}(R_n) = \{f \mid f \in S'(R_n) ,\quad ||f||^{\phi}_{B^s_{p,q}(R_n)} = ||2^{sj}F^{-1}[\phi_j Ff]||_{\ell_q(L_p(R_n))} < \infty \text{ for all } \phi \in \mathcal{A}^c\} . \tag{7}$$

(*ii*) *If* $0 < p < \infty$, *then*

$$F^s_{p,q}(R_n) = \{f \mid f \in S'(R_n),\ ||f||^{\phi}_{F^s_{p,q}(R_n)} = ||2^{sj}F^{-1}[\phi_j Ff]||_{L_p(R_n,\ell_q)} < \infty \text{ for all } \phi \in \mathcal{A}^c\} , \tag{8}$$

$$F^s_{\infty,q}(R_n) = \{f \mid f \in S'(R_n),\ \exists \phi \in \mathcal{A}^0,\ \exists \{f_j\}_{j=0}^{\infty} \subset L_\infty(R_n) \text{ such that } ||2^{sj}f_j||_{L_\infty(R_n,\ell_q)} < \infty \text{ and } f \underset{S'}{=} \sum_{j=0}^{\infty} F^{-1}[\phi_j Ff_j]\} . \tag{9}$$

REMARK 4. These are the spaces in the sense of the above Principle. We used the modified decomposition in the sense of (4) only in the case of the spaces $F^s_{\infty,q}(R_n)$, formula (9). Of course, one could try to use this modified decomposition also in (7) and (8). As we remarked above, in that case one must restrict p to $p \geq 1$. One can prove that spaces of type $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ defined with the help of modified decompositions in the sense of (4) and (9) coincide with $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$, respectively, if $1 < p < \infty$ and $1 < q < \infty$, cf [11], 2.5.1. Hence the modified decomposition in the sense of (9) seems to be of interest mainly for limiting cases. Maybe spaces with $p = 1$ are also of some interest, but this is not yet clear at this moment. On the other hand, if one extends (8) to $p = \infty$ then the resulting spaces do not satisfy the above Criterion and do not coincide with $F^s_{\infty,q}(R_n)$ from (9), cf. [11], 2.1.4. (How these spaces are quasi-normed will be explained below.)

REMARK 5. $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ are isotropic non-homogeneous spaces. In the same way one can define isotropic homogeneous spaces. $\dot{\mathcal{A}}^c$ is the set of all systems $\phi = \{\phi_j(x)\}_{j=-\infty}^{\infty} \subset S(R_n)$ with (2) and (3) for all $j = 0,\pm 1,\pm 2,\ldots$. Furthermore, $\phi \in \dot{\mathcal{A}}^0$ if $\phi \in \dot{\mathcal{A}}^c$ and $\sum_{j=-\infty}^{\infty} \phi_j(x) = 1$ for all $x \in R_n - \{0\}$. In (5) and (6) one must replace $\sum_{j=0}^{\infty}$ by $\sum_{j=-\infty}^{\infty}$. Then $\dot{B}^s_{p,q}(R_n)$ and $\dot{F}^s_{p,q}(R_n)$ can be defined similarly as in (7), (8) and (9) ($j = 0,1,2,\ldots$ must be replaced by $j = 0,\pm 1,\pm 2,\ldots$). Because the origin plays a peculiar role, there are some not very serious technical difficulties, cf. [11], Chapter 3.

Theorem 1. (*i*) *If* $-\infty < s < \infty$, $0 < p \leqq \infty$ *and* $0 < q \leqq \infty$, *then* $B^s_{p,q}(R_n)$, *equipped with the quasi-norm* $\|f\|^{\phi}_{B^s_{p,q}(R_n)}$, $\phi \in \mathcal{A}^0$, *is a quasi-Banach space. All these quasi-norms are mutually equivalent.* ($B^s_{p,q}(R_n)$ *is a Banach spaces if* $1 \leqq p \leqq \infty$ *and* $1 \leqq q \leqq \infty$.)
(*ii*) *If* $-\infty < s < \infty$, $0 < p < \infty$ *and* $0 < q \leqq \infty$, *then* $F^s_{p,q}(R_n)$, *equipped with the quasi-norm* $\|f\|^{\phi}_{F^s_{p,q}(R_n)}$, $\phi \in \mathcal{A}^0$, *is a quasi-Banach space. All these quasi-norms are mutually equivalent.*
($F^s_{p,q}(R_n)$ *is a Banach space if* $1 \leqq p < \infty$ *and* $1 \leqq q \leqq \infty$.)
(*iii*) *If* $-\infty < s < \infty$ *and* $1 < q \leqq \infty$, *then* $F^s_{\infty,q}(R_n)$, *equipped with the norm* $\inf \|2^{sj} f_j\|_{L_\infty(R_n,\ell_q)}$, *where the infimum is taken over all admissible representations, is a Banach space.*

(iv) All the spaces from (i), (ii), and (iii) satisfy the Criterion from Section 3.

