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Generalized ordinary differential equations - a survey

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GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS – A SURVEY

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This is an account of the work which started in 1957 by the paper [13] of J. Kurzweil. The creation of the generalized (ordinary) differential equations is the result of the effort to explain some convergence effects in the classical theory of ordinary differential equations.

We settle some matters of notation. R denotes the real numbers, R^n is the n-dimensional euclidean space, $G = R^n \times [a,b]$ where $[a,b]$ is an interval in R.

If $F : G \to R^n$ then the symbol

(GDE) $\quad \dfrac{dx}{d\tau} = DF(x,t)$

is used for the generalized differential equation.

Definition. A function $x : [\alpha,\beta] \to R^n$ is called a solution of (GDE) on $[\alpha,\beta]$ if

$$(x(s),s) \in G\ , \quad s \in [\alpha,\beta]$$

and

$$x(s_2)-x(s_1) = \int_{s_1}^{s_2} DF(x(\tau),t), \qquad s_1 s_2 \in [\alpha,\beta],$$

where the expression on the right hand side is the generalized Perron integral in the sense of J. Kurzweil [13],[16].

Definition. $\int_{s_1}^{s_2} DF(x(\tau),t) \in R^n$ is the generalized Perron integral of $F(x(\tau),t)$ over the interval $[s_1,s_2]$ if for every $\varepsilon > 0$ there exists a $\delta : [s_1,s_2] \to (0,+\infty)$ (called a gauge on $[s_1,s_2]$) such that

$$(1) \qquad \Big| \sum_{i=1}^{k} [F(x(\tau_i),\alpha_i)-F(x(\tau_i),\alpha_{i-1})] - \int_{s_1}^{s_2} DF(x(\tau),t) \Big| < \varepsilon$$

for every partition $s_1 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = s_2$ of $[s_1,s_2]$ with $\tau_i \in [\alpha_{i-1},\alpha_i]$, $i=1,2,\ldots,k$ which is δ-fine, i.e. which satisfies

$$(2) \qquad [\alpha_{i-1},\alpha_i] \subset [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)]\ , \quad i=1,2,\ldots,k .$$

Remark. It is worth mentioning that if $F(x(\tau),t) = g(\tau).t$ for

τ, $t \in [s_1, s_2]$ then the generalized Perron integral defined above leads to a certain concept of integral $\int_{s_1}^{s_2} g(t)dt = \int_{s_1}^{s_2} D[g(\tau).t]$ which appeared to be exactly the nonabsolutely convergent Perron integral of the function g over $[s_1, s_2]$. Since in this case the corresponding integral sums in (1) assume the form of Riemann sums $\sum' g(\tau_i).(\alpha_i - \alpha_{i-1})$, the integral introduced by J. Kurzweil in [13] was in fact a new Riemann type definition of the general nonabsolutely convergent Perron integral. The only difference in comparison with the original concept of B. Riemann is the notion of the refinement of the partition which is in our case determined in (2) by a gauge δ. If only constants are admitted in the role of a gauge then our definition of the integral coincides with that of Riemann.

A completely different approach conduced R. Henstock in 1961 to the same definition of the Perron integral. The Kurzweil-Henstock definition has given rise to an important and growing field in integration theory which demonstrates the mutual influence of diff. equations and integral calculus as was the case in the times of I. Newton, A.L. Cauchy, H. Lebesgue and C. Carathéodory (see e.g. the contribution [20] of J. Mawhin to this point of the history of mathematics).

The following result forms a link between GDE's and classical ordinary differential equations (ODE's), see [26].

Theorem 1. Assume that $f : G \to R^n$ satisfies the Carathéodory conditions and set

(3) $$F(x,t) = (L)\int_{t_0}^{t} f(x(s),s)ds, \quad t_0 \in [a,b], \quad (x,t) \in G$$

with the Lebesgue integral on the right hand side.

