# SUBHARMONIC BIFURCATION IN EQUIVARIANT SYSTEMS

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1. INTRODUCTION. We know from equivariant bifurcation theory (see e.g. [4] or [6]) that the presence of symmetry can considerably change the typical bifurcation behaviour of a system. In this note we discuss a problem – namely subharmonic bifurcation – where the presence of symmetry not only leads to different answers, but forces us even to ask different questions.

Consider the differential equation

$$\dot{x} = f(x,\lambda) \ , \tag{1}$$

with $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^k$ (the parameter space), and $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ smooth. Suppose that (1) has for $\lambda=0$ a periodic solution $x_0(t)$, with minimal period $T_0>0$. The problem of subharmonic bifurcation is then the following : describe, for some given $q \geq 2$ and for all $\lambda$ near zero, all periodic solutions of (1) with minimal period near $qT_0$ and with orbit nearby $\kappa_0 := \{x_0(t) \mid t \in \mathbb{R}\}$. Such solutions can be found in all neighborhoods of $\kappa_0$ only if $x_0(t)$ has some Floquet multiplier $\mu$ with $\mu^q = 1$ and $\mu^{q'} \neq 1$ for $q' < q$. Generically period-doubling (i.e. the case q=2) can happen in one-parameter problems (k=1), while subharmonic bifurcation with $q \geq 3$ requires two parameters (k=2) and leads to so-called Arnol'd tongues; these are thin horn-like regions in parameter space corresponding to parameter values for which (1) has a subharmonic solution. The standard method to study subharmonic bifurcation consists in finding q-periodic points of a Poincaré map associated to $x_0(t)$ (see [1]).

Now suppose that our system (1) is symmetric, i.e. there exists a compact group $\Gamma \subset L(\mathbb{R}^n)$ such that

$$f(\gamma x,\lambda) = \gamma f(x,\lambda) \quad , \quad \forall \gamma \in \Gamma \ ; \tag{2}$$

we also say that (1) is $\Gamma$-equivariant. Without loss of generality we may assume that $\Gamma \subset O(n)$; moreover, $\Gamma$ forms a Lie group, and we will denote by $L(\Gamma)$ its Lie algebra, i.e. the tangent space to $\Gamma$ at the identity. An immediate consequence of (2) is that instead of just a periodic orbit $\kappa_0$ we now have a compact invariant manifold

$$M_0 := \{\gamma x_0(t) \mid t \in \mathbb{R}, \ \gamma \in \Gamma\} \ , \tag{3}$$

foliated with periodic orbits of $(1)_{\lambda=0}$; this manifold is invariant under both the flow and the group action. It is then clear that the questions we should ask are the following :

(i) can we continue the manifold $M_0$, i.e. does there exist a family of compact manifolds $M_\lambda$, depending in some sense smoothly on $\lambda$, coinciding with $M_0$ for $\lambda=0$, and such that $M_\lambda$ is invariant under the group action of $\Gamma$ and under the flow of $(1)_\lambda$?

(ii) are there bifurcations of such manifolds?

(iii) if there are bifurcations, how does the dimension of the manifolds and the flow on them change?

The following (intuitive) example shows that this is not the same as asking for the continuation and the bifurcations of the periodic orbit $\kappa_0$.

Suppose $\Gamma \cong SO(2)$ and $f(x,\lambda) = f_0(x) + \lambda f_t(x)$, where $f_0$ and $f_t$ are $\Gamma$-equivariant, and such that $f_t(x)$ is for each $x \in \mathbb{R}^n$ tangent to the group orbit $\Gamma x$. Suppose $f_0$ has a periodic orbit $\kappa_0$ such that $M_0$ is a 2-torus. Then $M_0$ is still invariant for $\lambda\neq 0$, but the orbits of $f_\lambda$ on $M_0$ will either not be periodic at all, or periodic with a very large period (if $\lambda$ is small). We see that although we can continue the manifold $M_0$, we cannot continue the periodic orbit $\kappa_0$.

2. <u>A POINCARE MAP</u>. We will now combine some ideas of Chossat and Golubitsky [2] with a construction introduced by Fiedler [3] to obtain a Poincaré map which forms the basic tool for studying the bifurcation problem described above. This Poincaré map will reflect the symmetry of the periodic orbit $\kappa_0$, which we can describe as follows (see [3]).

