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ON A CLASS OF SCALAR CONSERVATION LAWS WITH LOCALLY UNBOUNDED SOLUTIONS

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1. Introduction

Conservation laws which depend explicitly on the space variables, namely

$$\partial_t u + \sum_{i=1}^{n} \partial_{x_i}[\varphi_i(x,u)] = 0 \tag{1.1}$$

($x \equiv (x_1, \ldots, x_n)$), occur is several situations (e.g., flood waves [Wh], mathematical biology [SKT], exploitation of oil reservoirs [Ew]). This note is concerned with the existence and uniqueness of solutions to the Cauchy problem for scalar conservation laws of this kind, i.e.

$$\begin{cases} \partial_t u + \partial_x[\varphi(x,u)] = 0 & \text{in } \mathbf{R}_+ \times \mathbf{R} \\ u(0,x) = u_0(x) & \text{in } \mathbf{R} \end{cases} , \tag{1.2}$$

where $\varphi(0,u) = 0$.

The Cauchy problem has been investigated by the vanishing viscosity method for a class of conservation laws of the form (1.1) (see in particular [Ol], [Kr]). To our knowledge, however, the assumptions encountered in the literature guarantee the a priori boundedness of the weak entropy solutions (uniformly on the compact subsets of $\mathbf{R}_+$). An assumption of this kind reads for problem (1.2) (see [Kr]):

$$\sup_{x\in\mathbf{R}, u\in\mathbf{R}} [-\varphi_{xu}(x,u)] \leq \text{ const.} \tag{1.3}$$

Interesting phenomena can arise when (1.3) is not satisfied, even in very simple cases. As a matter of fact, it is easy to see that the Cauchy problem for the equation

$$\partial_t u - \frac{1}{2}\partial_x(xu^2) = 0 \qquad \text{in } \mathbf{R}_+ \times \mathbf{R} \tag{1.4}$$

has solutions which blow up in finite time at $x = 0$ (see Section 2).

As (1.4) suggests, the problem (1.2) can be investigated in weighted spaces. This approach enables to prove the existence and uniqueness of weak entropy solutions to (1.2) (see Section 3), if the nonlinearity φ is of the form

$$\varphi(x,u) = -\sum_{k=1}^{n} |x|^{m_k - 1} x \frac{u^{p_k}}{p_k} , \tag{1.5}$$

where $m_k > 0$, $p_k \in \mathbf{N}$ satisfy the inequality

$$m := \max_{k=1,\dots,n} \frac{m_k}{p_k} \leq \min_{k=1,\dots,n} \frac{m_k}{p_k - 1} =: M \ . \tag{1.6}$$

2. A model equation

Let us denote by [0,T] the maximal interval of existence of regular solutions to the Cauchy problem for (1.1). If $T < \infty$, then either (i) $|u|_\infty(t) \to \infty$ or (ii) $|u_t|_\infty(t) + |u_x|_\infty(t) \to \infty$ as $t \to T^-$ (see [Ma]). If (i) is ruled out (e.g., by assuming (1.3)), then a loss of regularity occurs at $t = T$ and discontinuous solutions have to be considered.

It is easily seen that for equation (1.4) both phenomena (i.e., blow-up in finite time and formation of shock waves) can occur. Let $u_0 \geq 0$ be a regular Cauchy data with compact support. By obvious symmetry arguments, we can restrict ourselves to the quadrant $t \geq 0$, $x \geq 0$. As an elementary calculation shows, the characteristic x_α for (1.4) starting at $x = \alpha \geq 0$ is given by

$$x_\alpha(t) = \alpha \left(1 - \frac{u_0(\alpha)}{2} t\right)^2 \ ; \tag{2.1}$$

moreover,

$$u_\alpha(t) := u(t, x_\alpha(t)) = \frac{u_0(\alpha)}{1 - \frac{u_0(\alpha)t}{2}}.$$

Hence u blows up in $x = 0$ at the time $t_1 = [u_0(\alpha)/2]^{-1}$, if shock waves haven't appeared at an earlier time. On the other hand, it follows from (2.1) that characteristics issued at α, respectively $\alpha + d\alpha$, can only intersect at the time $t_2 = [u_0(\alpha)/2 + \alpha u_0'(\alpha)]^{-1}$. Thus we have blow-up if $\alpha u_0'(\alpha) \leq 0$ for any $\alpha > 0$, or earlier appearance of shock waves otherwise.

