ON THE COUPLING OF FINITE ELEMENTS AND BOUNDARY ELEMENTS FOR TRANSONIC POTENTIAL FLOW COMPUTATIONS

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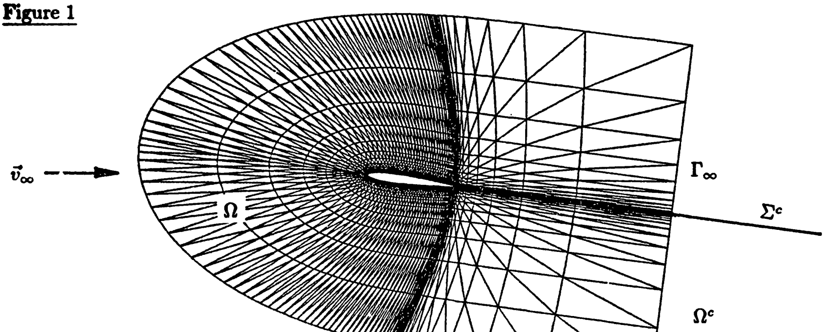
For the improved treatment of the far field boundary conditions in nonlinear compressible transonic flow computations around airfoil profiles we suggest the following method. Assuming that the far field is subsonic, the full potential equation may be linearized to the Prandtl-Glauert equation in the exterior of a bounded region Ω containing the supersonic flow regions. The solution of the linear equation in the exterior can easiliy be reduced to a boundary integral equation on the boundary Γ_∞ (see Figure 1). This leads to a coupled finite element/boundary element method for the flow problem. The numerical results show a considerable improvement in comparison with the commonly used method of taking the normal projection of the far field flow on Γ_∞ [3], [5], [7].With our method we obtain results which show that the computational FEM domain can be chosen much smaller when the coupling is used. In the following, we describe briefly the method, a more detailed description including error analysis will be given in [4].
A simple model for transonic flows with weak shocks is the full potential equation

$$div\left(\rho(|\nabla u|^2)\,\nabla u\right) = 0\ . \tag{1}$$

The density function is obtained under the assumption of isentropic flow from Bernoulli's law as $\rho(s) = \rho_0\left(1 - \frac{\kappa - 1}{2a_0^2}s\right)^{\frac{1}{\kappa-1}}$ where κ denotes the adiabatic exponent and a_0 the speed of sound in the motionless gas. In order to have a flow potential in Ω we have to introduce a slit Σ across which we assume ∇u to be continuous, whereas the potential has a finite constant jump β, which gives the circulation of the flow and which is determined by the Kutta-Joukowski condition. This implies a smooth flow with a stagnation point at TE (see [6]). For convenience we take the slit from the trailing edge point TE. On the profile, we take the non

Figure 1



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penetration condition $\partial_n u \ := \ \nabla u \cdot \vec{n} \ = \ 0$ on Γ_P where $\vec{n}$ denotes the normal vector. The simplest approximation of the exterior boundary condition is

$$\rho(|\nabla u|^2)\, \partial_n u \ = \ \rho(|\vec{v}_\infty|^2)\, \vec{v}_\infty \cdot \vec{n} \ \ on \ \ \Gamma_\infty. \tag{2}$$

This condition was used in [3], [5] and [7].
Boundary condition (2) will be compared to the following coupling procedure. For the linear Prandtl-Glauert equation we introduce a perturbed potential

$$\varphi \ := \ u - \vec{v}_\infty \cdot \vec{x} - \frac{\beta}{2\pi}\, arctan\left(\sqrt{1-M_\infty^2}\; x_2 \,/\, x_1\right) \text{in } \Omega^c. \tag{3}$$

This leads to the following transmission problem.
Find the functions u, φ and the constant β satisfying the
Interior full potential problem,

$$\begin{array}{rcll} div\,(\rho(|\nabla u|^2)\,\nabla u) & = & 0 & in\ \Omega\ , \\ \partial_n u & = & 0 & on\ \Gamma_P\ , \\ u^+ - u^- & = & \beta & on \Sigma, \\ \partial_n u^+ - \partial_n u^- & = & 0 & '', \\ F(\beta)\ =\ |\nabla u^+|^2_{TE}\ -\ |\nabla u^-|^2_{TE} & = & 0, & \end{array} \tag{4}$$

