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ON A BOUNDARY VALUE PROBLEM WITH GENERAL LINEAR CONDITIONS

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1. Introduction

We shall consider the boundary value problem (BVP for short)

$$(1_\lambda) \qquad x^{(n)}+p_1(t)x^{(n-1)}+ \dots +p_n(t)x-\lambda x = f(t,x)$$

$$(2) \qquad l_i(x) = 0, \qquad i=1,\dots,n ,$$

where $n\geq 1$ is a natural number, $a<b$ are real numbers, $p_k\in C([a,b],R)$, $k=1,\dots,n$, $f\in C([a,b]\times R,R)$, $l_i : C^{n-1}([a,b],R)\to R$ is a linear continuous functional, $i=1,\dots,n$, and λ is a real parameter.

The BVP (1_λ), (2) is of the form

$$(3_\lambda) \qquad L(x)-\lambda x = F(x)$$

and the existence and bifurcation of a solution to (3_λ) will be investigated under the assumptions that the resolvent of the operator L is completely continuous at a point $\lambda_0\in R$ and that the nonlinear operator F is sublinear at infinity.

2. The general theory

Let $(X,\|.\|)$ be an infinitely dimensional real Banach space. In this space the following lemma holds.

Lemma 1. Let $L : D(L)\subset X\to X$ be a linear mapping such that
(H_1) for some $\lambda_0\in(-\infty,\infty)$ the operator $L-\lambda_0 I$ (I is the identity on X) is one-to-one and onto X, $(L-\lambda_0 I)^{-1}$ is completely continuous on X.

Denote by $\{\lambda_n\}$ the sequence of all eigenvalues of $(L-\lambda_0 I)^{-1}$ (it may be finite or even void, 0 is the only accumulation point of it if there is any) and by $\{x_n\}$ the corresponding sequence of eigenvectors of $(L-\lambda_0 I)^{-1}$ where each term λ_n occurs in the sequence $\{\lambda_n\}$ so many times as its multiplicity indicates.

Then the following statements are true:

(i) The operator L is closed, its resolvent set $\varrho(L)$ is non void and for each $\lambda\in\varrho(L)$ the resolvent $(L-\lambda I)^{-1}$ is a completely continuous operator defined everywhere on X.

(ii) The spectrum $\sigma(L)$ consists of the eigenvalues

$$\mu_n = \lambda_0 + \frac{1}{\lambda_n}$$

of L only and x_n are the corresponding eigenvectors. $\{\mu_n\}$ has no finite accumulation point.

(iii) L is a Fredholm mapping of index zero.

(iv) If $P : X \to X$ and $Q : X \to X$ are arbitrary linear continuous projectors such that

$\operatorname{Im} P = \ker L$, $\ker Q = L$ and $X = \ker L \oplus \ker P$, $X = \operatorname{Im} Q \oplus \operatorname{Im} L$
and

$L_P = L_{|D(L)\cap \ker P}$, $K_P : \operatorname{Im} L \to D(L)\cap \ker P$ is the inverse of L_P, then:

(a) the operator L_P is closed;

(b) the operator $K_P : \operatorname{Im} L \subset X \to X$ is completely continuous.

(v) If $\operatorname{Im} L \cap \ker L = \{0\}$, then

$X = \ker L \oplus \operatorname{Im} L$.

Proof. The statements (i), (ii), (v) and the statement (iii) under additional hypothesis $\operatorname{Im} L\cap \ker L = \{0\}$ have been proved in Theorem 1, [4], pp. 555-558. Keeping the notation from the proof of that theorem, in the general case it suffices to consider the case that $Z_1 = \ker L \oplus Z_{12}$. If $\dim Z_1 = n$, $\dim \ker L = k$, then $\dim Z_{12} = n-k$. Since $L_{1|Z_{12}}$ is one-to-one, $\dim L_1(Z_{12}) = \dim Z_{12} = n-k$. As $L_1(Z_{12}) \subset Z_1$, we can write $Z_1 = Z_{13} \oplus \operatorname{Im} L_1$ whereby Z_{13} is a suitable vector subspace of Z_1 and $\dim Z_{13} = k$. Then

$$Z = Z_1 \oplus \operatorname{Im} L_2 = Z_{13} \oplus \operatorname{Im} L_1 \oplus \operatorname{Im} L_2 = Z_{13} \oplus \operatorname{Im} L$$

and $\dim Z_{|\operatorname{Im} L} = \dim Z_{13} = k = \dim \ker L$. The statement (iv) has been proved in [4], pp. 554-558.