We call

$$H_0 := \{\gamma \in \Gamma \mid \gamma(\kappa_0)=\kappa_0\} \qquad (4)$$

the *orbital symmetry* of $\kappa_0$, and

$$K_0 := \{\gamma \in \Gamma \mid \gamma\bar{x}_0=\bar{x}_0\} \qquad (5)$$

the *spatial symmetry* of $\kappa_0$. In (5) $\bar{x}_0$ is any point of $\kappa_0$; the definition is independent of the choice of $\bar{x}_0 \in \kappa_0$. It is easy to see that $K_0$ is a normal subgroup of $H_0$, and that $H_0/K_0$ is cyclic, i.e. we have either $H_0/K_0 \cong S^1$ or $H_0/K_0 \cong \mathbb{Z}_m$ for some $m\geq 1$. If $H_0/K_0 \cong S^1$ then we say that $\kappa_0$ corresponds to a *rotating wave* solution; in that case $M_0$ consists of just one single group orbit, i.e. $M_0 = \Gamma\bar{x}_0$, and we say that $\bar{x}_0$ is a relative equilibrium. For more information on this case we refer to recent work of Krupa [5] (see also [7]). Here we will restrict to the case where $H_0/K_0 \cong \mathbb{Z}_m$ for some $m\geq 1$; we put $\bar{x}_0 := x_0(0)$, and we fix $\delta \in H_0$ such that $x_0(t+T_0/m) = \delta x_0(t)$. We have then $K_0 = \mathrm{Fix}(\bar{x}_0)$, $\delta\kappa_0\delta^{-1} = K_0$ and $\delta^m \in K_0$.

The tangent space at $\bar{x}_0$ to $M_0$ is given by

$$T_{\bar{x}_0}M_0 = \mathrm{span}\{\dot{x}_0(0)\} \oplus \{\eta\bar{x}_0 \mid \eta\in L(\Gamma)\} . \qquad (6)$$

We define $Y_0 := (T_{\bar{x}_0}M_0)^\perp$ and $S_0 := \{\bar{x}_0+y_0 \mid y_0 \in Y_0\}$. Remark that $T_{\bar{x}_0}M_0$, $Y_0$ and $S_0$ are invariant under $K_0$. When we start at a point $x = \bar{x}_0+y_0 \in S_0$ $(y_0 \in Y_0)$

sufficiently close to $\bar{x}_0$ then the flow $\phi(t;x,\lambda)$ of (1) will in general not intersect $S_0$ after some time near $T_0$, but it will intersect $S_0$ *modulo a symmetry operator*, i.e. it will intersect $\Gamma S_0$. This is the basic idea behind the $K_0$-equivariant Poincaré map $\Pi : Y_0 \times \mathbb{R}^k \to Y_0$ which we construct as follows.

The group $\Gamma$ acts on $L(\Gamma)$ via the adjoint action $(\gamma,\eta) \mapsto \gamma\eta\gamma^{-1}$; using this action we see that $L(K_0)$ is $K_0$-invariant, and hence there exists a $K_0$-invariant subspace U of $L(\Gamma)$ such that $L(\Gamma) = L(K_0) \oplus U$. A simple application of the implicit function theorem then shows that for each sufficiently small $(y_0,\lambda) \in Y_0 \times \mathbb{R}^k$ there exists a unique $(\tau,\eta) = (\tilde{\tau}(y_0,\lambda),\tilde{\eta}(y_0,\lambda)) \in \mathbb{R}\times U$ near $(T_0,0)$ such that $e^{-\eta}\phi(\tau;\bar{x}_0+y_0,\lambda) \in S_0$. We define then

$$\Pi(y_0,\lambda) := e^{-\tilde{\eta}(y_0,\lambda)}\phi(\tilde{\tau}(y_0,\lambda),y_0,\lambda) - \bar{x}_0 \; ; \tag{7}$$

It is easy to check that for each $\sigma \in K_0$ we have

$$\begin{aligned} &\tilde{\tau}(\sigma y_0,\lambda) = \tilde{\tau}(y_0,\lambda) \;,\; \tilde{\eta}(\sigma y_0,\lambda) = \sigma\tilde{\eta}(y_0,\lambda)\sigma^{-1} \\ &\text{and } \Pi(\sigma y_0,\lambda) = \sigma\Pi(y_0,\lambda) \;, \end{aligned} \tag{8}$$

i.e. $\Pi$ is $K_0$-equivariant. We have $\Pi(0,0) = 0$, and the eigenvalues of $D_1\Pi(0,0)$ will be the Floquet multipliers of $x_0(t)$ which are not forced to be 1 by the flow and the symmetry. For example, if the multiplicity of 1 as a multiplier of $x_0(t)$ equals $\dim T_{\bar{x}_0}M_0 = 1 + \dim U$, then 1 will not be an eigenvalue of $D_1\Pi(0,0)$ and $\Pi$ will have for each sufficiently small $\lambda$ a unique $K_0$-invariant fixed point $\bar{y}_0(\lambda)$, corresponding to a continuation of $M_0$.

