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# LOCAL SUPERCONVERGENCE ANALYSIS OF THE APPROXIMATE BOUNDARY-FLUX CALCULATION

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**1. Introduction.** In many applications such as potential **flow,** heat and mass transfer or elasticity problems the quantity of interest is the flux across the boundary. Carey, Chow and Seager [1] proposed a natural procedure for computing the boundary flux for two-dimensional problems based on the previous ideas in Wheeler [2] and Carey [3]. In [4] Lazarov *et al* prove that the method of [1] gives $O(h^{3/2})$ accurate boundary-flux calculations for both the consistent and lumped mass procedures. These superconvergence estimates are derived under the assimptions of quasiuniformity of the partition and high regularity of the solution. But this is almost never satisfied in practice! Then it is much more natural to consider an approach using local analysis. In this paper we introduce a subdomain $\Omega_0$, which includes a flat portion of the boundary of the domain $\Omega$, and where the solution has the required regularity. We prove that the flux across $\partial\Omega_0 \cap \partial\Omega$ can be estimated with a superconvergent order of accuracy plus the solution error and the flux error in weaker norms over the slightly larger subdomain $\Omega_1$. The last two terms measure the effects from outside of $\Omega_1$.

Our investigations are based on the papers of Nitsche and Schatz [6] and Wahlbin [7] and the previous investigations of superconvergence subdomain estimates up to the boundary [8,9].

**2. Notations and Problem Formulation.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with a boundary $\Gamma$. The standard notations for Sobolev spaces and associated norms are implied throughout. We consider the Dirihlet problem : Find $u \in \mathbb{H}^1_0(\Omega)$ such that

$$a(u,v) \equiv \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x)\, \partial_i u\, \partial_j v\, dx = \int_\Omega f\cdot v\, dx \equiv f(v) \qquad (1)$$

for each $v \in \mathbb{H}_0^1(\Omega)$, where the bilinear form $a(\circ,\circ)$ is $\mathbb{H}_0^1(\Omega)$-elliptic.

The normal flux across the boundary is defined by

$$q = - \sum_{i,j=1}^{2} a_{ij}(x)\ \partial_i u \cos(\vec{n},x_j) \quad , \quad x \in \Gamma \quad , \tag{2}$$

where $\vec{n}$ is the outward normal to the boundary $\Gamma$. Then the following ralation for the flux holds :

$$- \langle q,v \rangle_{o,\Gamma} = a(u,v) - f(v) \quad \text{for any} \quad v \in \mathbb{H}^1(\Omega) \ , \tag{3}$$

where $\langle q,v \rangle_{o,\Gamma} = \int_\Gamma qv \ ds$ .

The finite element space $\mathcal{X}_h$ is defined by introducing a piecewise polynomial basis on a discretization $T_h$ of $\Omega$ comprised of elements K. Then

$$\mathcal{X}_h = \{ \ v \in \mathbb{C}^0(\Omega) \ : \ v \big|_K \in \mathbb{P}_1(K) \ , \ K \in T_h \ \} \ ,$$

where $\mathbb{P}_1(K)$ is the set of polynomials of first degree. Next, we define the following subspaces of $\mathcal{X}_h$ :

$$\mathbb{V}_h = \{ \ v \in \mathcal{X}_h \ : \ v = 0 \text{ at the corners of } \Omega \ \} \ ,$$

$$\mathring{\mathbb{V}}_h = \{ \ v \in \mathcal{X}_h \ : \ v = 0 \text{ on } \Gamma \ \} \ . \tag{4}$$

Then the corresponding to (1) discrete problem reads as follows: Find $u_h \in \mathring{\mathbb{V}}_h$ such that

$$a_h(u_h,v) = f_h(v) \quad \text{for any} \quad v \in \mathring{\mathbb{V}}_h \ , \tag{5}$$

where sign h in $a_h(\circ,\circ)$ and $f_h(\circ)$ indicates numerical integration / see [5] /.

