Nikolai Sergeevich Bachvalov; A. V. Knyazev

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A NEW ITERATIVE ALGORITHM FOR SOLVING THE FICTITIOUS FLUXES METHOD PROBLEMS FOR ELLIPTIC EQUATIONS

BAKHVALOV N.S., KNYAZEV A.V., MOSKVA, USSR

We consider a variant of the fictitious domain method for solving the Dirichlet problem

$$\frac{\partial}{\partial x_i}\left(K_{\mathcal{D}}\frac{\partial u_{\mathcal{D}}}{\partial x_i}\right)=f_{\mathcal{D}}\,,\ u_{\mathcal{D}}\in \overset{\circ}{W}{}_2^1(\mathcal{D}),\ f_{\mathcal{D}}\in W_2^{-1}(\mathcal{D}) \qquad (1)$$

with the diffusion coefficient $K_{\mathcal{D}}\in L_\infty(\mathcal{D}),\ K_{\mathcal{D}}\geq \underline{k}>0$ in a complex shaped domain $\mathcal{D}$. Let us complement $\mathcal{D}$ by "fictitious" domain $\mathcal{D}^\perp$ so, that in the resulting domain $\square$ (in fact $\overline{\square}=\overline{\mathcal{D}}\cup\overline{\mathcal{D}^\perp}$) the equation of the type $\Delta u=f\,,\ u\in \overset{\circ}{W}{}_2^1(\square)\,,\ f\in W_2^{-1}(\square)$ could be efficiently solved. In $\square$ we construct the Dirichlet problem

$$\frac{\partial}{\partial x_i}\left(K\frac{\partial u}{\partial x_i}\right)=f\,,\ u\in \overset{\circ}{W}{}_2^1(\square)\,,\ f\in W_2^{-1}(\square), \qquad (2)$$

where

$$K=\begin{cases} K_{\mathcal{D}} & \text{in } \mathcal{D}\,,\\ k & \text{in } \mathcal{D}^\perp\,, \end{cases} \qquad f=f_{\mathcal{D}} \text{ in } \mathcal{D}$$

and k is a large (may be infinitely large) constant. Let $\mathcal{D}$ and $\square$ be bounded Lipschitzian domains. The inequalities hold

$$\|\operatorname{grad} u-\operatorname{grad} u_{\mathcal{D}}\|_{L_2(\mathcal{D})}\leq c\,k^{-1},\ \|\operatorname{grad} u\|_{L_2(\mathcal{D}^\perp)}\leq c\,k^{-1} \qquad (3)$$

and if $\mathcal{D}$ is simply connected then

$$\|u-u_{\mathcal{D}}\|_{W_2^1(\mathcal{D})}\leq c\,k^{-1}\,,\ \|u\|_{W_2^1(\mathcal{D}^\perp)}\leq c\,k^{-1}.$$

In the case $k=+\infty$ one can speak about fictitious fluxes in $\mathcal{D}^\perp$ which provide, that $\operatorname{grad} u=0$ in $\mathcal{D}^\perp$, so that $\operatorname{grad} u = \operatorname{grad} u_{\mathcal{D}}$ in $\mathcal{D}$

We introduce the flux vector $p \equiv K \,\mathrm{grad}\, u$ and rewrite the equation (2) in the form

$$\begin{cases} \mathrm{div}\, p = f, & p \in \mathbb{L}_2(\square),\ f \in W_2^{-1}(\square), \\ K^{-1}p - \mathrm{grad}\, u = 0, & u \in \overset{\circ}{W}{}_2^1(\square). \end{cases} \qquad (4)$$

Let now $Q \equiv \mathrm{grad}\, \Delta^{-1}\mathrm{div} : \mathbb{L}_2(\square) \to \mathbb{L}_2(\square)$, where $\Delta \equiv \mathrm{div}\,\mathrm{grad} : \overset{\circ}{W}{}_2^1(\square) \to W_2^{-1}(\square)$. We can eliminate the unknown function u from (4) and obtain the equations

$$Q\,p = q, \quad Q^{\perp}K^{-1}p = 0, \quad q \equiv \mathrm{grad}\, \Delta^{-1} f, \quad Q^{\perp} \equiv I - Q \qquad (5)$$

and u could be computed, for example, by the formula $u = \Delta^{-1}\mathrm{div}\, K^{-1}p$ if the (approximate) solution p of (5) had been found before. The main point here is that the operator Q is the orthoprojector in $\mathbb{L}_2(\square)$ on the subspace $\mathbb{Q} \equiv \mathrm{grad}\, \overset{\circ}{W}{}_2^1(\square) \subset \mathbb{L}_2(\square)$. We need (5) exept (2) first of all to treat the case $k = +\infty$, when $K^{-1} \equiv 0$ in $\mathcal{D}^{\perp}$. In this case the uniqueness of p in (5) lost and we mean by p the normal (with minimal $\mathbb{L}_2(\square)$ norm) solution of (5).