If $x : [\alpha, \beta] \to R^n$ is the pointwise limit of finite step functions then

$$\int_{s_1}^{s_2} DF(x(\tau),t) = (L)\int_{s_1}^{s_2} f(x(s),s)ds, \quad s_1, s_2 \in [\alpha, \beta].$$

Consequently, we have

Theorem 2. $x : [\alpha, \beta] \to R^n$ is a solution of (GDE) with F given by (3) if and only if

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} f(x(s),s)ds, \quad s_1, s_2 \in [\alpha, \beta] \subset [a,b],$$

i.e. if x is a solution of

(ODE) $\quad x' = f(x,t)$

in the sense of C. Carathéodory.

V.D. Mil'man, A.D. Myshkis and A.M. Samojlenko in the sixties started to study the so called systems with impulses (see e.g. the recent book [22] of A.M. Samojlenko and N.A. Perestjuk).

Given a finite sequence of points $a = t_0 \leqq t_1 < t_2 < \dots < t_k \leqq \leqq t_{k+1} = b$ and mappings $I_i : R^n \to R^n$, $i=1,2,\dots,k$, the system

(ODE) $\quad x' = f(x,t), t \neq t_i, \quad i=1,\dots,k$

(I) $\quad \Delta x|_{t_i} = x(t_i+)-x(t_i) = I_i(x(t_i))$, $\quad i=1,\dots,k$

is called a <u>system with impulses</u> if $f : R^n \times [a,b] \to R^n$ satisfies the Carathéodory conditions. A function $x : [\alpha,\beta] \to R^n$, $[\alpha,\beta] \subset \subset [a,b]$ is called a solution of (ODE)+(I) if x satisfies (ODE) almost everywhere on every interval $(t_{i-1},t_i) \cap [\alpha,\beta]$, $i=1,\dots,k+1$ and if the "interface condition" (I) holds for every i such that $t_i \in [\alpha,\beta]$.

For the system with impulses (ODE)+(I) let us set

$$(4) \qquad F(x,t) = \int_{t_0}^{t} f(x,s)ds + \sum_{i=1}^{k} I_i(x)H_{t_i}(t)$$

with $H_\tau(t) = 0$ for $t \leqq \tau$, $H_\tau(t) = 1$ for $t > \tau$. Then it is known (see [25]) that (GDE) with F given by (4) is equivalent to the system (ODE)+(I) in the same way as was stated above in Theorem 2 for the equivalence of (ODE) and (GDE) with F given by (3).

This result shows that the concept of (GDE) is more general than the concept of a classical ODE because the solutions are allowed to be discontinuous.

The fundamental concepts of the theory of GDE's depend on the function F of the right hand side of (GDE). A fairly wide class of such functions is the class $F(G,h,\Omega)$ defined as follows. Given $G = R^n \times [a,b]$, $h : [a,b] \to R$ nondecreasing, continuous from the left and $\Omega : [0,+\infty) \to R$ increasing, continuous with $\Omega(0) = 0$, we have

Definition. A function $F : G \to R^n$ belongs to the class $F(G,h,\Omega)$ if

(i) $\quad \|F(x,t_2)-F(x,t_1)\| \leqq |h(t_2)-h(t_1)|$,

(ii) $\quad \|F(x,t_2)-F(x,t_1)-F(y,t_2)+F(y,t_1)\| \leqq \Omega(\|x-y\|)|h(t_2)-h(t_1)|$

provided $x,y \in R^n$, $t_1,t_2 \in [a,b]$.

The fundamental results are the following.

Theorem 3. If $F \in F(G,h,\Omega)$, $(x_0,t_0) \in G$ then the initial value problem

(IVP) $\quad \frac{dx}{d\tau} = DF(x,t)$, $\quad x(t_0) = x_0$, $\quad t_0 \in [\alpha,\beta]$

has locally a solution.