REMARK 6. The theorem remains valid if one replaces $B^s_{p,q}(R_n)$ by $\dot{B}^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ by $\dot{F}^s_{p,q}(R_n)$, respectively. However, some technical modifications in comparison with the non-homogeneous spaces are convenient, in particular, $S'(R_n)$ should be replaced by another set of distributions. We shall not go into detail here and refer to [11], Chapter 3.

Theorem 2. *(i) If* $s > 0$ *then* $\mathcal{C}^s(R_n) = B^s_{\infty,\infty}(R_n)$.
(ii) If $s > 0$, $1 < p < \infty$ *and* $1 \leq q \leq \infty$, *then* $\Lambda^s_{p,q}(R_n) = B^s_{p,q}(R_n)$.
(iii) If $-\infty < s < \infty$ *and* $1 < p < \infty$, *then* $H^s_p(R_n) = F^s_{p,2}(R_n)$.
(iv) If $0 < p < \infty$, *then* $\dot{H}_p(R_n) = \dot{F}^0_{p,2}(R_n)$.
(v) BMO $= \dot{F}^0_{\infty,2}(R_n)$.

REMARK 7. By Proposition 1 and Proposition 2, Theorem 2 covers all the constructive spaces for which the Criterion is satisfied. Proofs of both the theorems can be found in [11]. In particular, in [11], Chapter 3, interpretations of (iv) and (v) are given.

5. Equivalent Quasi-Norms in the Spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$

The main bulk of [11] deals with properties of the spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$. Of peculiar interest are equivalent quasi--norm in those spaces. In this section we describe some new equivalent quasi-norms (partly hitherto unpublished, partly taken over from [12], cf. also the appendix in [11]).

Theorem 3. *(i) Let* $0 < p \leq \infty$, $0 < q \leq \infty$ *and* $s \geq n + 5 + \frac{6n}{p}$. *If* ℓ *is a natural number,* $\ell > s$, *then*

$$\|f\|_{L_p(R_n)} + \Big[\int_0^\infty r^{-sq} \Big\| \sup_{|h| < r} |(\Delta^{\ell}_h f)(x)| \Big\|^q_{L_p(R_n)} \frac{dr}{r}\Big]^{\frac{1}{q}} \tag{10}$$

(its modification if $q = \infty$ *) is an equivalent quasi-norm norm if* $p \geq 1$ *and* $q \geq 1$) *in* $B^s_{p,q}(R_n)$.

(*ii*) *Let* $0 < p < \infty$, $0 < q \leqq \infty$ *and* $s \geqq n + 5 + \frac{6n}{\min(p,q)}$. *If* ℓ *is a natural number,* $\ell > s$, *then*

$$||f||_{L_p(R_n)} + ||[\int_0^\infty r^{-sq}(\sup_{|h|<r} |(\Delta_h^\ell f)(x)|)^q \frac{dr}{r}]^{\frac{1}{q}}||_{L_p(R_n)} \tag{11}$$

(*its modification if* $q = \infty$) *is an equivalent quasi-norm (norm if* $p \geqq 1$ *and* $q \geqq 1$ *) in* $F^s_{p,q}(R_n)$.

REMARK 8. A proof of this theorem will be published elsewhere. The restrictions $s \geqq n + 5 + \frac{6n}{p}$, $s \geqq n + 5 + \frac{6n}{\min(p,q)}$ are somewhat artifical. They depend on the maximal technique used. The most striking assertion of the theorem is the fact that there is no essential difference between $p \geq 1$ and $p < 1$. Furthermore, we mention that I_σ ,

$$I_\sigma f = F^{-1}(1 + |\xi|^2)^{\frac{\sigma}{2}} Ff ,$$

where $-\infty < \sigma < \infty$, yields an isomorphic mapping from $B^s_{p,q}(R_n)$ (with $-\infty < s < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$) onto $B^{s-\sigma}_{p,q}(R_n)$, and from $F^s_{p,q}(R_n)$ (with $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leq \infty$, or with $-\infty < s < \infty$, $p = \infty$ and $1 < q \leq \infty$) onto $F^{s-\sigma}_{p,q}(R_n)$, respectively. Hence, with the help of I_σ , (10) and (11) , one obtains "explicit" quasi-norms for all spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ (with $p < \infty$ in the case of the F-spaces).

