# ON REGULARITY OF SOLUTIONS TO SOME PROBLEMS OF MATHEMATICAL PHYSICS

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Let $\Omega \subset R^m$ $(m \geq 2)$ be a bounded domain with $C^1$ boundary $\partial\Omega$. Consider a boundary value problem for the elliptic system

(1) $\quad Lu \equiv D_i a_i(x,Du) = 0, \quad x \in \Omega,$

(2) $\quad u(x) = g(x) \;, \; x \in \partial\Omega, \; g \in W_2^1(\Omega).$

Here $u(x) = \{u^1(x),\dots,u^N(x)\}$ is an unknown vector function, $Du = \{D_0 u, D_1 u,\dots,D_m u\}$, $D_0 u = u$, $D_i = \partial/\partial x_i$ $(i=1,\dots,m)$.

Until now we considered mostly the case when the coefficients $a_i(x,p) = \{a_i^1(x,p),\dots,a_i^N(x,p)\}$ were measurable with respect to x, continuously differentiable with respect to $p = \{p_0,\dots,p_m\}$ and for the matrix

$$(3) \qquad A = \frac{\partial a_i}{\partial p_k} \;, \; (i,k=0,\dots,m)$$

the inequalities

$$(4) \qquad \mu\|\xi\|^2 \leq A\xi\cdot\xi \leq \nu\|\xi\|^2 \;, \; \|A\| < C$$

took place with some positive constants $\mu$, $\nu$ and C. Suppose that A is symmetric. Denote as $\{\lambda_i(x,p)\}$ the set of all eigenvalues of A at the point $\{x,p\}$ and put $\lambda = \inf \lambda_i$ and $\Lambda = \sup \lambda_i$. It was proved in [1] that if the inequality

$$(5) \qquad \frac{\Lambda - \lambda}{\Lambda + \lambda}\left[1 + \frac{(m-2)^2}{m-1}\right]^{1/2} < 1$$

holds then the weak solution u of (1), (2) is locally Hölder-continuous in $\Omega$. It was also proved that the condition (5) is sharp.

In some important problems the differentiability of the coefficients $a_i$ with respect to p doesn't hold. It happens e.g. in case of the systems of equations for elasto-plastic media with hardening, the Maxwell systems for materials with ferromagnetic intrudings etc.

Suppose now that, instead of (4), the inequalities

$$(6) \qquad \begin{aligned} &(a(x,p) - a(x,q))\,(p-q) \geq \mu\|p-q\|^2 \\ &\|a(x,p) - a(x,q)\| \leq \nu\,\|p-q\| \end{aligned}$$

take place with some positive constants $\mu, \nu$ for almost all $x \in \Omega$ and all $p,q \in R^{(m+1)N}$. (The norms are calculated in $R^{(m+1)N}$.)

Denote

$$(7) \qquad K = \inf_{\varepsilon > 0} \sup \frac{\| p - q - \varepsilon [a(x,p) - a(x,q)] \|}{\| p - q \|} \quad .$$

THEOREM 1. If

$$(8) \qquad K \left[ 1 + \frac{(m-2)^2}{m-1} \right]^{1/2} < 1 ,$$

then $u$ is locally Hölder continuous on $\Omega$ . The condition (8) is sharp.

Consider now the equilibrium system of the theory of small deformations for a material with strenght condition (so called Hencky theory). Let $S = \{\sigma_{ij}\}$ and $D = \{\varepsilon_{ij}\}$ are the stress and strain tensors, respectively. Suppose

$$\sigma_{ik} = a_i^k (x; \varepsilon_{jl}) \ , \text{ where } \quad {}_{jl} = \frac{u^j}{x_l} + \frac{u^l}{x_j}$$

and $u$ is the displacement vector.

Then the system corresponding to (1) has the form

$$(1') \quad D_i a_i (x; \varepsilon_{jl}) - f = 0,$$

where $f$ is a vector of mass forces.

Denote

$$T^2 = \inf_{\varepsilon > 0} \sup_{x,p,q} \frac{\sum_{i,k=1}^{m} \left| (p_i^k + p_k^i) - (q_i^k + q_k^i) - \varepsilon \left[ a_k^i (x; p_j^l + p_l^j) - a_k^i (x; q_j^l + q_l^j) \right] \right|^2}{\sum_{i,k=1}^{m} \left| (p_i^k + p_k^i) - (q_i^k + q_k^i) \right|^2} .$$

THEOREM 2. If

$$2T \left[ 1 + \frac{(m-2)^2}{m-1} \right]^{1/2} \left[ 1 + \frac{m^{1/2}}{2} \left( 1 + \frac{m-2}{m+1} \right)^{1/2} \right] < 1,$$

then the displacement $u$ (the weak solution of the problem (1')-(2)) is locally Hölder-continuous on $\Omega$ .

As we have seen above, the conditions (5) and (8) are sharp. Nevertheless, there are the cases when the solution $u$ has some additional properties. Suppose that $\Gamma_s$ $(s=1,\dots,m-2)$ is a smooth s-dimensional manifold in $\Omega$ .

THEOREM 3. Let the conditions (4) be satisfied and let

$$\frac{\Lambda - \lambda}{\Lambda + \lambda} \left[ 1 + \frac{(m-2-s)^2}{m-1-s} \right]^{1/2} < 1 .$$

Then $u \varphi \in L_1 (\Gamma_s)$ for all $C_0^\infty$ and $\| u \varphi \|_{L_1(\Gamma_s)}$ is Hölder continuous with respect to the coordinates ortogonal to $\Gamma_s$ .

Consider now the parabolic system

$$P(u) \equiv \dot{u} - L(u) = 0$$

in the cylinder $Q = \Omega \times (0,T)$ with the initial condition $u|_{t=0} = 0$ and the condition (2) . Suppose that the conditions (4) are satisfied.

THEOREM 4. There exists such a constant $C(m) > 0$ that if the inequality

$$\frac{\Lambda - \lambda}{\Lambda + \lambda} C(m) < 1$$

holds, then the weak solution u of the problem

$$P(u) = 0, \ u|_{t=0} = 0, \ u|_{\partial\Omega} = g(x)$$

is for almost all $t \in [0,T]$ locally Hölder-continuous with respect to the variable x in $\Omega$ .

This theorem has some corollaries concerning the systems with hysteresis coefficients.

[1] A.I.Koshelev, Regularity of the solutions of elliptic equations and systems, Nauka, Moscow,1986.