Theorem 1. Let the operator $L : D(L) \subset X \to X$ be a linear mapping satisfying (H_1) and let the operator $F : X \to X$ fulfil the hypothesis:

(H_2) F is continuous, bounded (it maps bounded sets into bounded sets) and

$$\lim_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|} = 0 .$$

Then for each $\lambda \in \varrho(L)$ the set S of all solutions to the equation (3_λ) is nonempty and compact.

The proof follows from the facts that for $\lambda \in \varrho(L)$ (3_λ) is equivalent to the equation $x = (L-\lambda I)^{-1}.F(x)$, the operator $(L-\lambda I)^{-1}.F$ is completely continuous and for all possible solutions of $x = \alpha(L-\lambda I)^{-1}.F(x)$, $0 \leq \alpha \leq 1$ we have an apriori estimate.

Lemma 2. Let $\lambda \in \sigma(L)$ and let all assumptions of Theorem 1 be fulfilled. Then the following statement holds:

If there exists a sequence $\{\lambda_n\} \subset \varrho(L)$, $\lim_{n\to\infty} \lambda_n = \lambda$ and a bounded sequence of solutions x_n of (3_{λ_n}), $n=1,2,\dots$, then there exists a subsequence $\{x_{\bar{m}}\}$ of the sequence $\{x_n\}$ and a solution x of (3_λ) such that $\lim_{m \to \infty} x_m = x$.

Proof. If $P : X \to X$ and $Q : X \to X$ are linear continuous projections with the property $\operatorname{Im} P = \ker (L - \lambda I)$, $\ker Q = \operatorname{IM} (L - \lambda I)$, then each solution x of (3_{λ_n}) satisfies

$$(4) \qquad x = P(x) + K_P \circ (I-Q) \circ F(x) + (\lambda_n - \lambda) K_P \circ (I-Q)(x)$$

and the conclusion follows from the complete continuity of the operators $P + K_P \circ (I-Q) \circ F$, $K_P \circ (I-Q)$ and from the closedness of the operator L.

By the last lemma the following theorem is true.

Theorem 2. Assume that all assumptions of Theorem 1 are fulfilled. Then the following statement is true:

If $\lambda_0 \in \sigma(L)$ and the equation (3_{λ_0}) has no solution, then $\lim_{\lambda \to \lambda_0} \|x_\lambda\| = +\infty$, where x_λ is an arbitrary solution of (3_λ).

Apriori estimates for solutions x_λ of (3_λ) in a neighbourhood of $\lambda = 0$ are given by

Lemma 3. Suppose that $L : X \to X$ is a Fredholm operator of index zero, the mapping F satisfies (H_2) and that the following hypotheses:

(H_3) there exists a continuous positive definite bilinear form $\langle .,. \rangle : X \times X \to R$ such that

$$(5) \qquad \langle y,z \rangle = 0 \text{ for each } y \in \ker P \text{ and for each } z \in \ker L ,$$

where $P : X \to \ker L$ is a linear continuous projector;

(H_4) there exists a constant d, $0 < d < 1$, such that for each $y \in \ker L$, $\|y\| = 1$, each sequence $\{t_n\} \subset R$, $t_n \to \infty$ as $n \to \infty$, each sequence $\{y_n\} \subset \ker L$, $\|y_n\| = 1$ with $y_n \to y$ as $n \to \infty$ and each sequence $\{z_n\} \subset \ker P$ such that $\|z_n\| \leq d$

$$(6) \qquad \lim_{n \to \infty} \inf \langle L(t_n z_n) - F(t_n(y_n + z_n)), y \rangle < 0$$

$$\left((7) \qquad \lim_{n \to \infty} \sup \langle L(t_n z_n) - F(t_n(y_n + z_n)), y \rangle > 0 \right)$$

hold.

Then there exists an $R_0 > 0$ such that any solution x of (3_λ) satisfies $\|x\| < R_0$ as long as $0 \leq \lambda \leq d/8c$ $(-d/8c \leq \lambda \leq 0)$ where $c = \|K_P\| \cdot \|I-Q\| > 0$ and K_P, Q have the same meaning as in Lemma 1.