Theorem 4. (*i*) *Let*

either $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $s > 0$

or $0 < p < 1$, $0 < q \leq \infty$ and $s > n(\frac{1}{p} - 1)$.

If ℓ *is a natural number,* $\ell > s$, *then*

$$||f||_{L_p(R_n)} + [\int_{R_n} |h|^{-sq}||\Delta_h^\ell f||^q_{L_p(R_n)} \frac{dh}{|h|^n}]^{\frac{1}{q}} \tag{12}$$

(*its modification if* $q = \infty$) *is an equivalent quasi-norm in* $B^s_{p,q}(R_n)$

(*ii*) *Let* $1 \leq p < \infty$, $1 \leq q \leq \infty$ *and* $s \geqq n + 5 + \frac{6n}{\min(p,q)}$ *If* ℓ *is a natural number,* $\ell > s$, *then*

$$||f||_{L_p(R_n)} + ||[\int_{R_n} |h|^{-sq} |(\Delta_h^{\ell} f)(\cdot)|^q \frac{dh}{|h|^n}]^{\frac{1}{q}}||_{L_p(R_n)} \tag{13}$$

(its modification if $q = \infty$*) is equivalent norm in* $F^s_{p,q}(R_n)$.

REMARK 9. A proof of this theorem will be given in [12]. (12) with $1 < p < \infty$, $1 \leq q \leq \infty$ and $s > 0$ is well-known, cf. e.g. [5, 9]. It would be desirable to extend (13) to values $p < 1$.

6. Spaces on Domains and C^∞-Manifolds, Principles

Again we proceed in a somewhat heuristical way. As has been described in Section 3, our leitmotif for treating spaces on R_n are "good" properties which are closely connected with applications to partial differential equations (in particular, elliptic differential equations). So it seems reasonable (and in a certain sense necessary) to extend this point of view to spaces defined on domains in R_n . Since all the quasi-norms $||f||^{\phi}_{B^s_{p,q}(R_n)}$, $\phi \in \mathcal{A}^0$ are mutually equivalent, we write simply $||f||_{B^s_{p,q}(R_n)}$. Similarly, $||f||_{F^s_{p,q}(R_n)}$.

Definition 2. Let Ω *be a bounded* C^∞*-domain in* R_n .

(i) If $-\infty < s < \infty$, $0 < p \leq \infty$ *and* $0 < q \leq \infty$ *, then* $B^s_{p,q}(\Omega)$ *is the restriction of* $B^s_{p,q}(R_n)$ *to* Ω *, equipped with the quasi-norm*

$$||f||_{B^s_{p,q}(\Omega)} = \inf ||g||_{B^s_{p,q}(R_n)} ,$$

where the infimum is taken over all $g \in B^s_{p,q}(R_n)$ *with* $f = g$ *in* Ω *(in the sense of distributions* $D'(\Omega)$ *).*

(ii) If either $-\infty < s < \infty$, $0 < p < \infty$ *and* $0 < q \leq \infty$ *or* $-\infty < s < \infty$, $p = \infty$ *and* $1 < q \leq \infty$ *, then* $F^s_{p,q}(\Omega)$ *is the restriction of* $F^s_{p,q}(R_n)$ *to* Ω *, equipped with the quasi-norm*

$$||f||_{F^s_{p,q}(\Omega)} = \inf ||g||_{F^s_{p,q}(R_n)} ,$$

where the infimum is taken over all $g \in F^s_{p,q}(R_n)$ *with* $f = g$ *in* Ω *(in the sense of distributions* $D'(\Omega)$ *).*

REMARK 10. The spaces of Definition 2 are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$). This follows from Theorem 1. Furthermore, since Ω is a bounded smooth domain, in the cases described in Theorem 2 (i), (ii), and (iii), the above definition coincides with well-known other definitions, cf. [9], Chapter 4.

We are interested in properties of the spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$. Our goal is to apply these spaces to linear regular elliptic boundary value problems (extension of the well-known L_p-theory, $1 < p < \infty$, and the corresponding theory in Hölder spaces, cf. e. g. [9], Chapter 5). If $A(R_n)$ is a quasi-Banach space on R_n with $A(R_n) \subset S'(R_n)$, then $A(R_n^+)$ and $A(\Omega)$ denote the restriction of $A(R_n)$ to $R_n^+ = \{x | x_n > 0\}$ and to the bounded C^∞-domain Ω , respectively. If we want to replace the space L_p with $1 < p < \infty$ by a space A in the theory of regular elliptic boundary value problems, then a detailed examination of the L_p-theory (cf. e. g. [9], Chapter 5) shows that such a space A should satisfy a few basic principles :

Multiplier Property. $A(R_n)$ has the multiplier property if the Criterion from Section 3 is satisfied.