When $m>1$ then $\Pi$ has a more detailed structure which not only reflects the spatial symmetry $K_0$ but also the orbital symmetry $H_0$ of $\kappa_0$. To see this we set $Y_j := \delta^j(Y_0)$ and $S_j := \delta^j(S_0)$ for $j = 0,1,\ldots,m$. Each of the $Y_j$ and $S_j$ is $K_0$-invariant, since $\delta K_0\delta^{-1} = K_0$, while $Y_m = Y_0$ and $S_m = S_0$, since $\delta^m \in K_0$. In a similar way as above one defines then $K_0$-equivariant mappings $\hat{\Pi}_j : Y_j \times \mathbb{R}^k \to Y_{j+1}$ $(j=0,1,\ldots,m-1)$, corresponding to "partial" Poincaré maps. One can then easily check (see [3]) that $\hat{\Pi}_{j+1} = \delta\hat{\Pi}_j\delta^{-1}$, and when we define $\Pi_0 : Y_0 \times \mathbb{R}^k \to Y_0$ by $\Pi_0 := \delta^{-1}\hat{\Pi}_0$, then

$$\Pi = \delta^m \Pi_0{}^m \; . \tag{9}$$

The mapping $\Pi_0$ is not $K_0$-equivariant in the strict sense, but satisfies

$$\Pi_0(\sigma y_0,\lambda) = (\delta^{-1}\sigma\delta)\Pi_0(y_0,\lambda) \quad , \quad \forall\sigma \in K_0 \; . \tag{10}$$

That means : $\Pi_0$ is equivariant with respect to two different actions of $K_0$; it is however important to notice that the orbits of both actions coincide.

The bifurcation problem described in the introduction now reduces to the following : study the bifurcations from $\{0\}$ of compact manifolds in $Y_0$ which are invariant under $\Pi$ (or $\Pi_0$) and under the group action of $K_0$. Subharmonic bifurcation means in this context the bifurcation from $\{0\}$ of $K_0$-orbits which are invariant under some q-th iterate of $\Pi$. For example, if $M_0$ is a torus and $q=2$, then such bifurcation would correspond to a "torus-doubling". In general the flow on the bifurcating invariant manifolds will not be periodic, but quasi-periodic; therefore we should rather talk about quasi-subharmonic bifurcation.

In certain cases one can obtain sufficient conditions for such bifurcations by considering points with certain isotropy properties (see [2]). For example, if $m=1$ and the multiplier 1 of $\kappa_0$ has minimal multiplicity (given by $1 + \dim U$), then $\Pi=\Pi_0$ (we take $\delta=Id$), and we may without loss of generality assume that $\Pi(0,\lambda)=0$ for all $\lambda$. Modulo some generically satisfied transversality conditions one has then the following :

(i) if $-1$ is a $K_0$-simple multiplier of $\kappa_0$ (see [4] for the definition), with eigenspace $Z$, and if $k=1$, then there corresponds a quasi-period-doubling to each isotropy subgroup $\Sigma$ of $K_0$ such that $\dim Z^{\Sigma} = 1$;

(ii) let $\mu \in \mathbb{C}$ be a $K_0$-simple multiplier of $\kappa_0$ such that $\mu^q=1$ for some $q\geq 3$; on the corresponding (complex) eigenspace $Z$ we define an action of $K_0\times \mathbb{Z}_q$ by

$$(\sigma,j).z = \mu^j\sigma.z \quad , \quad \forall(\sigma,j) \in K_0\times \mathbb{Z}_q \ ; \tag{11}$$

(we take $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$); if $k=2$ then there corresponds a quasi-q-harmonic bifurcation to each isotropy subgroup $\Sigma$ of $K_0\times \mathbb{Z}_q$ such that $\dim_{\mathbb{C}} Z^{\Sigma} = 1$; to each such bifurcation there corresponds an Arnol'd tongue in parameter space.

We will give more details and examples in a forthcoming paper.


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