If $x : [\alpha,\beta] \to R^n$ is a solution of (GDE) on $[\alpha,\beta]$ with $F \in F(G,h,\Omega)$ then

$$\| x(s_2)-x(s_1) \| \le |h(s_2)-h(s_1)| \ , \quad s_1,s_2 \in [\alpha,\beta] \ .$$

Consequently, $x \in BV[\alpha,\beta]$ and $x(s-) = \lim_{\sigma \to s-} x(\sigma) = x(s)$ for every $s \in (\alpha,\beta]$.

Moreover, using the properties of the generalized Perron integral, it can be shown that

$$x(s+) = \lim_{\sigma \to s+} x(\sigma) = x(s)+F(x(s),s+)-F(x(s),s) \ , \quad s \in [\alpha,\beta)$$

for the solution $x : [\alpha,\beta] \to R^n$ of (GDE).

For $F \in F(G,h,\Omega)$ the problem of local uniqueness of solutions of (IVP) depends on the mutual interaction between the "modulus of continuity" Ω and the values of the function h at t_0 . In general local uniqueness can be derived only for increasing values of t . Elementary examples show that local uniqueness for decreasing values of t (i.e. for $t < t_0$) is not guaranteed even in very simple situations; this is of course caused by the possible discontinuities of solutions.

For example, if

$$\lim_{v \to 0+} \int_v^u \frac{1}{\Omega(r)} \, dr = +\infty \quad \text{for every} \quad u > 0$$

then the solutions of (IVP) are unique for $t > t_0$.

The creation of GDE's was strongly influenced by the following discovery (I.I. Gichman, M.A. Krasnoselskij and S.G. Krejn, J. Kurzweil and Z. Vorel (1952-1957)).

Let us have a sequence of ODE's

$$x' = f_k(x,t) \ , \quad k=1,2,\ldots \ ,$$

with solutions $x_k : [\alpha,\beta] \to R^n$ for which

(5) $$\lim_{k \to \infty} x_k(t) = x(t) \ , \quad t \in [\alpha,\beta] \ .$$

If

(6) $$\lim_{k \to \infty} \int_{t_0}^t f_k(x,s)ds = \lim_{k \to \infty} F_k(x,t) = F_0(x,t) = \int_{t_0}^t f_0(x,s)ds$$

with a certain suitable $f_0 : G \to R^n$ (e.g. f_0 has to be a Carathéodory function), then the limit function $x : [\alpha,\beta] \to R^n$ given by (5) is a solution of the ordinary differential equation

$$x' = f_0(x,t)$$

on the interval $[\alpha,\beta]$.

Further theorems of this type have been successively derived, the assumptions on the kind of convergence in (6) weakened, and finally it was realized that for results of this type only the knowledge of the "indefinite integrals" to the right hand sides of the ODE's in question is required. The right hand sides them-selves serve only for determining the concept of a solution. GDE's are in fact the result of the effort to determine the concept of a solution of an ODE in terms of the "indefinite integral" F (see (6)) of its right hand side.

The following result on continuous dependence on a parameter for GDE's is a relatively easy consequence of the dominated convergence theorem for the generalized Perron integral (see [26] or [16],[25]).

Theorem 4. Assume that $F_k \in F(G,h,\Omega)$, $k=0,1,\dots$ and

(7) $$\lim_{k\to\infty} F_k(x,t), = F_0(x,t) \ , \quad (x,t) \in G \ .$$

Then we have

Conclusion. If $x_k : [\alpha,\beta] \to R^n$, $k=1,2,\dots$ are solutions of

(GDE_k) $$\frac{dx}{d\tau} = DF_k(x,t) \ , \quad k=1,2,\dots$$

on $[\alpha,\beta]$ such that

(5) $$\lim_{k\to\infty} x_k(t) = x(t) \ , \quad t \in [\alpha,\beta]$$

then $x \in BV[\alpha,\beta]$ is a solution of

(GDE_0) $$\frac{dx}{d\tau} = DF_0(x,t)$$

on $[\alpha,\beta]$.

Using the equivalence of an ODE with a GDE a continuous dependence result for ODE's can be obtained via (3). Nevertheless, from the point of view of some applications the requirement that all F_k belong to the same class $F(G,h,\Omega)$ is too restrictive. In a special case we have the following result.

