ON THE THIRD ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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1. Have the differential equation of the third order of the form

(1) $u''' + q(t)\,u' + p(t)\,u^{\alpha} = 0$,

where $q(t)$, $p(t)$ are continuous functions of $t \in (a,\infty)$, a is a real number and α is an odd integer, but all results of this paper can be generalized to the case where $\alpha > 1$ is a ratio of odd integers.

There is a lot of papers devoted to the study of properties of solutions of the differential equation (1) or of a generalized form ([1] , [3] , [4] , [5] , [6] and the others).

In this paper some new results are introduced concerning oscillatory and nonoscillatory properties of solutions of the differential equation (1). In the proofs of this results the methods of the theory of linear third order differential equation are applied.

We restrict our considerations to those real solutions of (1) which exist on the interval $I \subset (a,\infty)$ and which are nontrivial for $t > \beta \in I$, for every $\beta \in I$.

Let $b > a$ is the right endpoint of the interval I. The solution u of (1) defined on I is oscillatory on I if it has a zero in the interval (β,b) for every $\beta \in I$. Otherwise it is called nonoscillatory on I.

Let u,v be the functions of the clase $C^3(I)$. Let $t_0 \in I$ and let

$$Lu = u''' + qu' + pu^{\alpha}$$

and

$$Mv = -v''' - qv' - (q' - pu^{\alpha-1})\,v.$$

Then there holds

(2) $$\int_{t_0}^{t} [vLu - uMv]\,d\tau = [vu'' - v'u' + (v'' + qv)\,u]_{t_0}^{t} .$$

Corollary 1. Let u be a solution of (1) defined on $I \subset (a,\infty)$ and v be a solution of the differential equation

(3) $v''' + qv' + [q' - pu^{\alpha-1}]\,v = 0$,

then from the relation (2) it follows

$$wu'' - v'u' + [v'' + qv]\, u = k,$$

where

$$k = v(t_0)u''(t_0) - v'(t_0)u'(t_0) + [v''(t_0) + q(t_0)v(t_0)]\, u(t_0).$$

Let u be a solution of (1) defined on $I \subset (a,\infty)$ and v be any solution of (3) corresponding to u, then the solutions u and v fulfil the following integral identities

$$(4)\quad u\,u'' - \frac{1}{2} u'^2 + \frac{1}{2} qu^2 + \int_{t_0}^{t} [pu^{\alpha-1} - \frac{1}{2} q']\, u^2 d\tau = \text{const.}$$

and

$$(5)\quad v\,v'' - \frac{1}{2} v'^2 + \frac{1}{2} qv^2 - \int_{t_0}^{t} [pu^{\alpha-1} - \frac{1}{2} q']\, v^2 d\tau = \text{const.}$$

where t_0, $t \in I$.
The identities (4) and (5) can be obtained as in the linear case [2]. From the integral identities (4) and (5) this corollaries follow:

Corollary 2. Let $p(t) \geq 0$, $q'(t) < 0$ [or $p(t) > 0$, $q'(t) \leq 0$] for $t \in (a,\infty)$. Let $u = u(t)$, $t \in I$ be a solution of (1) with the properties $u(t_0) = u'(t_0) = 0$, $u''(t_0) \neq 0$, $t_0 \in I$ and let $v = v(t)$ be a solution of (3) corresponding to u with the properties $v(t_0) = v'(t_0) = 0$, $v''(t_0) \neq 0$, $t_0 \in I$. Then there is $u(t) \neq 0$ for $t < t_0 \in I$ and $v(t) \neq 0$ for $t > t_0 \in I$.

Corollary 3. Let $p(t) \geq 0$, $q'(t) < 0$ [or $p(t) > 0$, $q'(t) \leq 0$ for $t \in (a,\infty)$. Then every solution of (1) and of (3) defined one some interva $I \subset (a,\infty)$ has at most one double zeropoint on I.

2. Oscillatory and nonoscillatory properties of solutions of the differential equation (1).

By means of the Theorem 2.4 [2] the following theorem can be proved.

Theorem 1. Let $q(t) \geq 0$, $q'(t) \leq 0$ and $p(t) \geq 0$ be continuous functions for $t \in (a,\infty)$ and let the differential equation

$$(6)\quad v'' + \frac{1}{4} q(t)v = 0$$

be oscillatory on (a,∞). Let $\bar{u}(t)$ be a solution of (1) defined on (a,∞) with the property

$$\bar{u}(t_0)\bar{u}''(t_0) - \frac{1}{2}\bar{u}'^2(t_0) + \frac{1}{2} q(t_0)\bar{u}^2(t_0) \leq 0$$

for $t_0 \in (a,\infty)$. Then $\bar{u}(t)$ is oscillatory on (t_0,∞).

By means of the Theorem 1.2 [3] the following theorem can be proved.

