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MULTIPOINT BOUNDARY VALUE PROBLEMS AT RESONANCE

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1. Introduction

We shall investigate the multipoint BVPs, where the number of points is greater than the order of a differential equation. For the differential equation of the second order such problem can have the form

(1) $$u'' = f(t,u,u'),$$

(2) $$u(a)=c_1,\ u(b)=u(t_0) + c_2,$$

where $a, t_0, b, c_1, c_2 \in R$, $a < t_0 < b$.

The questions of the existence of solutions of problem (1),(2) were studied by H.Dörner [3] and by I.Kiguradze and A.Lomtatidze [4] for linear differential equations, the nonlinear case was considered by A.Lomtatidze [5]. It is worth mentioning that the similar problem but for partial differential equations, which is known now as the Bitsadze-Samarskiĭ problem, was first stated and solved by A. Bitsadze and A. Samarskiĭ [1].

2. Differential equations of the second order

We are interested in the modifications and generalizations of problem (1),(2) turning it into problems at resonance. For example we consider the four-point condition

(3) $$u(a)-u(c)=A,\ u(b)-u(d)=B,$$

where $a < c \le d < t$, $a,b,c,d,A,B \in R$.

Problem (1),(3) is at resonance, so we have not the Green function for the corresponding homogenous problem. Thus we consider the consequence of the auxiliary equations

(4n) $$u'' = u/n + f(t,u,u')$$

and we find the conditions for the existence of solutions u_n of regular problems (4n),(3). Finally we prove that $u = \lim_{k\to\infty} u_k$ is a solution of (1),(3). Let us show a certain simplier modification of such existence theorems:

Theorem 1. Let f satisfy the local Carathéodory conditions on $[a,b]\times R^2$, let g_0 be a polynomial satisfying (3) and let there exist $r_1,r_2\in R$, $r_1\le r_2$, such that

(5) $f(t,r_1+g_0,g_0')\le g_0''$, $f(t,r_2+g_0,g_0')\ge g_0''$ for a.e. $t\in(a,b)$.

Further, let

(6) $f(t,x,y)\,\mathrm{sign}\,y\le h_1(t)+h_2(t)|y|+h_0y^2$
for a.e. $t\in(a,b)$ and for each $x\in[r_1,r_2]$, $|y|\ge 1$,

and

(7) $f(t,x,y)\,\mathrm{sign}\,y\ge -h_1(t)-h_2(t)|y|-h_0y^2$
for a.e. $t\in(a,c)$ and for each $x\in[r_1,r_2]$, $|y|\ge 1$,

where $h_1,h_2\in L(a,b)$, $h_0\in(0,+\infty)$.

Then problem (1),(3) has at least one solution u satisfying

$$r_1\le u(t)-g_0(t)\le r_2 \quad \text{for each } t\in[a,b].$$

Condition (5) of Theorem 1 can be replaced by the assumption of the existence of lower and upper functions σ_1, σ_2 for (1),(3) with $\sigma_1(t)\le\sigma_2(t)$ on $[a,b]$, [10]. Under the assumptions of Theorem 1 the polynomials r_1+g_0 and r_2+g_0 are the lower and the upper functions for (1),(3), respectively.

Using the Mawhin continuation theorem (see [6]) instead of the Schauder fixed point theorem, we have proved that inequality (6) can be changed on (c,d) by one-side inequality

$$f(t,x,y)\le h_1(t)+h_2(t)|y|+h_0y^2 .$$

3. Higher order differential equations

Now we will study the 2n-point BVP at resonance

(8) $u^{(n)}=f(t,u,u',\dots,u^{(n-1)})$,

(9) $u(a_{2j})-u(a_{2j-1})=A_j$, $j=1,\dots,n$,

where $-\infty<a=a_1<a_2<\dots<a_{2n}=b<+\infty$, $A_j\in R$, $j=1,\dots,n$.

Solving the boundary problems we often use theorems of the type of Conti [2] . These theorems guarantee the existence of solutions of boundary problems under the following assumptions:

(10) a non-linear part of a differential equation is bounded by an integrable function ,

(11) the corresponding homogenous problem has only the trivial solution (i.e. the BVP is regular).

Problem (8),(9) does not fulfil (11) and so we cannot use such theorems even though f satisfies (10). Therefore we have proved the existence proposition in which (11) is replaced by a sign condition:

Proposition. Let there exist $r \in (0,+\infty)$, $\lambda \in \{-1,1\}$ and a function $h \in L(a,b)$ such that on the set $[a,b] \times R^n$ there are satisfied the conditions

$$\lambda[f(t,x_1,\dots,x_n) \operatorname{sign} x_1] \geq 0 \quad \text{for} \quad |x_1| \geq r$$

and

$$|f(t,x_1,\dots,x_n)| \leq h(t) \text{ . } \quad \text{Let} \quad A_j=0, \; j=1,\dots,n.$$

Then problem (8),(9) has a solution v such that there exists $t_0 \in (a,b)$ with $|v(t_0)| \leq r$.

Now, by means of this proposition and the suitable lemmas on a priori estimates, we can prove various existence theorems:

Theorem 2. Let $g_0(t) = \sum_{i=1}^{n} d_i t^i$ be a polynomial satisfying (9), $\tau_k = \max\{|a_{2i}-a_{2i-2k+1}| : k \leq i \leq n$, $k=1,\dots,n-2$, $\tau_0=b-a$, $\tau_{n-1} = 1$. Further, let f satisfy the local Carathéodory conditions on $[a,b] \times R^n$ and let there exist $r \in (0,+\infty)$ and $\lambda \in \{-1,1\}$ such that on $[a,b] \times R^n$ the conditions

(12) $\quad \lambda[f(t,x_1,\dots,x_n) - n!d_n] \operatorname{sign} x_1 \geq 0 \quad$ for $|x_1| \geq r$,

(13) $\quad |f(t,x_1,\dots,x_n)| \leq \sum_{i=1}^{n} h_i(t)|x_i| + \omega(t, \sum_{i=1}^{n}|x_i|)$,

are satisfied, where $h_i \in L(a,b)$, $i=1,\dots,n$, are non-negative functions fulfilling

(14) $\quad \sum_{i=1}^{n} \tau_{n-1} \;\dots\; \tau_{i-1} \int_a^b h_i(t)dt < 1$

and ω, satisfying the local Carathéodory conditions on $[a,b] \times (0,\infty)$, is non-negative non-decreasing in its second argument and

$$\lim_{\rho \to \infty} \frac{1}{\rho} \int_a^b \omega(t,\rho)dt = 0 \text{ .}$$

Then problem (8),(9) has at least one solution.

Let us compare the existence conditions for the second order and for the n-th order.

a) The sign conditions. Comparing condition (5) and condition (12), we can see that (5) depends on r_1, r_2 and g_0 only, in contrast to (12) which has to be satisfied for each $|x_1| \geq r$ and for each $x_2, \dots, x_n \in R$.

b) The growth conditions. The functions h_1, h_2 of (6),(7) can be arbitrary Lebesgue-integrable functions, while h_i, $i=1,\dots,n$, of (13) satisfy (14), i.e. their greatness depends on $b-a$. Moreover (13) implies that f must not grow quickly in its variables $x_1,\dots,x_n$. In contradistinction to this f (of Theorem 1) can be arbitrary growing in x and not more than y^2 in y.

The uniqueness of (8),(9) can be proved under an appropriate Lipschitz condition, with sufficiently small Lipschitz constant (see [9]).

For similar k-point BVPs, $n < k < 2n$, see [7,8] .

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