Theorem 5. Assume that $F_k \in F(G,h,\Omega)$, $k=0,1,2,\dots$ where h_k : $[a,b] \to R$, $k=1,2,\dots$ is nondecreasing continuous from the left, $h_0 : [a,b] \to R$ is nondecreasing and continuous and

$$\limsup_{k\to\infty} (h_k(t_2)-h_k(t_1)) \leq h_0(t_2)-h_0(t_1) \ , \quad t_1 \leq t_2 \ .$$

Let

(7) $$\lim_{k\to\infty} F_k(x,t) = F_0(x,t) \ , \quad (x,t) \in G \ .$$

Then Conclusion of Theorem 4 holds.

Theorem 5 enables us to derive the following averaging result for GDE's.

Theorem 6. Assume that $G = R^n \times [0,+\infty)$, $F \in F(G,h,\Omega)$, h is nondecreasing and continuous from the left on $[0,+\infty)$. Assume further that

$$\limsup_{r \to 0} \frac{1}{r} (h(r+\alpha)-h(\alpha)) \leqq C = \text{const.} \quad \text{for every} \quad \alpha \geqq 0 ,$$

$$\lim_{r \to 0} \frac{1}{r} F(x,r) = F_0(x) , \quad x \in R^n$$

and that $y : [0,+\infty) \to R^n$ is a uniquely determined solution of the autonomous ordinary differential equation

$$y' = F_0(y)$$

on $[0,+\infty)$.

Then for every $\mu > 0$ and $L > 0$ there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ we have

$$\| x_\varepsilon(t) - y_\varepsilon(t) \| < \mu \quad \text{for} \quad t \in [0, \tfrac{L}{\varepsilon}] ,$$

where x_ε is a solution of

$$\frac{dx}{d\tau} = D[\varepsilon F(x,t)]$$

on $[0,L/\varepsilon]$ such that $x_\varepsilon(0) = y(0)$ and y_ε is a solution of the averaged equation

$$x' = \varepsilon F_0(x)$$

on $[0,L/\varepsilon]$ such that $y_\varepsilon(0) = y(0)$ (i.e. $y_\varepsilon(t) = y(\varepsilon t)$).

Remark. This result has evidently the form of the well-known Bogoljubov theorem on averaging and can be directly applied e.g. to systems with impulses for obtaining averaging in this case, too. Averaging results for systems with impulses have been derived by A.M. Samojlenko [21]; of course they have been achieved by different methods.

If the assumption of continuity of the function h_0 in Theorem 4 is omitted then the conclusion of this theorem is false in general.

Example. Let us have a "δ-sequence" of functions $\delta_k : [-1,1] \to R$, $k=1,2,\dots$; e.g.

$$\delta_k(t) = k \quad \text{for} \quad t \in (0,1/k] , \quad \delta_k(t) = 0 \quad \text{for} \quad t \in [-1,1] \setminus (0,1/k].$$

Assume that A, B are $n \times n$-matrices. Let us consider the sequence of linear ODE's

$$x' = [A + \delta_k(t).B]x \tag{8}$$

with the initial condition $x(-1) = x_0$. Using the facts mentioned above this sequence of systems of ODE's is equivalent to the sequence of GDE's

$$\frac{dx}{d\tau} = D[(At + \Delta_k(t)B)x] , \tag{9}$$

where $\Delta_k(t) = \int_{-1}^{t} \delta_k(s)ds$.