Multiplication Property. $A(R_n)$ has the multiplication property if there exists a positive number ρ such that $f \to gf$ (pointwise multiplication) yields a linear and bounded mapping from $A(R_n)$ into itself for all $g \in \mathcal{C}^\rho(R_n)$.

Diffeomorphism Property. Let $y = \psi(x)$ be an infinitely differentiable one-to-one mapping from R_n onto itself (diffeomorphism) such that $\psi(x) = x$ for large values of $|x|$. $A(R_n)$ has the diffeomorphism property if $f(x) \to f(\psi(x))$ (in the sense of distributions) is a one-to-one mapping from $A(R_n)$ onto itself.

Extension Property. $A(R_n)$ has the extension property if there exists a linear and bounded extension operator from $A(R_n^+)$ into $A(R_n)$ (i.e. a linear and bounded operator S from $A(R_n^+)$ into $A(R_n)$ such that the restriction of Sf with $f \in A(R_n^+)$ to R_n^+ coincides with f).

REMARK 11. These are the main basic principles which seem to be indispensable if one wants to deal with regular elliptic boundary value problems in the framework of the spaces $A(\Omega)$. Of course, for a successful attempt, one needs further properties, and also a detailed knowledge about traces of functions belonging to $A(\Omega)$ on the boundary $\partial\Omega$ of Ω (cf. Theorem 6 below).

Theorem 5. (i) If $-\infty < s < \infty$, $0 < p \leq \infty$ *and* $0 < q \leq \infty$, *then* $B^s_{p,q}(R_n)$ *has the multiplier property, the multiplication property, the diffeomorphism property and the extension property.*
(ii) If $-\infty < s < \infty$, $0 < p < \infty$ *and* $0 < q \leq \infty$, *then* $F^s_{p,q}(R_n)$ *has the multiplier property, the multiplication property and the diffeomorphism property.*

REMARK 12. Proofs of these assertions may be found in [11] and [12]. It is not clear (but probably true) whether $F^s_{p,q}(R_n)$ has also the extension property (in the cases $H^s_p(R_n) = F^s_{p,2}(R_n)$ with $1 < p < \infty$ this is known, cf. [9], p. 218). In other words, the consideration of regular elliptic boundary value problems must be restricted to the spaces $B^s_{p,q}(\Omega)$ at this moment.

REMARK 13. If a space $A(R_n)$ satisfies the multiplication property and the diffeomorphism property, then one can apply the well-known method of local charts (local coordinates). In particular, if one has a compact (n-1)-dimensional C^∞-manifold then one can define the corresponding spaces A on this manifold by standard prodedures. If $\partial\Omega$ is the boundary of a C^∞-domain Ω in R_n , then by Theorem 5 one can introduce $B^s_{p,q}(\partial\Omega)$ (where $-\infty < s < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$) and $F^s_{p,q}(\partial\Omega)$ (where $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leq \infty$). For details we refer to [9], p. 281, and [12].

Since we intend to apply the above spaces $B^s_{p,q}(\Omega)$ to boundary value problems (cf. Remark 12) we need an information about the traces of functions belonging to those spaces. As far as the spaces

$\Lambda^s_{p,q}(\Omega)$ (with $1 < p < \infty$ and $1 \leq q \leq \infty$) and $H^s_p(\Omega)$ (with $1 < p < \infty$) are concerned we refer to [9], p. 330. Let Ω be again a bounded C^∞-domain in R_n and let ν be the (outer) normal with respect to $\partial\Omega$ (boundary of Ω).

<u>*Theorem 6.*</u> *Let* $r = 0,1,2,\ldots$ *If* $0 < p \leq \infty$, $0 < q \leq \infty$ *and* $s > r + \frac{1}{p} + \max(0,(n-1)(\frac{1}{p} - 1))$ *then* R ,

$$Rf = \{f\Big|_{\partial\Omega}, \frac{\partial f}{\partial\nu}\Big|_{\partial\Omega}, \ldots, \frac{\partial^{\nu} f}{\partial\nu^{\nu}}\Big|_{\partial\Omega}\}, \tag{14}$$

yields a linear and continuous mapping from $B^s_{p,q}(\Omega)$ *onto* $\prod_{j=0}^{r} B^{s-\frac{1}{p}-j}_{p,q}(\partial\Omega)$.