If blow-up arises, it only takes place at $x = 0$. It is also easy to control the corresponding rate of divergence of u as $x \to 0^+$. In fact, $v(t,x) := \sqrt{x}\, u(t,x)$ satisfies the equation

$$\partial_t v - \frac{\sqrt{x}}{2} \partial_x (v^2) = 0 \qquad \text{in } \mathbf{R}_+ \times \mathbf{R}_+. \tag{2.2}$$

The characteristics of (2.2) are given again by (2.1) ($\alpha \geq 0$), but now $v = \sqrt{\alpha}\, u_0(\alpha) =$ const. along x_α. This is not surprising, since

$$\tilde{v}(t,x) := v\left(t, \frac{x^2}{4}\right) = \frac{x}{2} u\left(t, \frac{x^2}{4}\right) \qquad (x > 0)$$

satisfies the Bürger's equation in $(\mathbf{R}_+ \times \mathbf{R}_+)$ in the weak sense (in this connection, wee [Wa]).

In order to deal with the case (1.5) it is useful to consider a general weight $x^\rho (\rho > 0)$. In the present case $w(t,x) := x^\rho u(t,x) (\rho \neq 1/2)$ satisfies

$$w_t - \frac{1-2\rho}{2} x^{-\rho} w^2 - \frac{x^{1-\rho}}{2} (w^2)_x = 0 \qquad \text{in } \mathbf{R}_+ \times \mathbf{R}_+. \tag{2.3}$$

Observe that (2.3) can be rewritten as follows:

$$w_t - \frac{1}{2}\psi_x + \frac{\rho}{2x}\psi = 0 \qquad \text{in } \mathbf{R}_+ \times \mathbf{R}_+ \ , \tag{2.4}$$

where

$$\psi(t,x) := x^{1-\rho} w^2(t,x).$$

Singular scalar conservation laws of type (2.4) (where, however, ψ doesn't depend explicitly on x) have been studied in [Sch]. The same methods extend to (2.4), provided that $\rho \in [1/2, 1]$. Then, under this restriction, a unique (weak entropy) solution $w \in L^\infty(\mathbf{R}_+ \times \mathbf{R}_+)$ of (2.4) exists (see Section 3 for exact statements). Observe that w is infinitesimal of order $\rho - 1/2$ as $x \to 0^+$.

3. Results

Let us now consider the problem (1.2) with φ given by (1.5), and assume that (1.6) hold. The above discussion makes plausible the following definition.

DEFINITION. A measurable function u is a weak entropy solution of (1.2) (with φ as in (1.5)) if:

(i) $w := |x|^\rho u \in L^\infty(\mathbf{R}_+ \times \mathbf{R}_+) \quad (\rho > 0)$;

(ii) for any entropy pair (η, q) the inequality

$$\partial_t \eta(w) + \partial_x [q(x,w)] - [q_x(x,w) + \frac{\rho}{x}\eta'(w)\psi(x,w)] \leq 0$$

is satisfied in the sense of distributions. Here

$$\psi(x,w) := |x|^\rho \varphi(x, w/|x|^\rho) \ ,$$

(iii) there exists a set $\mathcal{E}$ of zero measure in $\mathbf{R}_+$ such that for any $t \in \mathbf{R}_+ \backslash \mathcal{E}$ the function w is defined a.e. in $\mathbf{R}$ and for any $\varepsilon > 0$

$$\lim_{\substack{t \to 0^+ \\ t \notin \mathcal{E}}} \int_{|x| \leq \varepsilon} |w(t,x) - |x|^\rho u_0(x)| dx = 0 \ .$$

We can prove the following result (see [NT]).

THEOREM. *Let (1.5), (1.6) hold. Assume moreover* $\rho \in [m,M]$. *Then for any bounded measurable Cauchy data with compact support there exists a unique weak entropy solution of (1.2).*

The existence proof makes use of the theory of compensated compactness [Ta]. The uniqueness follows by [Kr].

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