It is easy to check that the function $x_k : [-1,1] \to R$ given by

$$x_k(t) = \begin{cases} e^{A(t+1)}x_0 & \text{for } t \in [-1,0] , \\ e^{(A+kB)t}e^{A}x_0 & \text{for } t \in (0,1/k] , \\ e^{At}e^{A}x_0 & \text{for } t \in (1/k,1] \end{cases}$$

is a solution of this initial value problem and that

$$(10) \qquad \lim_{k\to\infty} x_k(t) = z(t) = \begin{cases} e^{A(t+1)}x_0 & \text{for } t \in [-1,0] , \\ e^{At}e^{B}e^{A}x_0 & \text{for } t \in (0,1] . \end{cases}$$

It is also easy to see that

$$\lim_{k\to\infty} \Delta_k(t) = H(t) = \begin{cases} 0 & \text{for } t \in [-1,0] , \\ 1 & \text{for } t \in (0,1] \end{cases}$$

and consequently, the right hand sides of the corresponding GDE's (8) satisfies

$$\lim_{k\to\infty} (At + \Delta_k(t)B)x = (At + H(t)B)x .$$

Nevertheless, the solution of GDE

$$\frac{dx}{d\tau} = D[(At + H(t))x]$$

with $x(-1) = x_0$ is given by

$$x(t) = \begin{cases} e^{A(t+1)}x_0 & \text{for } t \in [-1,0] , \\ e^{At}(I+B)e^{A}x_0 & \text{for } t \in (0,1] \end{cases}$$

and we can see that the limit $z(t)$ of the sequence of solutions given in (10) is different from $x(t)$ with the exception of the very special case when $I + B = e^B$ for the matrix B .

This limit behaviour of the sequence of ODE's (8) is caused by the emphatic influence of the term $\delta_k(t)B$ for large k on the support of the k-th term δ_k of the "δ-sequence". On the support of δ_k the ODE (8) behaves like the equation

$$y' = kB.y$$

and a simple transformation of the independent variable shows that the "jump" in the limit equation is determined by the increment of the solution of the ODE $z' = B.z$ over the interval $[0,1]$. Convergence effects of this type are explained by the notion of emphatic convergence which formally delineates the general situation corresponding to that occuring in this example (see e.g. [14] or [26]).

The limit $\dot{z}(t)$ of solutions of the initial value problems in this example given by (10) is a solution of the linear GDE

(11) $$\frac{dx}{d\tau} = D\left[(At + (e^B - I)H(t))x\right]$$

with $x(-1) = x_0$.

In [15] J. Kurzweil mentioned that one of the good reasons for introducing GDE's is to close the class of classical ODE's with respect to the convergence of "indefinite integrals" to their right hand sides, i.e. with respect to the convergence given by (7). The above example shows that the limit equation for a sequence of ODE's is not an ODE in general; indeed, (11) is a GDE which does not represent any classical ODE because of the discontinuity of the solution at the point $t=0$.

In certain cases GDE's form a good frame for developing abstract concepts. Let us shortly mention e.g. the concept of topological dynamics for nonautonomous equations. (It is easy to see that transforming an autonomous ODE into the corresponding GDE we do not obtain any relevant new information.)

Let us assume that **F** is a class of functions $F : R^n \times R \to R^n$ with the following properties:

For every compact $A \subset R^n$ there exist $\mu_A : (0,+\infty) \to (0,+\infty)$ and $N_A \geqq 0$ such that

1. F is continuous, $F(x,0) = 0$.
2. For $A \subset R^n$ compact there exist $l_{A,F}$, $k_{A,F} : R \to R$ nondecreasing, continuous, $l_{A,F}(0) = k_{A,F}(0) = 0$ and

$$\| F(x,s_2) - F(x,s_1) \| \leqq | l_{A,F}(s_2) - l_{A,F}(s_1) | ,$$

$$\| F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1) \| \leqq \| x-y \| . | l_{A,F}(s_2) - l_{A,F}(s_1) |$$

for $x,y \in R^n$, $s_1, s_2 \in R$,

where $|v| < \mu_A(\varepsilon)$ implies $|l_{A,F}(s+v) - l_{A,F}(s)| < \varepsilon$ for every $s \in R$ and $|k_{A,F}(s+1) - k_{A,F}(s)| \leqq N_A$ for $s \in R$.