Proof. Suppose that $x = x^1 + x^2$, $x^1 \in \ker L$, $x^2 \in \ker P$, is a solution of (3_λ). Then as in (4), $x^2 = K_P \circ (I-Q) \circ F(x^1 + x^2) + \lambda K_P \circ (I-Q)(x^1 + x^2)$ and

$$\|x^2\| \leq 2c(\|F(x^1+x^2)\| + |\lambda| \|x^1\|) \quad \text{for all } |\lambda| < \frac{1}{2c} .$$

Let $0 < \varepsilon < \frac{1}{4c}$ be arbitrary. By (H_2) there exists an $R > 0$ such that

$$\|x^2\| \leqq 2c(1-2c\varepsilon)^{-1}(|\lambda|+\varepsilon)\|x^1\| < 4c(|\lambda|+\varepsilon)\|x^1\| \quad \text{for} \quad x\in R .$$

Hence for $0<\varepsilon<d/8c$ and for $|\lambda| \leqq d/8c$ we obtain

(8) $\|x^2\| < d\|x^1\|$ for all $\|x\|>R$.

Now we put $x_n^1 = t_n y_n$, $t_n = \|x_n^1\|$, $y_n\in \ker L$, $x_n^2 = t_n z_n \in \ker P$ and by (8) we have $\|z_n\| < d$ and we continue as in the proof of Lemma 1 in [3].

Theorem 1 in [2] implies the following theorem.

Theorem 3. Assume that the hypotheses (H_1), (H_2) are fulfilled. Let further 0 be an eigenvalue of the linear operator L with odd algebraic multiplicity and let there exist a $\delta>0$ and an $R>0$ such that each possible solution x of (3_λ) for $-\delta\leqq\lambda\leqq 0$ (for $0\leqq\lambda\leqq \leqq\delta$) is such that $\|x\|<R$.

Then there exists an $\eta>0$ such that:

a) the equation (3_λ) has at least one solution for $-\eta\leqq\lambda\leqq 0$ (for $0\leqq\lambda\leqq\eta$) ;

b) the equation (3_λ) has at least two solutions for $0<\lambda\leqq\eta$ (for $-\eta\leqq\lambda<0$) .

Corollary 1. Let the assumptions (H_1) - (H_4) be fulfilled. Let, further, 0 be an eigenvalue of L with odd algebraic multiplicity. Then the statements a) and b) of Theorem 3 are valid with the only change that the statements in brackets should stand without brackets and conversely.

3. The boundary value problem

The theory developed in the preceding section can be applied to the problem (1_λ), (2). If we assume the hypotheses

(H_5) for some $\lambda_0\in(-\infty,\infty)$ the BVP (2),

(1_{λ_0}) $x^{(n)}+p_1(t)x^{(n-1)}+ \dots +p_n(t)x-\lambda_0 x = 0$

has only the trivial solution;

and

(H_6) $\lim\limits_{|x|\to\infty} \dfrac{|f(t,x)|}{|x|} = 0$ uniformly for all $t\in[a,b]$,

then the operator $L : D(L)\subset C\to C$ defined on $D(L) = \{x\in C : x^{(n)}\in C,$ x satisfies (2)$\}$ by

(9) $L(x)(t) = x^{(n)}(t)+p_1(t)x^{(n-1)}(t)+ \dots +p_n(t)x(t)$ for each $a\leqq t\leqq b$ and all $x\in D(L)$

satisfies (H_1) in $C = C([a,b],R)$ provided by the sup-norm and $F : C\to C$ determined by $F(x)(t) = f(t,x(t))$ for all $t\in[a,b]$ and all $x\in C$ satisfies (H_2) in C .

References

[1] I.T. Kiguradze: Kraevye zadači dlja sistem obyknovennych differencialnych uravnenij, Itogi nauki i tech., t. 30, Moskva 1987.

[2] J. Mawhin: Bifurcation from infinity and nonlinear boundary value problems, Proceed.Confer. on Ordinary and Partial Diff. Equations, Dundee, July 1988.

[3] J. Mawhin and K. Schmitt: Landesman-Lazer type problems at an eigenvalue of odd multiplicity, Results in Math. 14(1988), 138-146.

[4] V. Šeda: Some remarks to coincidence theory, Czech.Math.J. 38(113), (1988), 554-572.