Let us define the finite dimensional space $\mathbb{V}_h^\Gamma$ of functions which are restriction on the boundary $\Gamma$ of functions in $\mathbb{V}_h$ . Then following [1] the approximate flux across the boundary $\Gamma$ is constructed as a function $q_h \in \mathbb{V}_h^\Gamma$ such that

$$- \langle q_h,v \rangle_{o,\Gamma} = a_h(u_h,v) - f_h(v) \quad \text{for any} \quad v \in \mathbb{V}_h \ . \tag{6}$$

Let $\Omega_1 \subset \Omega$ be such that $\Omega_1$ include a flat portion of the boundary of $\Omega$. Moreover we assume that $\Omega_1$ can be covered with linear triangular elements exactly and $\Omega_1$ is a convex subdomain of $\Omega$.

Let $T_h(\Omega_1)$ be the partition of $\Omega_1$. We suppose that $T_h(\Omega_1)$ is regular / see [5] / and quasiuniform / see [4] /.

**3. 'Interior' Boundary Estimates for the Approximate Flux.** Let $\Gamma_o \subset \Gamma$. We introduce the following norm :

$$\|v\|^*_{o,\Gamma_o} = \Big( \sum_{e \subset \Gamma_o} \|v\|^2_{o,e} \Big)^{1/2} . \tag{7}$$

Our main result is :

**Theorem.** Let $\Omega_o \subset \Omega_1 \subseteq \Omega$, let $\Gamma_o = \partial\Omega_o \cap \Gamma$, $\Gamma_1 = \partial\Omega_1 \cap \Gamma$, let $q$ and $q_h$ be the flux and its approximation defined by (3) and (6). Let the following assumptions be fulfilled :

(i) $u \in \mathbb{H}^{3+\varepsilon}(\Omega_1)$ ;

(ii) $a_{ij} \in \mathbb{W}^{2,\infty}(\Omega_1)$, $1 \le i,j \le 2$, $f \in \mathbb{H}^2(\Omega_1)$ ;

(iii) $\Omega_1$ is covered by linear triangles exactly and $\Omega_1$ is convex subdomain of $\Omega$ ;

(iv) the partition $T_h(\Omega_1)$ is regular and quasiuniform ;

(v) the quadrature formula is exact for polynomials of first degree .

Then there exists $h_1 \in (0,1]$ such that the following holds : if $\text{dist}(\partial\Omega_o\backslash\Gamma, \partial\Omega_1\backslash\Gamma) > c_1 h_1$ then for all $h \in (0,h_1]$

$$\|q - q_h\|^*_{o,\Gamma_o} \le ch^{3/2} \Big( \|u\|_{3+\varepsilon,\Omega_1} + \|f\|_{2,\Omega_1} \Big)$$
$$+ ch^{-1/2}\|u - u_h\|_{-1,\Omega_1} + ch\,\|q - q_h\|_{o,\Gamma_1} . \tag{8}$$

**Proof.** The main idea of the analysis is the cancellation of the interpolation error for any two adjacent elements due to the regularity of the partition / see [10] /. □

**Corollary 1.** Let us suppose that the solution is sufficiently smooth, we cover the whole $\Omega$ by finite elements exactly and have an optimal error estimate in $\mathbb{H}^1(\Omega)$. Then

$$h\,\|q - q_h\|_{o,\Gamma_1} \le ch^{3/2}\,\|u\|_{2+\varepsilon,\Omega} .$$

and by the Aubin-Nitsche trick we get

$$h^{-1/2}\|u - u_h\|_{-1,\Omega_1} \le h^{3/2}\,\|u\|_{2,\Omega} .$$

Summarizing we conclude that if the partition of $\Omega$ is regular and the adjoint to (1) problem is regular / see [5] / then we obtain

$O(h^{3/2})$ estimate for the flux error across $\Gamma_0$ . □

**Corollary 2.** Let us consider the Poisson equation in L-shaped domain. As shown in [6] the best estimates we can derive are

$$\|u - u_h\|_{-1,\Omega_1} \le ch^{4/3-\varepsilon} \|u\|_{5/3-\varepsilon,\Omega} ,$$

$$\|u - u_h\|_{1,\Omega_1} \le ch^{2/3-\varepsilon} \|u\|_{5/3-\varepsilon,\Omega} .$$

Then by (8) the final estimate for the flux error is $O(h^{5/6-\varepsilon})$. But here we did not do anything to avoid singularities which arise from the reentrant corner. Proper mesh refinement or usage of singular trial functions lead us to the standard estimates both for the solution error and the gradient error. Further we proceed as in Corollary 1. □

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