For the class **F** we have the following results:

F is a topological space with the convergence given by the uniform convergence on compact subsets of $R^n \times R$.

The topological space **F** is compact.

Let us denote by $\Phi(t,y,F)$ the unique maximal solution of the initial value problem

$$\frac{dx}{d\tau} = DF(x,t) , \quad x(0) = y$$

with $F \in \mathbf{F}$, $y \in R^n$. Let $I_{(y,F)} \subset R$ be the maximal interval of definition of the solution Φ .

Define

(12) $\quad \pi(t,y,F) = (\Phi(t,y,F),F_t) \ , \quad (t,y,F)\in R\times R^n\times F,$

where $F_t(x,s) = F(x,t+s)-F(x,t)$, $x\in R^n$, $t,s\in R$.

Using the results on continuous dependence on parameters (e.g. in the form of Theorem 5) the following result can be derived.

Theorem 7. The mapping

$$\mathcal{T}\!\mathcal{C} : \{(t,y,F);(y,F)\in R^n\times F, t\in I_{(y,F)}\}\rightarrow R^n\times F$$

is a local flow on $R^n\times F$ in the sense of G.R. Sell (see e.g. [1],[30], [31] and others).

The compactness of the topological space F and the fact that $\mathcal{T}\!\mathcal{C}$ given above is a local flow open the access to methods of topological dynamics for GDE's. These methods are relevant for classical ODE's, too, due to the fact that the class of ODE's with respect to the topology of the space F of the corresponding "indefinite integrals" is not closed in general but the closure is contained in the class of GDE's. The first and main contribution to these problems belongs to Z. Artstein [1] (see also [26]).

To conclude this short survey let us mention some other fields where the ideas of J. Kurzweil concerning the concept of GDE's have been used.

A. There is an extensive theory of linear GDE's of the form

(13) $$\frac{dx}{d\tau} = D[A(t)x + f(t)],$$

where $A,f\in BV[a,b]$, $[a,b]\subset R$ being an interval. Interesting is the case when (13) is considered together with a side condition of the type

(14) $$\int_a^b d[K(s)]\,x(s) = r$$

with $K\in BV[a,b]$ (see e.g. [28],[29]).

It is perhaps of interest to mention that for the linear problem for the classical ODE

$$x' = a(t)x + g(t)$$

with the side condition (14) the methods of functional analysis lead to an adjoint problem which cannot be described by classical ODE's; the adjoint problem is a certain GDE with a side condition (see [28]). This fact and the common mathematical struggle for symmetry motivates the study of the problem from its very beginning in the framework of GDE's.

B. Kurzweil's concept of integration is very useful for studying Volterra-Stieltjes and Fredholm-Stieltjes integral equations of the form

$$x(t) - \int_a^b d\left[K(t,s)\right]x(s) = f(t) \ , \quad t \in [a,b]$$

in the space $BV[a,b]$ of functions of bounded variation on $[a,b]$ (see [29] for a complete Fredholm theory for such equations).

C. K. Kreith in [10] started to study the so called second order systems with strong impulses of the form

$$v' = \left[\frac{1}{m(t)} + \sum Q_i \, \delta(t-t_i)\right]z \quad ,$$

$$z' = \left[\frac{1}{p(t)} + \sum R_i \, \delta(t-t_i)\right]v$$

and obtained results on the "zeroes" of such systems. These problems can be also treated in the framework of GDE's with the aim of obtaining analogous results by other techniques developed in [29], for GDE's, see for example [6].

D. There is an interesting theory of "interface" type problems for ODE's (see e.g. [3], [7], [23]). Problems of this type fit into the framework of GDE's with right hand sides belonging to the class $F(G,h,\Omega)$ (see [23]).

E. Recently there have been new results on ODE's written in the form of an integral equation

$$x(t) = x(a) + (P)\int_a^t f(x(s),s)ds \ ,$$

where the Perron integral is used. The interest in this generalization of an ODE comes from new convergence theorems for the Perron integral (see e.g. [4], [5], [10], [17]).

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