<u>REMARK 14.</u> A proof of the theorem with R^+_n instead of Ω may be found in [11], 2.4. The imbedding on the boundary has to be understood in the same sense as in the usual theory of Sobolev-Besov spaces. The restrictions for s (in particular if $p < 1$) are natural, and cannot be improved. One can also prove the corresponding theorem for the spaces $F^s_{p,q}(R^+_n)$. The possibility to replace R^+_n by Ω is based on Theorem 5 and the method of local coordinates, cf. [11], Section 3. Furthermore, one can prove that there exists a linear and continuous operator S from $\prod_{j=0}^{r} B^{s-\frac{1}{p}-j}_{p,q}(\partial\Omega)$ into $B^s_{p,q}(\Omega)$ such that $R \circ S = E$ (identity in $\prod_{j=0}^{r} B^{s-\frac{1}{p}-j}_{p,q}(\partial\Omega)$) . In other words, S is an extension operator. One has also the corresponding counterpart for the spaces $F^s_{p,q}(\Omega)$.

7. <u>Distinguished and Quasi-Distinguished Spaces</u>

As we have remarked several times we consider spaces from the point of view of applications to partial differential equations. If one deals not only with linear, but with quasi-linear and general non-linear differential and integro-differential equations then some additional properties of the above spaces are desirable. We restrict ourselves to the spaces $B^s_{p,q}(R_n)$. With a few exceptions (e.g. statements about convolution algebras), the corresponding assertions hold also for the spaces $B^s_{p,q}(\Omega)$, where Ω is again a bounded C^∞-domain in R_n. Furthermore, some results of the type formulated below for the spaces $B^s_{p,q}(R_n)$ can also be obtained for the spaces $F^s_{p,q}(R_n)$ and $F^s_{p,q}(\Omega)$ (but sometimes not in such a final form).

(i) *Multiplication Algebras.* A space $B^s_{p,q}(R_n)$ is said to be a multiplication algebra if for any couple $f \in B^s_{p,q}(R_n)$ and $g \in B^s_{p,q}(R_n)$ the pointwise multiplication $f \cdot g$ belongs also to $B^s_{p,q}(R_n)$ and if there exists a constant c such that all those couples f and g satisfy

$$\|f \cdot g\|_{B^s_{p,q}(R_n)} \leq c \, \|f\|_{B^s_{p,q}(R_n)} \|g\|_{B^s_{p,q}(R_n)} .$$

(ii) *Convolution Algebras.* If f and g are two distributions, then $f * g$ denotes the convolution (if it exists). A space $B^s_{p,q}(R_n)$ is said to be a convolution algebra if for any couple $f \in B^s_{p,q}(R_n)$ and $g \in B^s_{p,q}(R_n)$ the convolution $f * g$ belongs also to $B^s_{p,q}(R_n)$ and if there exists a constant c such that all those couples f and g satisfy

$$\|f * g\|_{B^s_{p,q}(R_n)} \leq c \, \|f\|_{B^s_{p,q}(R_n)} \|g\|_{B^s_{p,q}(R_n)} .$$

(iii) *Schauder Bases.* A set $\{f_j\}_{j=1}^\infty \subset B^s_{p,q}(R_n)$ is said to be a Schauder basis in $B^s_{p,q}(R_n)$ if each element $f \in B^s_{p,q}(R_n)$ can be uniquely represented as

$$f = \sum_{j=1}^\infty \beta_j f_j \quad , \quad \beta_j \text{ complex numbers}$$

(convergence in $B^s_{p,q}(R_n)$). In separable Banach spaces this is the usual notation of a Schauder basis. Our spaces, in general, are only quasi-Banach spaces. As in the case of Banach spaces, one can prove that the linear operators P_N acting in $B^s_{p,q}(R_n)$,

$$P_N f = \sum_{j=1}^{N} \beta_j f_j \text{ , where } N = 1,2,3,\dots \text{ ,}$$

are uniformly bounded. In particular, $\beta_j = \beta_j(f)$ is a linear and continuous functional on $B^s_{p,q}(R_n)$. This shows that a quasi-Banach space with a Schauder basis must possess a sufficiently large dual space. If one replaces, for instance, $B^s_{p,q}(R_n)$ by $L_p(R_n)$ with $0 < p < 1$ then one has no chance to find a Schauder basis, because the topological dual of $L_p(R_n)$ consists of the 0-functional. On the other hand, the dual spaces of $B^s_{p,q}(R_n)$ are again spaces of such a type, cf. [11], 2.5.

<u>REMARK 15.</u> Comments and also more precise definitions (as far as the algebras are concerned) may be found in [11], 2.3.8 and 2.6.2. Schauder bases will be considered in [13].