Theorem 1. Let $1 \leq k \leq +\infty$. The unique solution p of (5) exists for every $q \in \mathbb{Q}$ and the inequality

$$\| p \|_{\mathbb{L}_2(\square)} \leq c \, \| q \|_{\mathbb{L}_2(\square)}, \quad c \neq c(k)$$

is true, i.e. problem (5) is well-posed uniformly in $k \geq 1$.

Theorem 2. Let p_k denote the solution p of (5) with $k \geq 1$ and p_∞ is the normal solution of (5) with $k = +\infty$. Then

$$\| p_k - p_\infty \|_{\mathbb{L}_2(\square)} \leq c\, k^{-1} \| q \|_{\mathbb{L}_2(\square)}.$$

The inequalities (3) are the consequences of this estimate.

The numerical solution of equations (5) can be obtained by the simple process

$$\frac{p^{n+1} - p^n}{\tau} + Q^{\perp}K^{-1}p^n = 0, \quad n = 0, 1, \dots, \quad p^0 = q. \qquad (6)$$

To study convergence properties of the method (6) we need the following statements.

Lemma 1. Let $\mathbb{N} \equiv \{ r \in \mathbb{L}_2(\square) : \mathrm{div}\, r = 0,\ r = \mathrm{grad}\, v \text{ in } \mathcal{D}^{\perp},$ for a $v \in \overset{\circ}{W}{}_2^1(\square) \}$

Then : (a) $\mathbb{N}$ is a subspace in $\mathbb{L}_2(\square)$,

(b) $\mathbb{N} = R(Q^{\perp}K^{-1})$ if $k = +\infty$,

(c) $\mathbb{N}$ is an invariant subspace for the operator $Q^{\perp}K^{-1}$ and $Q^{\perp}K^{-1} = Q^{\perp}K^{-1}Q^{\perp}$ on $\mathbb{N}$,

(d) for the solution p of (5) we have $p - q \in \mathbb{N}$

(e) the selfadjoint operator $Q^{\perp}K^{-1}Q^{\perp} : \mathbb{N} \to \mathbb{N}$

satisfies the inequalities

$$0 < cI \leq Q^{\perp}K^{-1}Q^{\perp} \leq \underline{k}^{-1}I \qquad \text{on } \mathbb{N}, \text{ i.e.}$$

$$0 < c\int_{\square} |r|^2 d\square \leq \int_{\square} K^{-1}|r|^2 d\square \leq \underline{k}^{-1}\int_{\square} |r|^2 d\square, \; r \in \mathbb{N}, \; r \neq 0$$

where $c \neq c(k)$ if $1 \leq k \leq +\infty$.

Now statements (a)-(c) enable us to prove the important

Lemma 2. Let $r^n \equiv p^n - q$. The the iterative method (6) is equivalent to $\dfrac{r^{n+1} - r^n}{\tau} + Q^{\perp}K^{-1}Q^{\perp}r^n = 0, \; n = 0, 1, \ldots, \; r^0 = 0$ and $r^n \in \mathbb{N}$.

From Lemma 1 (d), (e) and Lemma 2 follows immediately

Theorem 3. With appropriate choice of τ (for example, with $\tau = \underline{k}$) the iterative approximations p^n from (6) converge to the solution p of (5) as fast as geometric sequence uniformly in $k \geq 1$.

In the case $k = +\infty$, $K_{\mathcal{D}} \equiv 1$ we take $u^n \equiv \Delta^{-1} \operatorname{div} K^{-1} p^n$ and from (6) we have

$$\frac{u^{n+1} - u^n}{\tau} + u^n - \operatorname{div} K^{-1} \operatorname{grad} u^n = 0, \quad n = 0, 1, \ldots$$

Here $u^n \in \overset{\circ}{W}{}_2^1(\square)$, so the residuals will belong to $W_2^{-1}(\square)$ with support only on $\partial\mathcal{D}$.

There exist generalizations of the results. First of all, $K_{\mathcal{D}}$ and K might be matrices not functions. Secondly, one can substitute the original space $\overset{\circ}{W}{}_2^1(\square)$ for the space $W_2^1(\square)$ with other boundary conditions (for instance, with periodic boundary condition when $\square$ is a parallelepiped). Substituting in $W_2^1(\square)$ vector for scalar functions we get the method for solving elliptic systems. Some boundary value problems of the elasticity theory in displacements could also be solved if grad is changed to the matrix G, which transforms the displacements to an uppertriangle part of the deformation tensor, $-\operatorname{div}$ to G^T and $-\Delta$ to $G^T G$ with the boundary conditions of "rigit contact" on $\partial\square$: $u_{norm} = 0, \partial u_{tang} / \partial n = 0$ (in such a case the equation $G^T G u = \varphi$ could be solved by the Fourier Method).

References

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