Theorem 2. Let $q(t) \geqq 0$, $q'(t) \leqq 0$ and $p(t) \geqq 0$ be continuous functions of $t \in (a,\infty)$ and let the differential equation (6) be oscillatory on (a,∞). Let $u(t)$ be a solution of (1) defined on (a,∞). Then $u(t)$ is nonoscillatory on (a,∞) if and only if

$$u(t)\, u''(t) - \frac{1}{2} u'^2 (t) + \frac{1}{2} q(t) u^2(t) > 0$$

for $t \geqq t_0$, $t_0 > a$.

Corollary 4. From the theorems 1 and 2 it follows that a solution of (1) is either nonoscillatory and then it is defined on $(\bar{\alpha},\infty)$, $\bar{\alpha} \geqq a$, or it is oscillatory and then it is defined either on the interval $(\bar{\alpha},\infty)$, $\bar{\alpha} \geqq a$, or on a bounded interval $I \subset (a,\infty)$.

By means od the method of V.Šoltes [6] or Bobrowski [1] it can be proved.

Theorem 3. Let $q(t) \geqq 0$ and $p(t)$ be continuous functions of $t \in (a,\infty)$ and let $\int_{t_0}^{\infty} q(t)\, dt = \infty$, $t_0 \in (a,\infty)$. Let $u(t)$ be a solution of (1) with property

$$F(t_0) = u(t_0)\, u''(t_0) - \frac{1}{2} u'^2(t_0) + \frac{1}{2} q(t_0) u^2 (t_0) \leqq 0$$

and let the function $F(t)$ be decreasing for $t > t_0$. Then the solution $u(t)$ is oscillatory on (t_0,∞).

Corollary 5. Let $q(t) \geqq 0$, $q'(t) \leqq 0$, $p(t) > 0$ be continuous on (t_0,∞), $t_0 > a$ and let $\int_{t_0}^{\infty} q(t)dt = \infty$. Let $u(t)$ be a solution of (1) with the property $F(t_0) \leqq 0$.

Then the solution $u(t)$ is oscillatory on (t_0,∞).

Lemma 1. Let $p(t) < 0$, $q(t) \leqq 0$, $q'(t) \geqq 0$ be continuous functions of $t \in (a,\infty)$. Let $u(t)$ be a solution of (1) defined on $I \subset (a,\infty)$ with the property

$$u(t_0) \geqq 0, \quad u'(t_0) \geqq 0, \quad u''(t_0) > 0, \quad t_0 \in I.$$

Then there is $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$, $u'''(t) \geqq 0$ for $t > t_0 \in I$ and $\lim_{t \to b} u(t) = \lim_{t \to b} u'(t) = \infty$, where b is the right endpoint of the interval I.

Theorem 4. Let $p(t) < 0$, $q(t) \leqq 0$, $q'(t) \geqq 0$ be continuous functions of $t \in (a,\infty)$ and let $\int_{t_0}^{\infty} p(\tau)\, d\tau = -\infty$. Then every bounded solution of (1) defined on (a,∞) is either oscillatory on (a,∞) or it converges monotonly to zero for $t \to \infty$.

Theorem 5. Let $q(t) \gtreqless 0$, $q'(t) \gtreqless 0$ and $p(t) < 0$ be continuous functions of $t \in (a,\infty)$. Let $u(t)$ be a solution of (1) defined on $I \subset (a,\infty)$ with the property $F(t_0) = k \gtreqless 0$, $t_0 \in I$.

Then $u(t)$ has no zeropoints on the right of t_0 in I.

Theorem 6. Let $q(t) \geq 0$, $q'(t) \gtreqless 0$ and $p(t) < 0$ be continuous functions of $t \in (a,\infty)$ and let $\int_{t_0}^{\infty} q(t)\, dt = \infty$. Then the solution $u(t)$ of (1) defined on (t_0,∞) with the property $F(t) < 0$ for $t \in (t_0,\infty)$ $t_0 > a$, is either oscillatory on (t_0,∞) or it has the property $\liminf_{t \to \infty} |u(t)| = 0$.

References

[1] D.Bobrowski, Asymptotic behaviour of functionally bounded solutions of the third order nonlinear differential equation, Fasc. Math., Poznań, 10, 1978, 67-76.

[2] M.Greguš, Third order linear differential equations, D.Reidel Publishing Company, Dordrecht, Boston, Lancaster, Tokyo, 1987.

[3] I.W.Heidel, Qualitative behaviour of solutions of a third order nonlinear differential equation, Pacific J. Math. 27, 1968, 507-526.

[4] N.Parhi and S.Parhi, Nonoscillation and asymptotic behaviour for forced nonlinear third order differential equations, Bull. Insti. Acad. Sinica 13, 1985, 376-384.

[5] N.Parhi and S.Parhi, Oscillation and Nonoscillation theorems for nonhomogeneous third order differential equations, Bull. Insti. Acad. Sinica 11, 1983, 125-139.

[6] V.Šoltes, O koleblemosti rešenij neliniegnogo differencialnogo uravnenija tretiego porjadka, Math. Slovaca 26. 1976, 217-227.