<u>Theorem 7.</u> Let $-\infty < s < \infty$, $0 < p \leq \infty$ *and* $0 < q \leq \infty$.

(i) $B^s_{p,q}(R_n)$ *is a multiplication algebra if and only if*
either $0 < p \leq \infty$, $0 < q \leq \infty$, $s > \frac{n}{p}$
or $0 < p \leq \infty$, $0 < q \leq 1$, $s = \frac{n}{p}$.

(ii) $B^s_{p,q}(R_n)$ *is a convolution algebra if and only if* $0 < p \leq 1$, $0 < q \leq \infty$ *and* $s \geq n(\frac{1}{p} - 1)$.

(iii) If $-\infty < s < \infty$, $\infty > p > \frac{n}{n+1}$ *and* $0 < q < \infty$, *then* $B^s_{p,q}(R_n)$ *has a Schauder basis.*

<u>REMARK 16.</u> Proofs may be found in [11], 2.3.8 and 2.6.2, and [13]. The assertions (i) and (ii) are final, in contrast to (iii). In [13] we proved that the spaces $B^s_{p,q}(R_n)$, where (s,p) belongs to the interior of the area characterized by ≡ in Fig.2 and $0 < q < \infty$, have a Schauder basis consisting of Haar functions. (Beside limiting cases this assertion cannot be improved.) If one uses lifting properties then it follows that all spaces $B^s_{p,q}(R_n)$ with real s , $0 < q < \infty$ and $\infty > p > \frac{n}{n+1}$, have a Schauder basis. We conjecture

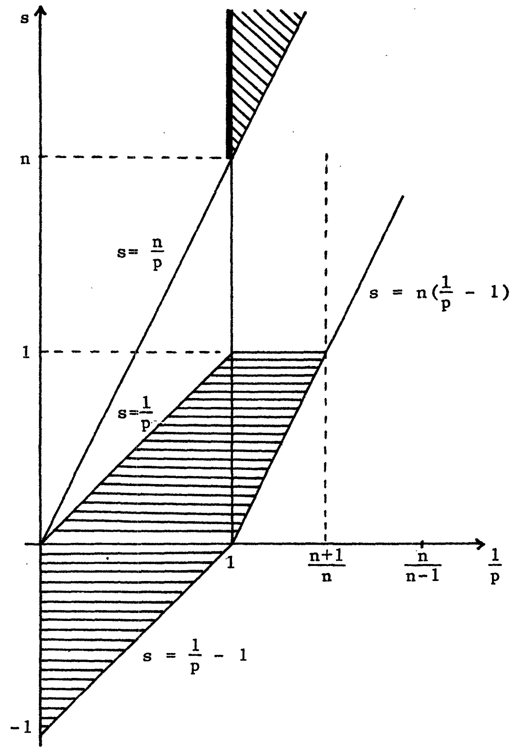

Fig. 2

$0 < q < \infty$ and $\infty > p > \frac{n}{n+1}$, have a Schauder basis. We conjecture that one can extend these considerations to higher spline functions and that one obtains in that way spline bases for all spaces $B^s_{p,q}(R_n)$ with real s, $0 < p < \infty$ and $0 < q < \infty$.

<u>*Definition 3.*</u> *(i)* $B^s_{p,q}(R_n)$ *is said to be a distinguished space if it is a Banach space, a multiplication algebra and a convolution algebra, and if it has a Schauder basis.*

(ii) $B^s_{p,q}(R_n)$ *is said to be a quasi-distinguished space if it is a multiplication algebra and a convolution algebra, and if it has a Schauder basis.*

REMARK 17. In any case, $B^s_{p,q}(R_n)$ is a quasi-Banach space. In other words, the difference between distinguished spaces and quasi-distinguished spaces is that the space in question is not only a quasi-Banach space but a Banach space. It seems reasonable that for applications Banach spaces are more convenient than quasi-Banach spaces, because one has a lot of theorems in Banach spaces which are useful for applications. On the other hand, one of the most striking facts of the theory of the spaces $B^s_{p,q}(R_n)$ is that $p = 1$ (which, roughly speaking, marks the border between Banach spaces and quasi-Banach spaces, which are not Banach spaces) does not play a peculiar role in many theorems. So it seems reasonable to extend assertions for Banach spaces to special classes of quasi-Banach spaces, which include e.g. the above quasi-distinguished spaces.

Theorem 8. *(i)* $B^s_{p,q}(R_n)$ *is a distinguished space if*

either $p = 1,\ 1 < q < \infty,\ s > n$

or $p = 1,\ q = 1,\ s \geqq n$.