Exterior Prandtl-Glauert problem,

$$\begin{array}{rcll} (1-M_\infty^2)\,\varphi_{x_1x_1}\ +\ \varphi_{x_2x_2} & = & 0 & in\ \Omega^c\ , \\ \nabla\varphi & = & o(1) & for\ |x| \to \infty\ , \\ \varphi^+ - \varphi^- & = & 0 & on \Sigma^c, \\ \partial_n\varphi^+ - \partial_n\varphi^- & = & 0 & '', \end{array} \tag{5}$$

Coupling conditions on Γ_∞,

$$\begin{array}{rcl} u & = & \varphi + \vec{v}_\infty \cdot \vec{x} + \dfrac{\beta}{2\pi}\, arctan\left(\sqrt{1-M_\infty^2}\; x_2 \,/\, x_1\right), \\ \rho(|\nabla u|^2)\,\partial_n u & = & \rho(|v_\infty|^2)\ \{((1-M_\infty^2)\,\varphi_{x_1}, \varphi_{x_2})\ \cdot \vec{n} + \vec{v}_\infty \cdot \vec{n} \\ & & +\dfrac{\beta}{2\pi}\,\nabla\ arctan\left(\sqrt{1-M_\infty^2}\; x_2 \,/\, x_1\right)\cdot \vec{n}\}. \end{array} \tag{6}$$

The exterior Prandtl-Glauert solution $u(x)$ is given by (3) where the perturbation potential φ is represented via Green's theorem as

$$\varphi(x) \ = \ \frac{2}{\sqrt{1-M_\infty^2}} \int_{\Gamma_\infty} \varphi(y) K(x,y) ds_y - \frac{2}{\rho_\infty\sqrt{1-M_\infty^2}} \int_{\Gamma_\infty} G(x,y)\ \lambda(y) ds_y \tag{7}$$

for $x \in \Omega^c$ in terms of the Cauchy data $\varphi|_{\Gamma_\infty}$ and

$$\lambda(y) = \rho_\infty \left\{(1-M_\infty^2)\varphi_{y_1} n_1(y) + \varphi_{y_2} n_2(y)\right\} \quad \text{for } y \in \Gamma_\infty. \tag{8}$$

Here $G(x,y)$ and $K(x,y)$ are given by

$$G(x,y) := -\frac{1}{2\pi}\, log\ \tilde{r}(x,y),\ \ K(x,y) := \frac{(x_1-y_1)n_1(y) + (x_2-y_2)n_2(y)}{2\pi\tilde{r}^2(x,y)}, \tag{9}$$

where

$$\tilde{r}^2(x,y) = \frac{(x_1-y_1)^2}{(1-M_\infty^2)} + (x_2 - y_2)^2.$$

For the coupled FEM-BEM procedure we introduce a family of quasi-regular triangulations of $\Omega\backslash\Sigma$ with maximum meshwidth h and associated piecewise linear continuous finite elements $V_h \subset W^{1,2}(\Omega\backslash\Sigma)$ satisfying additionally $v_h^+ - v_h^- = \beta$ on Σ. On Γ_∞ we consider a family of continuous piecewise linear periodic splines $\mathcal{S}_{\tilde{h}}^1$ subject to particions of maximum meshwidth $\tilde{h}$ of the arc length. The coupled boundary value problem (4), (5), (6) is then approximated by:
Find $u_h \in V_h, \lambda_{\tilde{h}} \in \mathcal{S}_{\tilde{h}}^1$, $\beta \in \mathbb{R}$ where u_h satisfies the speed condition $|\nabla u_h|^2 \leq s_0 < \frac{2a_0^2}{k-1}$ and a modified entropy condition

$$-\int_\Omega \nabla u_h \cdot \nabla \pi_h \psi dx \leq K \int_\Omega \pi_h \psi dx + o(1) \max_{x\in\bar{\Omega}} |\psi(x)|$$