(ii) $B^s_{p,q}(R_n)$ *is a quasi-distinguished space if*

either $\frac{n}{n+1} < p \leqq 1,\ 1 < q < \infty,\ s > \frac{n}{p}$

or $\frac{n}{n+1} < p \leqq 1,\ 0 < q \leqq 1,\ s \geqq \frac{n}{p}$.

REMARK 18. The area where $B^s_{p,q}(R_n)$ is a multiplication algebra as well as a convolution algebra has been marked in Fig.2 by . As we have mentioned above, the restriction $\frac{n}{n+1} < p \leqq 1$ in part (ii) of the theorem can probably be replaced by $0 < p \leqq 1$. The distinguished spaces have been characterized in Fig. 2 by a bold line.

REMARK 19. The best choice in Theorem 8 (i) seems to be $p = q = 1$, i.e. the spaces $B^s_{1,1}(R_n)$ with $s \geqq n$ satisfy almost all what one wants (or not?). There are nice explicit norms, cf. Theorem 3, and, in particular, Theorem 4. There is a lot of other equivalent norms, the full theory of equivalent norms as developed e.g. in [9], Chapter 4, is applicable. In particular, Δ_h^m in (12) can be replaced in the usual way by $\Delta_h^m D^\alpha$ etc. By Theorem 5, these spaces have the multiplier property, the multiplication property (which follows also from the fact that these spaces are multiplication algebras), the diffeomorphism property, and the extension property. Furthermore, by Theorem 6 (and Remark 14) one has a final knowledge about traces (Ω can be replaced by R_n^+). By Theorem 8, these spaces are multiplication algebras and convolution algebras. There exist Schauder bases (probably even spline bases). Some of these properties can be carried over immediately to the spaces $B_{1,1}^s(\Omega)$ with $s \geq n$, other properties are typical for spaces defined on R_n (e.g. multiplier properties, convolution algebras).

8. Linear Regular Elliptic Differential Equations

Let Ω be again a bounded C^∞-domain in R_n . We want to apply the theory of the spaces $B_{p,q}^s(\Omega)$ to linear regular elliptic differential equations. Proofs may be found in [12] (a short summary which coincides partly with the description below, has been given in the appendix of [11]). First we recall some well-known definitions, cf. also [9], pp. 361-364.

A differential operator A

$$(Af)(x) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha f(x) \;,$$

is said to be properly elliptic, if

$$a(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \neq 0 \quad \text{for all} \quad R_n \ni \xi \neq 0 \quad \text{and all} \quad x \in \overline{\Omega} \;,$$

and if for all couples $\xi \in R_n$ and $\eta \in R_n$ of linearly independent vectors and all $x \in \overline{\Omega}$ the polynomial $a(x,\xi+\tau\eta)$ in the complex variable τ has exactly m roots $\tau_k = \tau_k(x,\xi,\eta)$ with $\mathrm{Im}\tau_k > 0$ (including multiplicities). Here $a_\alpha(x)$ are infinitely differentiable complex-valued functions in $\overline{\Omega}$. Let

$$a^{+}(x,\xi,\eta,\tau) = \prod_{j=1}^{m} (\tau - \tau_k) .$$

Let B_j,

$$(B_j f)(x) = \sum_{|\alpha| \leqq m_j} b_{j,\alpha}(x) D^{\alpha} f(x) ,$$

$j = 1,\ldots,m$, be m differential operators, where $b_{j,\alpha}(x)$ are infinitely differentiable complex-valued functions on $\partial\Omega$. Then $\{B_j\}_{j=1}^{m}$ is said to be a normal complemented system (with respect to the above operator A) if $0 \leqq m_1 < m_2 < \ldots < m_m \leqq 2m-1$, if

$$\sum_{|\alpha|=m_j} b_{j,\alpha}(x) \nu_x^{\alpha} \neq 0 , \quad j = 1,\ldots,m ,$$

for every normal vector $\nu_x \neq 0$ on $\partial\Omega$ with $x \in \partial\Omega$, and if for all $x \in \partial\Omega$, the corresponding normal vector $\nu_x \neq 0$ and every tangential vector $\xi_x \neq 0$ at the point x with respect to $\partial\Omega$, the polynomials in τ

$$b_j(x,\xi_x + \tau\nu_x) = \sum_{|\alpha|=m_j} b_{j,\alpha}(x)(\xi_x + \tau\nu_x)^{\alpha}$$

are linearly independent modulo $a^{+}(x,\xi_x,\nu_x,\tau)$. If all these assumptions are satisfied then $\{A, B_1, \ldots, B_m\}$ is said to be a regular elliptic problem.