For $h \to 0$ and for all $\psi \in C_0^\infty(\Omega)$ with $\psi \geq 0$ such that the variational equations

$$\begin{aligned} a(u_h|u_h, v_h) - \langle \lambda_{\tilde{h}}, v_h \rangle &= \left\langle \rho_\infty(\vec{v}_\infty + \frac{\beta}{2\pi}\nabla arctan\left(\sqrt{1-M_\infty^2}\, x_2 \,/\, x_1\right)) \cdot \vec{n}, v_h \right\rangle, \\ \langle G\lambda_{\tilde{h}}, \psi_{\tilde{h}} \rangle &= -\left\langle [I-K]\left(u_h - \vec{v}_\infty \cdot \vec{x} - \frac{\beta}{2\pi} arctan\left(\sqrt{1-M_\infty^2} x_2 \,/\, x_1\right)\right) \;,\; \psi_{\tilde{h}} \right\rangle \end{aligned} \quad (10)$$

and the Kutta-Joukowski condition

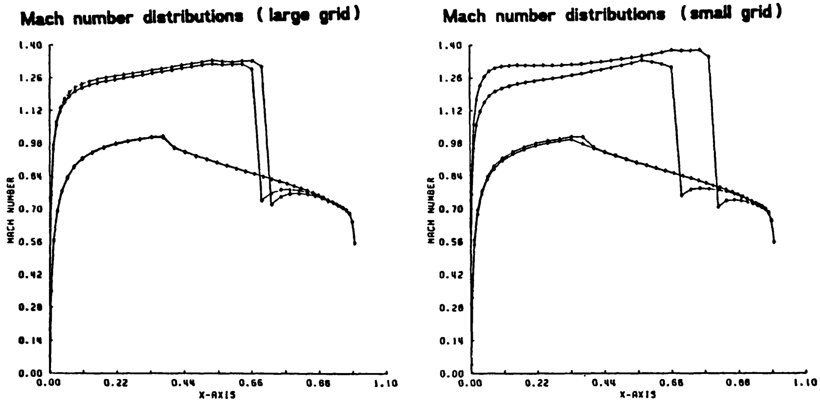
$$|\nabla u_h^+|_{TE}^2 = |\nabla u_h^-|_{TE}^2 \quad (11)$$

are satisfied for all $v_h \in V_h$ and all $\psi_{\tilde{h}} \in \mathcal{S}_{\tilde{h}}^1(\Gamma_\infty)$. Here we have used the notations

$$\begin{aligned} \langle v, \psi \rangle &:= \textstyle\int_{\Gamma_\infty} v(x)\psi(x)\, ds_x, \qquad a(u|u,v) := \int_\Omega \rho(|\nabla u|^2)\nabla u \cdot \nabla v dx, \\ G\lambda(x) &:= 2\int_{\Gamma_\infty} \lambda(y)\, G(x,y)\, ds_y \,/\, \left(\rho_\infty\sqrt{1-M_\infty^2}\right), \\ K\varphi\,(x) &:= 2\int_{\Gamma_\infty} \varphi(y)\, K(x,y)\, ds_y \,/\, \sqrt{1-M_\infty^2}. \end{aligned}$$

π_h denotes the interpolation operator associated with V_h. The above coupled problem is an extension of the linear coupling method of Johnson/Nedelec [8] to nonlinear equations. For the solution of the nonlinear equations we use an improvement by Berger [2] of the conjugate gradient method by Glowinski/Pironneau [7]. Some error analysis for this coupled FEM-BEM method will be presented in [4].
For two standard test cases of flows around the NACA-0012 profile, we made some numerical computations. We compare the use of the boundary condition (2) in the coupling method described above. Two different sized C-grids were used, a large grid with 115 by 15 nodes and Γ_∞ with 6 chord lengths distance from the profile and a smaller grid with 111 by 13 nodes and Γ_∞ 3 chord lengths distance. The lift coefficients c_a were calculated from the pressure distribution along the profile for a purely subsonic flow with $M_\infty = 0.63, \alpha = 2^o$. We obtained 0.3392 for the large and 0.3403 for the small grid versus 0.3333 due to Kroll/Jain [9] with a potential and an Euler code. (Without coupling 0.3455 for the large and 0.3639 for the small grid.) For a transonic flow with $M_\infty = 0.8, \alpha = 1.25^o$ we obtained 0.4455 for c_a on the large and 0.4485 on the small grid via lift coefficients between 0.5 and 1.1 due to Rizzi/Viviant [10] and between 0.35 and 0.37 due to AGARD [1] obtained with Euler equations.

Figure 2



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