We put $\mathcal{C}^s(\Omega) = B^s_{\infty,\infty}(\Omega)$ if s is a real number, cf. Theorem 2 (i). Furthermore, for our purpose the following additional assumption is convenient.

Hypothesis. *If* $f(x) \in C^{\infty}(\overline{\Omega})$ *with* $(Af)(x) = 0$ *for* $x \in \overline{\Omega}$ *and* $(B_j f)(x) = 0$ *for* $x \in \partial\Omega$ *and* $j = 1,\ldots,m$, *then* $f(x) \equiv 0$.

REMARK 20. In other words, the origin belongs to the resolvent set if $\{A, B_1, \ldots, B_m\}$ is considered as a mapping between appropriate function spaces.

Theorem 9. Let $\{ A, B_1, \ldots, B_m\}$ *be a regular elliptic problem and let the above Hypothesis be satisfied.*

(i) If

$$\text{either} \quad 1 \leqq p < \infty \quad , 0 < q \leqq \infty \quad , s > \frac{1}{p} - 1$$

$$\text{or} \quad \frac{n-1}{n} < p < 1 \quad , 0 < q \leqq \infty \quad , s > n\left(\frac{1}{p} - 1\right) ,$$

then $\{A, B_1, \ldots, B_m\}$ *yields an isomorphic mapping from*

$$B^{s+2m}_{p,q}(\Omega) \quad \text{onto} \quad B^{s}_{p,q}(\Omega) \times \prod_{j=1}^{m} B^{s+2m-k_j-\frac{1}{p}}_{p,q}(\partial\Omega) .$$

(ii) If $s > -1$ *, then* $\{A, B_1, \ldots, B_m\}$ *yields an isomorphic mapping from* $\mathcal{C}^{s+2m}(\Omega)$ *onto* $\mathcal{C}^{s}(\Omega) \times \prod_{j=1}^{m} \mathcal{C}^{s+2m-k_j}(\partial\Omega)$.

REMARK 21. It is not clear at this moment whether the restriction $p > \frac{n-1}{n}$ is natural or not. Part (i) with $1 < p < \infty$ and $1 \leqq q \leqq \infty$ follows essentially from the L_p-theory by interpolation, cf. [9], Chapter 5. Part (ii) with $s > 0$ is one of the main assertions of the famous theory of regular elliptic differential equations in Hölder spaces. Hence the most interesting assertion of the theorem is that for $p \leqq 1$. As mentioned above, proofs are given in [12] .

REMARK 22. Of course, the assertions of the theorem include the corresponding a-priori estimates.

References

[1] R. A. ADAMS: Sobolev Spaces. Academic Press; New York, San Francisco, London, 1975.

[2] O. V. BESOV, V. P. IL'IN, S. M. NIKOL'SKIJ: Integral Representations of Functions and Imbedding Theorems. Nauka, Moskva, 1975 (Russian).

[3] C. FEFFERMAN, E. M. STEIN: H^p spaces of several variables. Acta Math. 129(1972), 137-193.

[4] A. KUFNER, O, JOHN, S. FUČÍK: Function Spaces. Academia, Prague, 1977.

[5] S. M. NIKOL'SKIJ: Approximation of Functions of Several Variables and Imbedding Theorems. Nauka, Moskva 1969 (Russian) (English translation, Springer-Verl., Berlin, Heidelberg, New York, 1975.)

[6] J. PEETRE: New Thoughts on Besov Spaces. Duke Univ. Math. Series, Duke Univ., Durham, 1976.

[7] E. M. STEIN: Singular Integrals and Differentiability Properties of Functions. Princeton Univ. Press, Princeton, 1970.

[8] E. M. STEIN, G. WEISS: Introduction to Fourier Analysis on Euclidean Spaces. Princeton Univ. Press, Princeton, 1971.

[9] H. TRIEBEL: Interpolation Theory, Function Spaces, Differential Operators. VEB Deutscher Verl. Wissenschaften, Berlin 1978.

[10] H. TRIEBEL: Fourier Analysis and Function Spaces. Teubner - Texte Math., Teubner, Leipzig, 1977.

[11] H. TRIEBEL: Spaces of Besov - Hardy - Sobolev Type. Teubner - Texte Math., Teubner, Leipzig, 1978.

[12] H. TRIEBEL: On Besov - Hardy - Sobolev spaces in domains and regular elliptic boundary value problems. The case $0 < p \leq \infty$ (to appear).

[13] H. TRIEBEL: On Haar bases in Besov spaces. (To appear.)

[14] A. ZYGMUND: Trigonometric Series I, 2nd Ed. Cambridge Univ. Press, Cambridge, London, New York, Melbourne, 1977.